ЧЕБЫШЕВСКИЙ СБОРНИК
Том 14 Выпуск 2 (2013)

УДК 519.14

THE ATKINSON TYPE FORMULA FOR THE PERIODIC ZETA-FUNCTION

S. Černigova (Vilnius), A. Laurinčikas (Vilnius)

Abstract

In the paper an explicit formula for the error term in the average mean square formula for the periodic zeta-function with rational parameter in the critical strip is obtained.

Keywords: Atkinson formula, generalized divisor function, periodic zeta-function.

ФОРМУЛА ТИПА АТКИНСОНА ДЛЯ ПЕРИОДИЧЕСКОЙ ДЗЕТА-ФУНКЦИИ

С. Чернигова (Вильнюс), А. Лауринчикас (Вильнюс)

Аннотация

В статье получена явная формула для остаточного члена в формуле для усредненного второго момента периодической дзета-функции с рациональным параметром в критической полосе.

Ключевые слова: периодическая дзета-функция, обобщенная функция делителей, формула Аткинсона.

1. Introduction

Denote, as usual, by $\zeta(s)$, $s = \sigma + it$, the Riemann zeta-function. In the theory of the function $\zeta(s)$, the moment problem occupies an important place. It consists of finding the asymptotic behavior for

$$\int_0^T |\zeta(\sigma+it)|^{2k} dt,\ \sigma \geq \frac{1}{2},\ k > 0,$$

as $T \to \infty$. Many attention is devoted to the mean square

$$J_\sigma(T)=\int_0^T |\zeta(\sigma+it)|^2 dt,$$

of $\zeta(s)$ for $\frac{1}{2}\leq\sigma\leq 1$. The asymptotics of $J_\sigma(T)$ as $T\to\infty$ is well-known. Let γ_0 denote the Euler constant, and

$$E(T)=J_{\frac{1}{2}}(T)-T\log\frac{T}{2\pi}-(2\gamma_0-1)T.$$

In [1], F. V. Atkinson obtained an interesting explicit formula for the error term $E(T)$ in the formula for $J_{\frac{1}{2}}(T)$. Let $0<c_1<c_2$ be two fixed constants such that $c_1T<N<c_2T$, and

$$N_1=N(T)=\frac{T}{2\pi}+\frac{N}{2}-\sqrt{\left(\frac{N^2}{4}+\frac{NT}{2\pi}\right)}.$$

Moreover, as usual, denote by $d(m)$, $m\in\mathbb{N}$, the divisor function, and define

$$\operatorname{arsinh}(x)=\log(x+\sqrt{1+x^2})$$

and

$$f(T,m)=2T\operatorname{arsinh}\left(\sqrt{\frac{\pi m}{2T}}\right)+\sqrt{2\pi mT+\pi^2m^2}-\frac{\pi}{4}.$$

Then Atkinson proved [1] that

$$\begin{aligned}E(T)=&\frac{1}{\sqrt{2}}\sum_{m\leq N}\frac{(-1)^m d(m)}{\sqrt{m}}\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi m}{2\pi}}\right)\right)^{-1}\left(\frac{T}{2\pi m}+\frac{1}{4}\right)^{-\frac{1}{4}}\cos(f(T,m))\\&-2\sum_{m\leq N_1}\frac{d(m)}{\sqrt{m}}\left(\log\frac{T}{2\pi m}\right)^{-1}\cos\left(T\log\frac{T}{2\pi m}-T+\frac{\pi}{4}\right)+O(\log^2 T).\end{aligned}\qquad(1)$$

The proof of the Atkinson formula is also given in [4]. The papers [5], [6], [14], [15], are devoted to modified versions of formula (1).

K. Matsumoto [11] and jointly with T. Meurman [12] obtained the analogue of the Atkinson formula in the critical strip. The second author [9], [10] gave a version of the Atkinson formula near the critical line.

The Atkinson formula is very useful in the theory of $\zeta(s)$. This formula allows to obtain various estimates for the error term $E_\sigma(T)$ in the formula for $J_\sigma(T)$, to study the mean square of $E_\sigma(T)$ and to continue other investigations of $J_\sigma(T)$. In [3], the Atkinson formula has been applied to obtain an estimate for the twelfth power moment of $\zeta(s)$.

Analogues of the Atkinson formula are also known for other zeta-function, for example, for Dirichlet L-function [13], and for the periodic zeta-function $\zeta_\lambda(s)$ [7], [8]. The function $\zeta_\lambda(s)$, $\lambda \in \mathbb{R}$, is defined, for $\sigma > 1$, by the series

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}}{m^s},$$

and by analytic continuation elsewhere. For $\lambda \in \mathbb{Z}$, the function $\zeta_\lambda(s)$ reduces to the Riemann zeta-function. In view of the periodicity of the coefficients $e^{2\pi i\lambda m}$, we may suppose that $0 < \lambda \leq 1$. In the above mentioned papers [7] and [8], the Atkinson type formula has been studied for the error term of

$$\sum_{a=1}^{q} \int_0^T |\zeta_\lambda(\sigma+it)|^2 dt,$$

where $\lambda = \frac{a}{q}$ with integers a and q, $1 \leq a \leq q$. In [8], the case $\sigma = \frac{1}{2}$ has been investigated, while the paper [7] deals with the case $\frac{1}{2} < \sigma < 1$. Let, for $\frac{1}{2} < \sigma < 1$,

$$E_\sigma(q,T) = \sum_{a=1}^{q} \int_0^T |\zeta_{\frac{a}{q}}(\sigma+it)|^2 dt - q\zeta(2\sigma)T - \frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}(qT)^{2-2\sigma}.$$

Then, in [7], an explicit formula for $E_\sigma(q,T)$ with a certain error term has been obtained. However, the error term of that formula with respect to q is not right. Therefore, the present paper is devoted to a more precise Atkinson type formula for $E_\sigma(q,T)$, and removes some inaccuracies of [7]. We limit ourselves to the case $\frac{1}{2} < \sigma < \frac{3}{4}$. We note that the method of investigation is analogical to that used for the Riemann zeta-function, however, some new problems arise from the involving of the parameter q.

Let $c_1 T < N < c_2 T$ with some positive constants $c_1 < c_2$. Define

$$N_1 = N_1(q,N,T) = q\left(\frac{T}{2\pi} + \frac{qN}{2} - \left(\left(\frac{qN}{2}\right)^2 + \frac{qNT}{2\pi}\right)^{\frac{1}{2}}\right),$$

denote by $\sigma_\alpha(m)$, $\alpha \in \mathbb{C}$, $m \in \mathbb{N}$, the generalized divisor function, i.e.,

$$\sigma_\alpha(m) = \sum_{d|m} d^\alpha,$$

and let

$$\sum\nolimits_1(q,T) = 2^{\sigma-1}q^{1-\sigma}\left(\frac{T}{\pi}\right)^{\frac{1}{2}-\sigma}\sum_{m\le N}\frac{(-1)^{qm}\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2T}}\right)\right)^{-1}\times$$
$$\times\left(\frac{T}{2\pi qm}+\frac{1}{4}\right)^{-\frac{1}{4}}\cos\left(2T\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2T}}\right)+\sqrt{2\pi qmT+T^2q^2m^2}-\frac{\pi}{4}\right)$$

and

$$\sum\nolimits_2(q,T) = -2q^{1-\sigma}\left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma}\sum_{m\le N_1}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\log\left(\frac{qT}{2\pi m}\right)\right)^{-1}$$
$$\times\cos\left(T\log\left(\frac{qT}{2\pi m}\right)-T+\frac{\pi}{4}\right).$$

Theorem 1. *Suppose that $\frac{1}{2}<\sigma<\frac{3}{4}$. Then, for $q\le T$,*

$$E_\sigma(q,T) = \sum\nolimits_1(q,T)+\sum\nolimits_2(q,T)+R(q,T),$$

where $R(q,T) = O(q^{\frac{7}{4}-\sigma}\log T)$, with the O - constant depending only on σ.

If $q = 1$, then we have the Atkinson formula for the Riemann zeta-function obtained in [11].

2. Lemmas

Lemma 1. *Let $\alpha\ne 1$, β, γ and $T\in\mathbb{R}_+$, $k\in\mathbb{R}$, $|k|\ge 1$, $0<a<\frac{1}{2}$, $a<\frac{T}{8\pi|k|}$ and $b\ge T$. Then, for every $\varepsilon>0$,*

$$\int_a^b \frac{\exp\{iT\log\frac{1+y}{y}+2\pi kiy\}dy}{y^\alpha(1+y)^\beta(\log\frac{1+y}{y})^\gamma} =$$
$$= \delta(k)(2k\sqrt{\pi})^{-1}T^{\frac{1}{2}}V^{-\gamma}U^{-\frac{1}{2}}\left(U-\frac{1}{2}\right)^{-\alpha}\left(U+\frac{1}{2}\right)^{-\beta}$$
$$\times\exp\left\{iTV+2\pi ikU-\pi ik+\frac{\pi i}{4}\right\}+O(a^{1-\alpha}T^{-1})+O(b^{\gamma-\alpha-\beta}|k|^{-1})+R(T,k)$$

uniformly for $|\alpha - 1| > \varepsilon$, *where*

$$U = \left(\frac{T}{2\pi k} + \frac{1}{4}\right)^{\frac{1}{2}},$$

$$V = 2\operatorname{arsinh}\left(\sqrt{\frac{\pi k}{2T}}\right),$$

$$R(T,k) = \begin{cases} T^{\frac{\gamma-\alpha-\beta}{2}-\frac{1}{4}}|k|^{-\frac{\gamma-\alpha-\beta}{2}-\frac{5}{4}} & \textit{if} \quad |k| \ll T, \\ T^{-\frac{1}{2}-\alpha}|k|^{\alpha-1} & \textit{if} \quad |k| \gg T, \end{cases}$$

and

$$\delta(k) = \begin{cases} 1 & \textit{if} \quad k > 0, \\ 0 & \textit{if} \quad k < 0. \end{cases}$$

The lemma is Lemma 2 of [1], see also Lemma 15.1 of [4]. In the above form, the lemma is stated in [11].

For a, b, $\alpha \in \mathbb{R}_+$, and m, $q \in \mathbb{N}$, define

$$I\left(a, b; \pm, \frac{m}{q}, \alpha\right) =$$

$$= \int_a^b x^{-\alpha}\left(\operatorname{arsinh}\left(x\sqrt{\frac{\pi q}{2T}}\right)\right)^{-1}\left(\frac{T}{2\pi x^2} + \frac{1}{4}\right)^{\frac{1}{4}}\left(\left(\frac{T}{2\pi x^2} + \frac{1}{4}\right)^{\frac{1}{2}} + \frac{1}{2}\right)^{-1} \times$$
$$\times \exp\left\{i\left(\pm 4\pi x\sqrt{\frac{m}{q}} - 2T\operatorname{arsinh}\left(x\sqrt{\frac{\pi}{2T}}\right) - \left(2\pi x^2 T + \pi^2 x^4\right)\right)\right\} dx.$$

Lemma 2. *Let* $c_1\sqrt{qT} < a < c_2\sqrt{qT}$ *with fixed* $0 < c_1 < c_2$. *Then*

$$I\left(a, b; \pm, \frac{m}{q}, \alpha\right) = 4\pi\delta T^{-1}\left(\frac{m}{q}\right)^{\frac{\alpha-1}{2}}\left(\log\left(\frac{Tq}{2\pi m}\right)\right)^{-1}\left(\frac{T}{2\pi} - \frac{m}{q}\right)^{\frac{3}{2}-\alpha}$$
$$\times \exp\left\{i\left(T - T\log\left(\frac{Tq}{2\pi m}\right) - \frac{2\pi m}{q} + \frac{\pi}{4}\right)\right\}$$

$$+ O\left(\delta\left(\frac{m}{q}\right)^{\frac{\alpha-1}{2}}\left(\frac{T}{2\pi}-\frac{m}{q}\right)^{1-\alpha}T^{-\frac{3}{2}}\right)$$
$$+ O\left(T^{-\frac{\alpha}{2}}\min\left(1,\left|a-\left(a^2-\frac{2T}{\pi}\right)^{\frac{1}{2}}\pm 2\sqrt{\frac{m}{q}}\right|^{-1}\right)\right)$$
$$+ O\left(b^{-\alpha}\left(\frac{n}{q}\right)^{\frac{1}{2}}+O\left(\frac{T}{b}\right)^{-1}\right)$$
$$+ O\left(e^{-CT-C\sqrt{\frac{mT}{q}}}\right)$$

with a large constant $C>0$, *where*

$$\delta=\begin{cases}1 & \text{if} \quad m\le \frac{Tq}{2\pi},\ ma^2\le\left(\frac{Tq^2}{2\pi}-mq\right)^2\le mb^2 \\ & \quad \text{and the double sign takes } +, \\ 0 & \text{otherwise.}\end{cases}$$

The lemma is a slight modification of Lemma 3 from [1], see also Lemma 15.2 of [4]. The statement of the lemma follows that of Lemma 4 of [11].

The next lemmas are related to the function $\sigma_{1-2\sigma}(m)$. Let

$$D_\sigma(x)=\sum_{m\le x}{}' \sigma_{1-2\sigma}(m),$$

where the sign "$'$"means that the last term in the sum is to be halved if $x\in\mathbb{N}$. Define $\Delta_{1-2\sigma}(x)$ by

$$\Delta_{1-2\sigma}(x)=D_\sigma(x)-\zeta(2\sigma)x-\frac{\zeta(2-2\sigma)x^{2-2\sigma}}{2-2\sigma}+\frac{\zeta(2\sigma-1)}{2}.$$

Lemma 3. *For every* $\varepsilon>0$,

$$\Delta_{1-2\sigma}(x)=O(x^{\frac{1}{4\sigma+1}+\varepsilon}).$$

The lemma is Lemma 2 from [11].

Lemma 4. *We have*

$$\Delta_{1-2\sigma}(x)=\frac{x^{\frac{3}{4}-\sigma}}{\sqrt{2}\pi}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\times$$
$$\times\left(\cos\left(4\pi\sqrt{mx}-\frac{\pi}{4}\right)-(32\pi\sqrt{mx})^{-1}(16(1-\sigma)^2-1)\sin\left(4\pi\sqrt{mx}-\frac{\pi}{4}\right)\right)+$$
$$+O(x^{-\frac{1}{4}-\sigma}),$$

the series being boundedly convergent in any fixed finite interval of x.

The lemma is Lemma 1 of [11], and is a result of [16] and [2].

3. A formula for $E_\sigma(q,T)$

Let u and v be complex variables, $Re u>1$ and $Re v>1$. Then we have

$$\sum_{a=1}^{q}\zeta_{\frac{a}{q}}(u)\zeta_{-\frac{a}{q}}(v)=\sum_{a=1}^{q}\sum_{m=1}^{\infty}\frac{2\pi i\frac{a}{q}m}{m^u}\sum_{n=1}^{\infty}\frac{e^{-2\pi i\frac{a}{q}n}}{n^v}$$
$$=q\zeta(u+v)+\sum_{a=1}^{q}\sum_{m=1}^{\infty}\sum_{\substack{n=1\\ m\neq n}}^{\infty}\frac{e^{2\pi i\frac{a}{q}(m-n)}}{m^u n^v}. \qquad (2)$$

Since

$$\sum_{a=1}^{q}e^{2\pi\frac{a}{q}(m-n)}=\begin{cases} q & \text{if} \quad m\equiv n \pmod q,\\ 0 & \text{if} \quad m\not\equiv n \pmod q,\end{cases}$$

we have from (2) that

$$\sum_{a=1}^{q}\zeta_{\frac{a}{q}}(u)\zeta_{-\frac{a}{q}}(v)=q(\zeta(u+v)+f_q(u,v)+f_q(v,u)), \qquad (3)$$

where

$$f_q(u,v)=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{1}{m_1^u(m_1+qm_2)^v}.$$

Using the Poisson summation formula and properties of the gamma-function $\Gamma(s)$, we find that, for $Re(u+v)>2$ and $Re u<0$,

$$f_q(u,v)=\frac{\zeta(u+v-1)\Gamma(u+v-1)\Gamma(1-u)}{q^{u+v-1}\Gamma(v)}+g_q(u,v), \qquad (4)$$

where

$$g_q(u,v)=\frac{2}{q^{u+v-1}}\sum_{m=1}^{\infty}\sigma_{1-u-v}(m)\int_0^{\infty}\frac{\cos(2\pi mqy)dy}{y^u(1+y)^v}.$$

We need the analytics continuation for $g_q(u,v)$ to a certain region lying in $0<Re u<1$, $0<Re v<1$. Suppose that we have such an analytic continuation. Then, in view of (3) and (4), we find that

$$\sum_{a=1}^{q} \zeta_{\frac{a}{q}}(u)\zeta_{-\frac{a}{q}}(v) = q\left(\zeta(u+v) + \frac{\zeta(u+v-1)\Gamma(u+v-1)\Gamma(1-u)}{q^{u+v-1}\Gamma(v)}\right.$$
$$\left. + \frac{\zeta(u+v-1)\Gamma(u+v-1)\Gamma(1-v)}{q^{u+v-1}\Gamma(u)}\right) + g\left(g_q(u,v) + g_q(v,u)\right).$$

In the latter equality, we take $u = \sigma + it$ and $v = 2\sigma - u = \sigma - it$. Then, using the estimate [12]

$$\int_0^T \left(\frac{\Gamma(1-\sigma-it)}{\Gamma(\sigma-it)} + \frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}\right) dt = \frac{\sin(\pi\sigma)}{1-\sigma} T^{2-2\sigma} + O(T^{-2\sigma}),$$

we obtain that

$$\sum_{a=1}^{q} \int_0^T |\zeta_{\frac{a}{q}}(\sigma+it)|^2 dt = q\zeta(2\sigma)T + \frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}(qT)^{2-2\sigma}$$
$$- iq \int_{\sigma-iT}^{\sigma+iT} g_q(u, 2\sigma-u)du + O(qT^{-2\sigma}). \qquad (5)$$

Now we consider the function $g_q(u, 2\sigma - u)$. Define

$$h(u,x) = 2\int_0^\infty \frac{\cos(2\pi xy)dy}{y^u(1+y)^{2\sigma-u}}.$$

Then, by the definition of $g_q(u,v)$,

$$g_q(u, 2\sigma-u) = \frac{1}{q^{2\sigma-1}} \sum_{m=1}^{\infty} \sigma_{1-2\sigma}(m)h(u, mq). \qquad (6)$$

Suppose that $N \in \mathbb{N}$, and let $X = N + \frac{1}{2}$. Then, by the definition of $D_{1-2\sigma}(x)$ and $\Delta_{1-2\sigma}(x)$, we have that

$$\sum_{m>N} \sigma_{1-2\sigma}(m)h(u,mq) = \int_X^\infty h(u,qx)dD_{1-2\sigma}(x)$$
$$= \int_X^\infty (\zeta(2\sigma) + \zeta(2-2\sigma)x^{1-2\sigma})h(u,qx)dx$$
$$+ \int_X^\infty h(u,qx)d\Delta_{1-2\sigma}(x)$$
$$= -\Delta(X)h(u,qX) - \int_X^\infty \Delta_{1-2\sigma}(x)\frac{\partial h(u,qx)}{\partial x}$$
$$+ \int_X^\infty (\zeta(2\sigma) + \zeta(2-2\sigma)x^{1-2\sigma})h(u,qx)dx.$$

This and (6) show that

$$\begin{aligned} g_q(u, 2\sigma - u) &= \frac{1}{q^{2\sigma-1}} \sum_{m \le N} \sigma_{1-2\sigma}(m) h(u, mq) - \frac{1}{q^{2\sigma-1}} \Delta_{1-2\sigma}(X) h(u, qX) \\ &- \frac{1}{q^{2\sigma-1}} \int_X^\infty \Delta_{1-2\sigma}(x) \frac{\partial h(u, qx)}{\partial x} dx \\ &+ \frac{1}{q^{2\sigma-1}} \int_X^\infty (\zeta(2\sigma) + \zeta(2-2\sigma) x^{1-2\sigma}) h(u, qx) dx \\ &\stackrel{\text{def}}{=} g_{q,1}(u) - g_{q,2}(u) - g_{q,3}(u) + g_{q,4}(u). \end{aligned} \tag{7}$$

By the definition, the function $h(u, x)$ is analytic in the $Re u < 1$. Therefore, the functions $g_{q,1}(u)$ and $g_{q,2}(u)$ also are analytic in the latter region.

Using Lemma 3 and estimate [1]

$$\frac{\partial h(u, x)}{\partial x} = O(x^{Re u - 2}),$$

we obtain that

$$\frac{1}{q^{2\sigma-1}} \int_X^\infty \Delta_{1-2\sigma}(x) \frac{\partial h(u, qx)}{\partial x} dx \ll q^{Re u - 2\sigma} \int_X^\infty x^{Re u + \frac{1}{4\sigma+1} - 2 + \varepsilon} dx,$$

and the integral is convergent for $Re u < 1 - \frac{1}{4\sigma+1}$. Since $1 - \frac{1}{4\sigma+1} > \sigma$ for $\sigma < \frac{3}{4}$, we have that the function $g_{q,3}(u)$ is analytic in the region including the line $Re u = \sigma$.

It is easily seen that

$$\begin{aligned} g_{q,4}(u) &= \frac{1}{q^{2\sigma-1}} \int_X^\infty (\zeta(2\sigma) + \zeta(2-2\sigma) x^{1-2\sigma}) \\ &\times \left(\int_0^{i\infty} \frac{e^{2\pi i q x y} dy}{y^u (1+y)^{2\sigma-u}} + \int_0^{-i\infty} \frac{e^{-2\pi i q x y} dy}{y^u (1+y)^{2\sigma-u}} \right) dx. \end{aligned}$$

Suppose that $Re u < 0$. Then

$$\begin{aligned} &\frac{1}{q^{2\sigma-1}} \int_X^\infty \left((\zeta(2\sigma) + \zeta(2-2\sigma) x^{1-2\sigma}) \int_0^{i\infty} \frac{e^{2\pi i q x y} dy}{y^u (1+y)^{2\sigma-u}} \right) dx \\ &= \frac{1}{2\pi i q^{2\sigma}} (\zeta(2\sigma) + \zeta(2-2\sigma) x^{1-2\sigma}) \int_0^{i\infty} \frac{e^{2\pi i q x y} dy}{y^{u+1} (1+y)^{2\sigma-u}} \Big|_X^\infty \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2\pi q^{2\sigma}}\int_X^\infty\left(\left(\zeta(2-2\sigma)(1-2\sigma)x^{-2\sigma}\right)\int_0^\infty\frac{e^{2\pi iqxy}dy}{y^{u+1}(1+y)^{2\sigma-u}}\right)dx\\
&=-\frac{1}{2\pi iq^{2\sigma}}\left(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma}\right)\int_0^{i\infty}\frac{e^{2\pi iqXy}dy}{y^{u+1}(1+y)^{2\sigma-u}}\\
&-\frac{\zeta(2-2\sigma)(1-2\sigma)}{2\pi iq^{2\sigma}}\int_X^\infty dx\int_0^{i\infty}\frac{e^{2\pi iqy}dy}{y^{u+1}(x+y)^{2\sigma-u}}\\
&=-\frac{1}{2\pi iq^{2\sigma}}\left(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma}\right)\int_0^{\infty}\frac{e^{2\pi iqXy}dy}{y^{u+1}(1+y)^{2\sigma-u}}\\
&-\frac{\zeta(2-2\sigma)(1-2\sigma)X^{1-2\sigma}}{2\pi iq^{2\sigma}(2\sigma-u-1)}\int_0^{\infty}\frac{e^{2\pi iqXy}dy}{y^{u+1}(1+y)^{2\sigma-u-1}}.
\end{aligned}
$$

Similarly, we find that

$$
\begin{aligned}
&\frac{1}{q^{2\sigma-1}}\int_X^\infty\left(\left(\zeta(2\sigma)+\zeta(2-2\sigma)x^{1-2\sigma}\right)\int_0^{-i\infty}\frac{e^{-2\pi iqxy}dy}{y^{u+1}(1+y)^{2\sigma-u}}\right)dx\\
&=\frac{1}{2\pi iq^{2\sigma-2}}\left(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma}\right)\int_0^{\infty}\frac{e^{-2\pi iqXy}dy}{y^{u+1}(1+y)^{2\sigma-u}}\\
&+\frac{\zeta(2-2\sigma)(1-2\sigma)X^{1-2\sigma}}{2\pi iq^{2\sigma}(2\sigma-u-1)}\int_0^{\infty}\frac{e^{-2\pi iqXy}dy}{y^{u+1}(1+y)^{2\sigma-u-1}}.
\end{aligned}
$$

The later two qualities yield

$$
\begin{aligned}
g_{q,4}(u)=&-\frac{1}{\pi q^{2\sigma}}\left(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma}\right)\int_0^{\infty}\frac{\sin(2\pi qXy)dy}{y^{u+1}(1+y)^{2\sigma-u}}\\
&-\frac{\zeta(2-2\sigma)(1-2\sigma)X^{1-2\sigma}}{\pi q^{2\sigma-2}(2\sigma-u-1)}\int_0^{\infty}\frac{\sin(2\pi qXy)dy}{y^{u+1}(1+y)^{2\sigma-u-1}}. \qquad (8)
\end{aligned}
$$

The above integrals are convergent absolutely for $Re u < 1$. Thus, we have analytic continuation for $g_{q,4}(u)$ to the suitable region. Consequently, (5) is true for $\frac{1}{2} < \sigma < \frac{3}{4}$.

From (5) we find that, for $\frac{1}{2} < \sigma < \frac{3}{4}$,

$$
E_\sigma(q,T)=-iq\int_{\sigma-iT}^{\sigma+iT}g_q(u,2\sigma-u)+O(qT^{-2\sigma}).
$$

Therefore, in view of (7),

$$
E_\sigma(q,T)=-iq^{2-2\sigma}(G_{q,1}-G_{q,2}-G_{q,3}+G_{q,4})+O(qT^{-2\sigma}), \qquad (9)
$$

where

$$G_{q,1} = 2\sum_{m\le N}\sigma_{1-2\sigma}(m)\int_0^\infty\left(\frac{\cos(2\pi qmy)}{(1+y)^{2\sigma}}\int_{\sigma-iT}^{\sigma+iT}\left(\frac{1+y}{y}\right)^u du\right)dy$$
$$= 4i\sum_{m\le N}\sigma_{1-2\sigma}(m)\int_0^\infty\frac{\cos(2\pi qmy)\sin(T\log\frac{1+y}{y})dy}{y^\sigma(1+y)^\sigma\log\frac{1+y}{y}},$$

$$G_{q,2} = 4i\Delta_{1-2\sigma}(X)\int_0^\infty\frac{\cos(2\pi qXy)\sin(T\log\frac{1+y}{y})dy}{y^\sigma(1+y)^\sigma\log\frac{1+y}{y}},$$

$$G_{q,3} = 4i\int_X^\infty\Delta_{1-2\sigma}(x)\frac{\partial}{\partial x}\left(\int_0^\infty\frac{\cos(2\pi qxy)\sin(T\log\frac{1+y}{y})dy}{y^\sigma(1+y)^\sigma\log\frac{1+y}{y}}\right)dx$$

$$= 4i\int_X^\infty\Delta_{1-2\sigma}(x)\frac{\partial}{\partial x}\left(\int_0^\infty\frac{\cos(2\pi qy)\sin(T\log\frac{x+y}{y})dy}{y^\sigma(x+y)^\sigma x^{1-2\sigma}\log\frac{x+y}{y}}\right)dx$$
$$= 4i\int_X^\infty\Delta_{1-2\sigma}(x)\int_0^\infty\frac{\cos(2\pi qy)}{y^\sigma}\left(\frac{(2\sigma-1)x^{2\sigma-2}\sin(T\log\frac{x+y}{y})}{(x+y)^\sigma\log\frac{x+y}{y}}\right.$$
$$\left.+\frac{x^{2\sigma-1}T\cos(T\log\frac{x+y}{y})}{(x+y)^{\sigma+1}}-\frac{\sigma x^{2\sigma-1}\sin(T\log\frac{x+y}{y})}{(x+y)^{\sigma+1}\log\frac{x+y}{y}}-\frac{x^{2\sigma-1}\sin(T\log\frac{x+y}{y})}{(x+y)^{\sigma+1}\log^2\frac{x+y}{y}}\right)dxdy$$
$$= 4i\int_X^\infty\frac{\Delta_{1-2\sigma}(x)}{x}\left(\int_0^\infty\frac{\cos(2\pi qxy)}{y^\sigma(1+y)^{\sigma+1}\log\frac{1+y}{y}}\left(T\cos\left(T\log\frac{1+y}{y}\right)\right.\right.$$
$$\left.\left.+\sin\left(T\log\frac{1+y}{y}\right)\left((2\sigma-1)(1+y)-\sigma-\frac{1}{\log\frac{1+y}{y}}\right)\right)dy\right)dx,$$

$$G_{q,4} = -\frac{2i}{\pi q}(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma})\int_0^\infty\frac{\sin(2\pi qXy)\sin(T\log\frac{1+y}{y})}{y^{\sigma+1}(1+y)^\sigma\log\frac{1+y}{y}}$$
$$+\frac{(1-2\sigma)\zeta(2-2\sigma)X^{1-2\sigma}}{\pi q}\int_0^\infty\left(\frac{\sin(2\pi qXy)}{y(1+y)^{2\sigma-1}}\int_{\sigma-it}^{\sigma+iT}\frac{(\frac{1+y}{y})^u du}{u-2\sigma+1}\right)dy.$$

4. Proof of Theorem 1

By (9), it suffices to evaluate $G_{q,1}-G_{q,4}$. For evaluation of $G_{q,1}$, we apply Lemma 1 with $\alpha=\beta=\sigma$, $\gamma=1$, $k=qm$ and $k=-qm$. Then taking $T\ll n\ll T$ gives

$$G_{q,1} = 2^{\sigma-1}q^{\sigma-1}\left(\frac{\pi}{T}\right)^{\sigma-\frac{1}{2}} i\times$$
$$\times \sum_{m\le N} \sigma_{1-2\sigma}(m)m^{\sigma-1}V^{-1}U^{-\frac{1}{2}}\sin(TV + 2\pi qmU - \pi qm + \frac{\pi}{4})+$$
$$+O(\max(T^{\frac{1}{4}-\sigma}q^{-\frac{7}{4}+\sigma}, T^{-\frac{1}{2}})) =$$
$$= 2^{\sigma-1}q^{\sigma-1}\left(\frac{\pi}{T}\right)^{\sigma-\frac{1}{2}} i\sum_{m\le N}(-1)^{qm}\sigma_{1-2\sigma}(m)m^{\sigma-1}\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi mq}{2\pi}}\right)\right)^{-1}\times$$
$$\times\left(\frac{T}{2\pi mq}+\frac{1}{4}\right)^{-\frac{1}{4}}\cos\left(2T\operatorname{arsinh}\left(\sqrt{\frac{\pi mq}{2\pi}}\right) + 2\pi qm\left(\frac{T}{2\pi qm}+\frac{1}{4}\right)^{\frac{1}{2}} - \frac{\pi}{4}\right)+$$
$$+O\left(\max\left(T^{\frac{1}{4}-\sigma}q^{-\frac{7}{4}+\sigma}, T^{-\frac{1}{2}}\right)\right). \tag{10}$$

For $G_{q,2}$, it is sufficient to obtain an estimate. Lemma 1 implies that

$$G_{q,2} = O(\Delta_{1-2\sigma}(X)q^{\sigma-1}T^{\frac{1}{2}-\sigma}X^{\sigma-1}\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qX}{2T}}\right)\right)^{-1}$$
$$\times\left(\frac{T}{2\pi Xq}+\frac{1}{4}\right)^{-\frac{1}{4}} + O(\Delta_{1-2\sigma}(X)T^{-\frac{3}{2}}q^{\sigma-1}).$$

Therefore, in view of Lemma 3,

$$G_{q,2} = O\left(T^{\frac{1-4\sigma}{2(4\sigma+1)}+\varepsilon}q^{\sigma-1}(\log q)^{-1}\right) + O\left(T^{\frac{1}{1-4\sigma}-\frac{3}{2}+\varepsilon}q^{\sigma-1}\right) = O\left(T^{\frac{1-4\sigma}{2(4\sigma+1)}+\varepsilon}q^{\sigma-1}\right). \tag{11}$$

Now we will deal with $G_{q,4}$. First we observe that, in virtue of the residue theorem, for $0 < y \le 1$,

$$\int_{\sigma-iT}^{\sigma+iT}\frac{\left(\frac{1+y}{y}\right)^u du}{u-2\sigma+1} = 2\pi i Res_{u=2\sigma-1}(...)-$$
$$-\left(\int_{\sigma+iT}^{-\infty+iT}+\int_{-\infty-iT}^{\sigma-iT}\right)\left(\frac{1+y}{y}\right)^u\frac{du}{u-2\sigma+1} = 2\pi i\left(\frac{1+y}{y}\right)^{2\sigma-1} + O(T^{-1}y^{-\sigma}).$$

Moreover, for $y \ge 1$,

$$\int_{\sigma-iT}^{\sigma+iT}\left(\frac{1+y}{y}\right)^u\frac{du}{u-2\sigma+1} = O\left(\int_{\sigma-iT}^{\sigma+iT}\left|\frac{du}{u-2\sigma+1}\right|\right) = O(\log T).$$

Thus,

$$\int_0^\infty \left(\frac{\sin(2\pi qXy)}{y(1+y)^{2\sigma-1}} \int_{\sigma-iT}^{\sigma+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u-2\sigma+1}\right) dy = \left(\int_0^1 + \int_1^\infty\right)(...)dy$$
$$= 2\pi i \int_0^1 \frac{\sin(2\pi qXy)}{y^{2\sigma}} dy + O\left(T^{-1}\int_0^1 \frac{|\sin(2\pi qXy)|dy}{y^{\sigma+1}}\right)$$
$$+ \int_1^\infty \left(\frac{\sin(2\pi qXy)}{y(1+y)^{2\sigma-1}} \int_{\sigma-iT}^{\sigma+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u-2\sigma+1}\right) dy.$$

We have that

$$2\pi i \int_0^1 \frac{\sin(2\pi qXy)dy}{y^{2\sigma}} = 2\pi i \int_0^\infty \frac{\sin(2\pi qXy)}{y^{2\sigma}} dy + O(T^{-1}q^{-1}) =$$
$$= 2\pi i (2\pi qX)^{2\sigma-1} \int_0^\infty \frac{\sin y dy}{y^{2\sigma}} + O(T^{-1}q^{-1}) =$$
$$= (2\pi)^{2\sigma}(qX)^{2\sigma-1} i \frac{\pi}{2\Gamma(2\sigma)\sin(\pi\sigma)} + O(T^{-1}q^{-1}),$$

$$T^{-1}\int_0^1 \frac{\sin(2\pi qXy)dy}{y^{\sigma+1}} =$$
$$= O\left(T^{-1}qX \int_0^{(qX)^{-1}} \frac{dy}{y^\sigma}\right) + O\left(T^{-1}\int_{(qX)^{-1}}^\infty \frac{dy}{y^{\sigma+1}}\right) = O(q^\sigma T^{\sigma-1}),$$

and, in view of the estimate

$$\int_{\sigma-iT}^{\sigma+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u-2\sigma+1} = O(\log T),$$

$$\int_1^\infty \left(\frac{\sin(2\pi qXy)}{y(1+y)^{2\sigma-1}} \int_{\sigma-iT}^{\sigma+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u-2\sigma+1}\right) dy$$
$$= \left(-\frac{\cos(2\pi qXy)}{2\pi qXy(1+y)^{2\sigma-1}} \int_{\sigma-iT}^{\sigma+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u-2\sigma+1}\right)\Big|_1^\infty$$
$$- \int_1^\infty \left(\frac{\cos(2\pi qXy)}{2\pi qXy^2(1+y)^{2\sigma-1}} \int_{\sigma-iT}^{\sigma+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u-2\sigma+1}\right) dy$$
$$+ (1-2\sigma)\int_1^\infty \left(\frac{\cos(2\pi qXy)}{2\pi qXy(1+y)^{2\sigma}} \int_{\sigma-iT}^{\sigma+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u-2\sigma+1}\right) dy$$
$$- \int_1^\infty \left(\frac{\cos(2\pi qXy)}{2\pi qXy(1+y)^{2\sigma-1}} \int_{\sigma-iT}^{\sigma+iT} \left(\frac{1+y}{y}\right)^{u-1} \frac{du}{y^2(u-2\sigma+1)}\right) dy$$
$$= O(q^{-1}T^{-1}\log T).$$

All these estimates show that the second term in the formula for $G_{q,4}$ is

$$i\pi(2\pi)^{2\sigma-1}(1-2\sigma)q^{2\sigma-2}\frac{1}{\Gamma(2\sigma)\sin(\pi\sigma)}+O(q^{\sigma-1}T^{1-\sigma}). \tag{12}$$

For the evaluation of the first term of $G_{q,4}$, we apply the second mean value theorem and Lemma 1. We write the integral as

$$\int_0^\infty (...)dy = \left(\int_0^{(2qX)^{-1}} + \int_{(2qX)^{-1}}^\infty\right)(...)dy.$$

Then

$$\begin{aligned}\int_0^{(2qX)^{-1}} (...)dy &\le 2\pi qX\int_0^\beta \frac{\sin(T\log\frac{1+y}{y})y^{1-\sigma}(1+y)^{1-\sigma}}{y(1+y)\log\frac{1+y}{y}}dy\\ &= \frac{2\pi qX\beta^{1-\sigma}(1+\beta)^{1-\sigma}}{\log\frac{1+\beta}{\beta}}\int_\alpha^\beta \frac{\sin(T\log\frac{1+y}{y})dy}{y(1+y)}\\ &= \frac{2\pi qX\beta^{1-\sigma}(1+\beta)^{1-\sigma}}{\log\frac{1+\beta}{\beta}}\left(T^{-1}\cos\left(T\log\frac{1+y}{y}\right)\right)\Big|_\alpha^\beta = O(q^\sigma T^{\sigma-1}),\end{aligned}$$

where $0\le\alpha\le\beta\le(2qX)^{-1}$. Moreover, an application of Lemma 1 gives the estimate

$$\int_{(2qX)^{-1}}^\infty (...)dy = O(q^\sigma T^{\sigma-1}).$$

From these estimates and (12), we obtain that

$$G_{q,4} = i\pi(2\pi)^{2\sigma-1}(1-2\sigma)q^{2\sigma-2}\frac{1}{\Gamma(2\sigma)\sin(\pi\sigma)}+O(q^{\sigma-1}T^{\sigma-1}). \tag{13}$$

The most complicated is the integral $G_{q,3}$. We apply Lemma 1 again and find that, for $x\gg T$,

$$\int_0^\infty \frac{\cos(2\pi qxy)}{y^\sigma(1+y)^{\sigma+1}\log\frac{1+y}{y}}\left(T\cos\left(T\log\frac{1+y}{y}\right)\right)+\sin\left(T\log\frac{1+y}{y}\right)$$

$$\times\left((2\sigma-1)(1+y)-\sigma-\left(\log\frac{1+y}{y}\right)^{-1}\right)dy = i2^{2\sigma-1}\pi^{\sigma-\frac{1}{2}}q^{\sigma-1}x^{\sigma-1}T^{\frac{3}{2}-\sigma}$$
$$\times\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qx}{2T}}\right)\right)^{-1}\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{-\frac{1}{4}}\left(\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{\frac{1}{2}}+\frac{1}{2}\right)^{-1}$$
$$\times\cos\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qx}{2T}}\right)+2\pi qx\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{\frac{1}{2}}-\pi qx+\frac{\pi}{4}\right)+O(q^{\sigma-1}T^{\frac{1}{2}-\sigma}x^{\sigma-1}).$$

Hence,

$$G_{q,3} = i2^{\sigma-1}\pi^{\sigma-\frac{1}{2}}q^{\sigma-1}T^{\frac{3}{2}-\sigma}\int_X^\infty\frac{\Delta_{1-2\sigma}(x)}{x^{2-\sigma}}\left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qx}{2T}}\right)\right)^{-1}$$

$$\times\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{-\frac{1}{4}}\left(\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{\frac{1}{2}}+\frac{1}{2}\right)^{-1}$$
$$\times\cos\left(2T\operatorname{arsinh}\left(\sqrt{\frac{\pi qx}{2T}}\right)+2\pi qx\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{-\frac{1}{2}}-\pi qx+\frac{\pi}{4}\right)dx$$
$$+O\left(q^{\sigma-1}T^{\frac{1}{2}-\sigma}\int_X^\infty\frac{\Delta_{1-2\sigma}(x)}{x}dx\right). \qquad (14)$$

In remains to evaluate and estimate the latter integrals.

Using Lemma 3 and the restriction $\frac{1}{2}<\sigma<\frac{3}{4}$, we obtain that

$$q^{\sigma-1}T^{\frac{1}{2}-\sigma}\int_X^\infty\frac{\Delta_{1-2\sigma}(x)dx}{x^{2-\sigma}} = O(q^{\sigma-1}T^{\frac{1-4\sigma}{2(1+4\sigma)}+\varepsilon}). \qquad (15)$$

For the evaluation of the first integral in (14), we apply Lemma 4 and the argument proposed in [11] to avoid the problem arising from the bounded convergence of the series in Lemma 4. Thus, by (14) and (15),

$$G_{q,3} = iq^{\sigma-\frac{3}{4}}\left(\frac{T}{2\pi}\right)^{\frac{3}{2}-\sigma}\lim_{b\to\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\int_{\sqrt{qX}}^b x^{-\frac{3}{2}}\left(\cos\left(4\pi x\sqrt{\frac{m}{q}}-\frac{\pi}{4}\right)\right.$$
$$\left.-\left(32\pi x\sqrt{\frac{m}{q}}\right)^{-1}\times\left(16(1-\sigma)^2-1\right)\sin\left(4\pi x-\sqrt{\frac{m}{q}}-\frac{\pi}{4}\right)\right)$$
$$\times\left(\operatorname{arsinh}\left(x\sqrt{\frac{\pi}{2T}}\right)\right)^{-1}\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^{-\frac{1}{4}}\left(\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^2+\frac{1}{2}\right)^{-1}$$
$$\times\cos\left(2T\operatorname{arsinh}\left(x\sqrt{\frac{\pi}{2T}}\right)+(2\pi x^2T+\pi^2x^4)^{\frac{1}{2}}-\pi x^2+\frac{\pi}{4}\right)dx$$
$$+O\left(q^{\sigma-1}T^{\frac{1-4\sigma}{2(1+4\sigma)}+\varepsilon}\right).$$

In the notation of Lemma 2, this can be rewritten in the form

$$
\begin{aligned}
G_{q,3} = {} & iq^{\sigma-\frac{3}{4}}\left(\frac{T}{2\pi}\right)^{\frac{3}{2}-\sigma}\lim_{b\to\infty}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}} \\
& \times\left(ReI\left(\sqrt{qX}, b; -, \frac{m}{q}, \frac{3}{2}\right) + ImI\left(\sqrt{qX}, b; +, \frac{m}{q}, \frac{3}{2}\right) + \left(32\pi x\sqrt{\frac{m}{q}}\right)^{-1}\right. \\
& \left.\times\left(16(1-\sigma)^2-1\right)\left(Im\left(\sqrt{qX}, b; +, \frac{m}{q}, \frac{5}{2}\right) + ReI\left(\sqrt{qX}, b; +, \frac{m}{q}, \frac{5}{2}\right)\right)\right) \\
& + O\left(q^{\sigma-1}T^{\frac{1-4\sigma}{2(1+4\sigma)}+\varepsilon}\right). \qquad (16)
\end{aligned}
$$

Define

$$
Z = q\left(\frac{T}{2\pi}+\frac{qX}{2}\right) - \left(\left(\frac{qX}{2}\right)^2 + \frac{qXT}{2\pi}\right)^{\frac{1}{2}}.
$$

Then an application of Lemma 2 with $\alpha = \frac{3}{2}$ and $\alpha = \frac{5}{2}$, and $a = \sqrt{qX}$ for (16) yields

$$
\begin{aligned}
G_{q,3} = {} & iq^{\sigma-\frac{3}{4}}\left(\frac{T}{2\pi}\right)^{\frac{3}{2}-\sigma}\lim_{b\to\infty}\left(4\pi q^{-\frac{1}{4}}T^{-1}\sum_{m\le Z}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\right. \\
& \times\left(\log\left(\frac{Tq}{2\pi m}\right)^{-1}\right)\cos\left(T\log\left(\frac{Tq}{2\pi m}\right) - T + \frac{\pi}{4}\right) \\
& + O\left(q^{-\frac{1}{4}}T^{-1}\sum_{m\le Z}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\log\left(\frac{Tq}{2\pi m}\right)\right)^{-1}\left(\frac{T}{2\pi}-\frac{m}{q}\right)^{-1}\right) \\
& + O\left(q^{-\frac{1}{4}}T^{-\frac{3}{2}}\sum_{m\le Z}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\frac{T}{2\pi}-\frac{m}{q}\right)^{-\frac{1}{2}}\right) \\
& + O\left(b^{-\frac{3}{2}}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\left(\left(\frac{m}{q}\right)^{\frac{1}{2}} + O\left(\frac{T}{b}\right)\right)^{-1}\right) \\
& + O\left(e^{-CT}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}e^{-C\sqrt{\frac{mT}{q}}}\right) \\
& \left.+ O\left(T^{-\frac{3}{4}}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\min\left(1, \left|(q,X)^{\frac{1}{2}} - \left(qX+\frac{2T}{\pi}\right)^{\frac{1}{2}} + 2\sqrt{\frac{m}{q}}\right|^{-1}\right)\right)\right) \\
& + O\left(q^{\sigma-1}T^{\frac{1-4\sigma}{2(1+4\sigma)}+\varepsilon}\right). \qquad (17)
\end{aligned}
$$

Since $\frac{1}{2} < \sigma < \frac{3}{4}$, we have that

$$b^{-\frac{3}{2}} \sum_{m=1}^{\infty} \frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}}} \left(\left(\frac{m}{q} \right)^{\frac{1}{2}} + \left(\frac{T}{b} \right) \right)^{-1} \to 0 \tag{18}$$

as $b \to \infty$.

From the definition of Z, it follows that $Z \ll T$. Thus, $\frac{Tq}{2\pi} - Z \gg Tq$. Therefore,

$$\begin{aligned} & T^{-1} q^{-\frac{1}{4}} \sum_{m \le Z} \frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \left(\log \left(\frac{Tq}{2\pi m} \right) \right)^{-1} \left(\frac{T}{2\pi} - \frac{m}{q} \right)^{-1} \\ & \ll T^{-2} q^{-\frac{1}{4}} \sum_{m \le Z} \frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \ll T^{\sigma-2} q^{-\frac{1}{4}} \end{aligned} \tag{19}$$

in view of the estimate

$$\sum_{m \le x} \sigma_{1-2\sigma}(x) \ll x, \ \ x > 0. \tag{20}$$

Similary, we find that

$$T^{-\frac{3}{2}} q^{-\frac{1}{4}} \sum_{m \le Z} \frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \left(\frac{T}{2\pi} - \frac{m}{q} \right)^{-\frac{1}{2}} \ll T^{\sigma-2} q^{-\frac{1}{4}}. \tag{21}$$

Since $q \ll T$,

$$e^{CT} \sum_{m=1}^{\infty} \frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}} e^{-C\sqrt{\frac{Tm}{q}}} = O(e^{-c_1} T) \tag{22}$$

with some $c_1 > 0$. We have that

$$\left(\frac{1}{2} \sqrt{q^2 X + \frac{2Tq}{\pi}} - \frac{1}{2} \sqrt{q^2 X} \right)^2 = \frac{q^2 X}{2} + \frac{Tq}{2\pi} - q \sqrt{\frac{q^2 T^2}{4} + \frac{qXT}{2\pi}} = Z.$$

Thus,

$$T^{-\frac{3}{4}}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\min\left(1,\left|(q,X)^{\frac{1}{2}}-\left(qX+\frac{2T}{\pi}\right)^{\frac{1}{2}}+2\sqrt{\frac{m}{q}}\right|^{-1}\right)$$
$$\ll q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\min(1,|\sqrt{m}-\sqrt{Z}|^{-1})$$
$$=q^{\frac{1}{2}}T^{-\frac{3}{4}}\left(\sum_{m\le\frac{Z}{2}}+\sum_{\frac{Z}{2}<m\le Z-\sqrt{Z}}+\sum_{Z-\sqrt{Z}<m\le Z+\sqrt{Z}}+\sum_{Z+\sqrt{Z}<m\le 2Z}+\sum_{m>2Z}\right)$$
$$\times\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\min(1,|\sqrt{m}-\sqrt{Z}|^{-1}). \tag{23}$$

Clearly, in view of (20) and $Z\ll T$,

$$q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{m\le\frac{Z}{2}}(...)\ll q^{\frac{1}{2}}T^{-\frac{5}{4}}\sum_{m\le\frac{Z}{2}}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}=q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}}, \tag{24}$$

$$q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{\frac{Z}{2}<m\le Z-\sqrt{Z}}(...)\ll q^{\frac{1}{2}}T^{-\frac{5}{4}}\sum_{\frac{Z}{2}<m\le Z-\sqrt{Z}}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}(\sqrt{Z}-\sqrt{m})^{-1}$$
$$\ll q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}}\sum_{\frac{Z}{2}<m\le Z-\sqrt{Z}}\sigma_{1-2\sigma}(m)(Z-m)^{-1}$$
$$\ll q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}}\sum_{\sqrt{Z}\le m\le\frac{Z}{2}}\sigma_{1-2\sigma}(Z-m)m^{-1}\ll q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}}\log T, \tag{25}$$

$$q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{Z-\sqrt{Z}<m\le Z+\sqrt{Z}}(...)\ll q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{Z-\sqrt{Z}<m\le Z+\sqrt{Z}}\frac{\sigma_{1-2\sigma}(m)}{m^{\frac{5}{4}-\sigma}}\ll q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}} \tag{26}$$

by using Lemma 3,

$$q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{Z+\sqrt{Z}<m\le 2Z}(...)\ll q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}}\log T, \tag{27}$$

and

$$q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{m>2Z}(...)\ll q^{\frac{1}{2}}T^{-\frac{3}{4}}\sum_{m>2Z}\frac{\sigma_{1-2\sigma}}{m^{\frac{7}{4}-\sigma}}\ll q^{\frac{1}{2}}T^{\sigma-\frac{3}{2}}. \tag{28}$$

Finally, combining (17) - (19) and (21) - (28), we obtain that

$$G_{q,3} = 2iq^{\sigma-1}\left(\frac{2\pi}{T}\right)^{\sigma-\frac{1}{2}} \sum_{m\le Z} \frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \left(\log\left(\frac{Tq}{2\pi m}\right)\right)^{-1}$$
$$\times \cos\left(T\log\left(\frac{Tq}{2\pi m}\right) - T + \frac{\pi}{4}\right) + O(q^{\sigma-\frac{1}{4}}\log T).$$

Thus, from this, (9) - (11) and (13), Theorem 1 follows because Z can be replaced by N_1 with a negligible error.

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Vilnius University
Поступило 3.05.2013

СВЕДЕНИЯ ОБ АВТОРАХ

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Авдеев Иван Федорович — кандидат физико-математических наук, доцент кафедры математического и информационного анализа экономических процессов факультета экономики и управления Орловского государственного университета
e-mail: ivan_avd@mail.ru

Авдеев Федор Степанович — доктор педагогических наук, профессор, ректор Орловского государственного университета
e-mail: ivan_avd@mail.ru

Авдеева Татьяна Константиновна — доктор педагогических наук, профессор кафедры геометрии и методики преподавания математики физико-математического факультета Орловского государственного университета
e-mail: ivan_avd@mail.ru

Архипова Людмила Геннадьевна — кандидат физико-математических наук, младший научный сотрудник механико-математического факультета Московского государственного университета имени М. В. Ломоносова, старший преподаватель Финансового университета при правительстве РФ
e-mail: arhiludka@mail.ru

Бояринов Роман Николаевич — доктор физико-математических наук, доцент кафедры математического анализа механико-математического факультета Московского государственного университета имени М. В. Ломоносова
e-mail: roma_boyarin@yahoo.com

Бурлакова Екатерина Анатольевна — кандидат физико-математических наук, доцент кафедры высшей математики факультета естественно-научного и гуманитарного образования Государственного университета — учебно-научно-производственного комплекса
e-mail: burlkat@rambler.ru

Вахитова Екатерина Васильевна — кандидат физико-математических наук, профессор, профессор кафедры цифровых технологий, Воронежский государственный университет
e-mail: algebraist@yandex.ru

Вахитова Светлана Рифовна — соискатель, Воронежский государственный университет
e-mail: algebraist@yandex.ru