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Особенные пространства для релятивистских полей

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Аннотация

Мы изучаем, как модифицируются модели квантовой теории при перепараметризации координат пространства-времени и одновременно некоторых преобразований полевой функции. Предъявлены преобразования, которые превращают действие массивного поля в пространстве-времени Минковского в действие безмассового поля в некотором искривлённом пространстве.

Ключевые слова: уравнение Клейна — Гордона — Фока, репараметризация пространства-времени, преобразование полевых функций

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Peculiar spaces for relativistic fields

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Abstract

We study how quantum field theory models are modified under the reparametrizations of the space-time coordinates and some simultaneous transformations of the field function. The transformations that turn the action of the massive field in the Minkowski space-time into the action of the massless field in some curved space are presented.

Keywords: Klein–Gordon–Fock equation reparametrization of the space-time transformation of field functions

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1. Introduction

In this paper, we study how the quantum field theory model is modified under the reparametrizations of the space-time coordinates x^μ and some simultaneous transformations of the field function $u(x)$. It is an important part of a more general problem to study the behaviour of functional integrals in quantum field theory under transformations of coordinates of the space-time and field functions.

It is known that the properties of the Wiener measure remain valid under reparametrizations of the time variable t. In this case, there is the invariant differential $(u(t))^{-2}\,dt$. We consider the class of transformations of the coordinates of the $d-$dimensional space and the simultaneous transformations of the field function $u(x)$ that leave invariant the differential

$$(u(x))^{2d-4}\,dx\,, \qquad x \in \mathbf{R}^d\,.$$

In this class, there are transformations that relate massive and massless theories. If we start with the action of the massive field in the Minkowski space-time then we get the action of the massless field in some curved space. The metric tensor of this space is determined by the mass of the initial field and the form of the transformations. The metric and the curvature are singular at some points of the space, although the determinant of the metric does not change: $\det G_{\mu\nu} = -1$.

2. Mass of relativistic field and deformation of the geometry of space-time

Consider the action of self-interacting scalar field in the ordinary Minkowski[1] space-time

$$2\mathcal{A} = \int \left[\left(\frac{\partial u}{\partial x^0}\right)^2 - \left(\frac{\partial u}{\partial x^i}\right)^2 - m^2u^2 + u^4 \right] d^4x\,. \qquad (1)$$

Representing the field $u(x)$ as the product

$$u(x) = v(x)\,\varphi(x)$$

and supposing that u and v vanish at infinity, we rewrite the action (1) in the form

$$2\mathcal{A} = \int \left[\left(\frac{\partial v}{\partial x^0}\right)^2 - \left(\frac{\partial v}{\partial x^i}\right)^2 \right] \varphi^2\,d^4x$$

[1]The similar scheme can also be carried out in the Euclidean space.

$$+\int v^4\,\varphi^4\,d^4x + \int \varphi\left[-\frac{\partial^2\varphi}{(\partial x^0)^2}+\frac{\partial^2\varphi}{(\partial x^i)^2}-m^2\varphi\right] v^2\,d^4x\,. \tag{2}$$

If the function $\varphi(x)$ satisfies the Klein-Gordon (KG) equation [1]

$$-\frac{\partial^2\varphi}{(\partial x^0)^2}+\frac{\partial^2\varphi}{(\partial x^i)^2}-m^2\varphi=0\,, \tag{3}$$

then the action (2) is reduced to the action of the massless field in some curved space

$$\int\left[G^{\mu\nu}\frac{\partial w}{\partial\xi^\mu}\frac{\partial w}{\partial\xi^\nu}+w^4\right]\sqrt{-G}\,d^4\xi\,, \tag{4}$$

where $v(x)=w(\xi(x))$. The geometry of the space is determined by the solution of the KG equation for $\varphi(x)$.

We demand the form of the interaction term to be invariant. In this case, the Jacobian of the substitution

$$\det\left(\frac{\partial\xi}{\partial x}\right)=\varphi^4(x)\,. \tag{5}$$

and

$$G=\det G_{\mu\nu}=-1\,.$$

However the metric tensor $G^{\mu\nu}$ is nontrivial.

There are various options for the function $\xi^\mu(x^\nu)$. First, let us consider a special case with $\varphi=\varphi(s)$, where $s=\sqrt{(x^0)^2-(x^i)^2}$. The KG equation is transformed to the ordinary differential equation

$$\varphi''+\frac{3}{s}\varphi'+m^2\varphi=0\,. \tag{6}$$

Its general solution is expressed in terms of the Bessel functions

$$\varphi(s)=\frac{1}{s}\left[C_1J_1(ms)+C_2Y_1(ms)\right]\,. \tag{7}$$

The values of the constants C_1, C_2 are determined by the boundary conditions. In particular, $s^{-1}J_1(ms)\sim const$ and $s^{-1}Y_1(ms)\sim C\left(\ln s-2m^{-2}s^{-2}\right)$ at $s\sim 0$.

A simple but nontrivial way to define the new coordinates is a dilatation of the old ones with the dilatation factor depending on s only

$$\xi^\mu=\rho(s)s^{-1}\,x^\mu\,,\quad \rho(0)=0\,.$$

In this case, the angles do not change and

$$\sigma\equiv\sqrt{(\xi^0)^2-(\xi^i)^2}=\rho(s)\,.$$

Evaluating the Jacobian of the substitution and taking into account (5) we get the equation for $\rho(s)$

$$\rho^3(s)\rho'(s)s^{-3}=\varphi^4(s)\,. \tag{8}$$

Thus,

$$\rho^4(s)=\int\limits_0^{s^4}\varphi^4(s)ds^4\,.$$

The points of the x space where $\varphi(s)=0$ correspond to some singular points in ξ space. To explain this statement in more detail, in the next section we consider a slightly different substitution $\xi^\mu(x^\nu)$ and find the metric tensor and scalar curvature [2] of the obtained space.

3. The structure of the peculiar spaces for some special diffeomorphisms of space-time coordinates

Now, we suppose that the function φ does not depend on the spatial variables: $\varphi = \varphi(t)$. In this case, the KG equation reduces to the equation for a harmonic oscillator with the solution

$$\varphi(t) = \mu \sin m(t - t_0)\,. \tag{9}$$

Consider the following transformations of space-time

$$\tau = f(t)\,, \quad \xi^i = f'(t)\, x^i\,, \tag{10}$$

and let $\varphi(t)$ be equal to $f'(t)$

$$u(t,x) = v(t,x)\,\varphi(t) = w\left(f(t),\, f'(t)\,x\right)\, f'(t)\,. \tag{11}$$

In order that the directions of the initial time t and the new time τ be the same, the function $\varphi(t) = f'(t)$ must be nonnegative. However, the solution (9) does not satisfy this condition. To overcome the problem, note that at the points t^* where $\varphi(t) = 0$ it may not be a solution of the equation (3). If the zeroes of the function $\varphi(t)$ form a discrete set then they do not give any contribution to integrals. So, we can take arbitrary solutions on the intervals between the zeroes and sew them together at these points.

For simplicity, we take

$$\varphi(t) = \mu |\sin m(t - t_0)|\,. \tag{12}$$

We denote by $g(\tau)$ the diffeomorphism inverse to $f(t)$

$$t = g(\tau) = f^{-1}\left(f(t)\right)\,,$$

and recall some useful relations

$$f'(t) = \frac{1}{g'(\tau)}\,, \quad \frac{f''(t)}{f'(t)} = -\frac{g''(\tau)}{\left(g'(\tau)\right)^2}\,.$$

The action (1) is written now in the form

$$\int \left[G^{\mu\nu} \frac{\partial w}{\partial \xi^\mu} \frac{\partial w}{\partial \xi^\nu} + w^4\right] \sqrt{-G}\, d^4\xi$$

$$= \int \left[\left(\frac{\partial w}{\partial \tau} - h(\tau)\, \xi^i \frac{\partial w}{\partial \xi^i}\right)^2 - \left(\frac{\partial w}{\partial \xi^i}\right)^2 + w^4\right] d\tau d^3\xi \tag{13}$$

where

$$h(\tau) = \frac{g''(\tau)}{g'(\tau)}\,.$$

Thus, the components of the metric tensor are

$$G^{00} = 1\,, \quad G^{0i} = -h(\tau)\, \xi^i\,, \quad G^{ij} = h^2(\tau)\, \xi^i \xi^j - \delta^{ij}\,. \tag{14}$$

By the direct evaluation, one can check that $\det G_{\mu\nu} = -1$ and get the expression for the Riemann tensor and the scalar curvature

$$R = 2\left[3h'(\tau) + \left(\xi_1^2 + \xi_2^2 + \xi_3^2\right)\left(h'(\tau)\right)^2 + \left(\xi_1^2 + \xi_2^2 + \xi_3^2\right) h(\tau) h''(\tau)\right]\,. \tag{15}$$

Now, let us find the explicit form of the dependence $\tau = f(t)\,,\ t = g(\tau)\,,\ h(\tau)\,$. From (12) it follows

$$\tau = f(t) = \frac{\mu}{m}\int\limits_0^t |\sin mt|\, m\, dt\,. \tag{16}$$

Here, we put the constant $t_0 = 0$ and assume that $\tau = 0$ at $t = 0\,$.

We integrate (16) over intervals $(k-1)\frac{\pi}{m} \le t \le k\frac{\pi}{m}$ separately and sew the results together.

At the interval $(k-1)\frac{\pi}{m} \le t \le k\frac{\pi}{m}\,,\quad k = 1, 2, \dots\,,$ the result looks like

$$\tau = \frac{\mu}{m}\Big((-1)^k \cos mt + 2k - 1\Big)\,, \quad 2(k-1)\frac{\mu}{m} \le \tau \le 2k\frac{\mu}{m}\,. \tag{17}$$

In this case,

$$t = g(\tau) = -\frac{1}{m}\arccos\Big(\tau - (2k-1)\frac{\mu}{m}\Big) + k\frac{\pi}{m}\,, \tag{18}$$

$$\frac{1}{g'(\tau)} = m\sqrt{\Big(\tau - (2k-1)\frac{\mu}{m}\Big)\,\Big(-\tau + 2k\frac{\mu}{m}\Big)}\,, \tag{19}$$

and

$$2h(\tau) = \frac{1}{\tau - (2k-1)\frac{\mu}{m}} - \frac{1}{2k\frac{\mu}{m} - \tau}\,. \tag{20}$$

Thus, the function $\tau = f(t)$ is a continuous monotone increasing one with the points of inflection at $t = k\frac{\pi}{2m}$. Note that it is valid for negative values of t and τ as well.

At some values of the new time variable, namely at $\tau = k\frac{\mu}{m}$, the components of the metric tensor G^{0i}, G^{ij} and the scalar curvature R tend to infinity. On the other hand, $\frac{1}{g'(\tau)}$, and hence the space coordinates ξ^i, vanish at these singular points. It makes sense to emphasize that the points $\tau = k\frac{\mu}{m}$, where the space collapses, in this case are determined only by the mass of the field m.

One can easily check that the form of the action of a gauge field is invariant at the transfer to the new space. In fact, in the Minkowski space the action of the electromagnetic field has the form

$$\mathcal{A}_{em} = -\frac{1}{4}\int \eta^{\alpha\gamma}\eta^{\beta\delta} f_{\alpha\gamma} f_{\beta\delta}\, d^4x\,, \qquad f_{\alpha\gamma} = \frac{\partial a_\alpha}{\partial x^\gamma} - \frac{\partial a_\gamma}{\partial x^\alpha}\,. \tag{21}$$

Using the following relations

$$\frac{\partial}{\partial x^\gamma} = \frac{1}{g'} V^\lambda_\gamma \frac{\partial}{\partial \xi^\lambda}\,, \qquad a_\alpha = \frac{1}{g'} V^\lambda_\alpha A_\lambda\,, \qquad V^\lambda_\gamma = g'\,\frac{\partial \xi^\lambda}{\partial x^\gamma}\,. \tag{22}$$

one gets

$$\mathcal{A}_{em} = -\frac{1}{4}\int G^{\mu\lambda} G^{\nu\sigma} F_{\mu\lambda} F_{\nu\sigma}\sqrt{-G}\, d^4\xi\,, \qquad F_{\mu\lambda} = \frac{\partial A_\mu}{\partial \xi^\lambda} - \frac{\partial A_\lambda}{\partial \xi^\mu}\,, \tag{23}$$

were

$$G^{\mu\lambda} = \eta^{\alpha\gamma}\, V^\mu_\alpha\, V^\lambda_\gamma\,. \tag{24}$$

The same result is valid for nonabelian gauge fields as well.

4. Conclusion

Although we do not yet know if the discussed peculiar spaces have any physical sense, the study of these spaces can help to understand the structure of functional integrals in quantum field theory in plane and in curved spaces.

Since the factor in the field function transformation (11) is singular at some points, the functional integrations in the theories in the initial and the new spaces are carried out over different functional spaces. Some simple examples of the modification of the functional spaces at nonlinear nonlocal substitutions in functional integrals related with interaction terms in the action were given in [3].

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