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Свободные прямоугольные n-кратные полугруппы

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Аннотация

n-кратной полугруппой называется непустое множество G, снабженное n бинарными операциями $\boxed{1}, \boxed{2}, ..., \boxed{n}$, удовлетворяющими аксиомам $(x\boxed{r}y)\boxed{s}z = x\boxed{r}(y\boxed{s}z)$ для всех $x, y, z \in G$ и $r, s \in \{1, 2, ..., n\}$. Это понятие рассматривал Н. А. Корешков в контексте теории n-кратных алгебр ассоциативного типа. Доппельполугруппы являются 2-кратными полугруппами. n-кратные полугруппы имеют связи с интерассоциативными полугруппами, димоноидами, триоидами, доппельалгебрами, дуплексами, g-димоноидами и рестриктивными биполугруппами. Если операции n-кратной полугруппы совпадают, то она превращается в полугруппу. Таким образом, n-кратные полугруппы являются обобщением полугрупп.

Класс всех n-кратных полугрупп образует многообразие. Недавно были построены свободная n-кратная полугруппа, свободная коммутативная n-кратная полугруппа, свободная k-нильпотентная n-кратная полугруппа и свободное произведение произвольных n-кратных полугрупп. Класс всех прямоугольных n-кратных полугрупп, то есть n-кратных полугрупп с n прямоугольными полугруппами, образует подмногообразие многообразия n-кратных полугрупп.

В этой статье мы строим свободную прямоугольную n-кратную полугруппу и характеризуем наименьшую прямоугольную конгруэнцию на свободной n-кратной полугруппе.

Ключевые слова: n-кратная полугруппа, свободная прямоугольная n-кратная полугруппа, свободная n-кратная полугруппа, полугруппа, конгруэнция.

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Free rectangular n-tuple semigroups

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Abstract

An n-tuple semigroup is a nonempty set G equipped with n binary operations $\boxed{1}, \boxed{2}, ..., \boxed{n}$, satisfying the axioms $(x\boxed{r}y)\boxed{s}z = x\boxed{r}(y\boxed{s}z)$ for all $x, y, z \in G$ and $r, s \in \{1, 2, ..., n\}$. This notion was considered by Koreshkov in the context of the theory of n-tuple algebras of associative type. Doppelsemigroups are 2-tuple semigroups. The n-tuple semigroups are related to interassociative semigroups, dimonoids, trioids, doppelalgebras, duplexes, g-dimonoids, and restrictive bisemigroups. If operations of an n-tuple semigroup coincide, the n-tuple semigroup becomes a semigroup. So, n-tuple semigroups are a generalization of semigroups.

The class of all n-tuple semigroups forms a variety. Recently, the constructions of the free n-tuple semigroup, of the free commutative n-tuple semigroup, of the free k-nilpotent n-tuple semigroup and of the free product of arbitrary n-tuple semigroups were given. The class of all rectangular n-tuple semigroups, that is, n-tuple semigroups with n rectangular semigroups, forms a subvariety of the variety of n-tuple semigroups.

In this paper, we construct the free rectangular n-tuple semigroup and characterize the least rectangular congruence on the free n-tuple semigroup.

Keywords: n-tuple semigroup, free rectangular n-tuple semigroup, free n-tuple semigroup, semigroup, congruence.

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1. Introduction

As a natural generalization of semigroups, n-tuple semigroups form an important variety of algebras arising from interassociative semigroups. Recall that an n-tuple semigroup [12] is a nonempty set G equipped with n binary operations $\boxed{1}, \boxed{2}, ..., \boxed{n}$, satisfying the axioms $(x\boxed{r}y)\boxed{s}z = x\boxed{r}(y\boxed{s}z)$ for all $x, y, z \in G$ and $r, s \in \{1, 2, ..., n\}$. The class of n-tuple semigroups causes the greatest interest from the point of view of applications in the theory of n-tuple algebras of associative type [12, 13, 14]. It turns out that $n > 1$ pairwise interassociative semigroups give rise to an n-tuple semigroup. Recall that two semigroups defined on the same set G are interassociative [6] provided that they satisfy the latter axioms for $r, s \in \{1, 2\}$. The notion of interassociativity for semigroups is of interest too (see, e.g., [3, 4, 6, 8, 9, 10]). It is known [22] that commutative dimonoids and commutative trioids provide subclasses in the variety of 2-tuple semigroups and 3-tuple semigroups, respectively. This fact allows us to study the classes of commutative dimonoids (trioids) via n-tuple semigroups. Recall that dimonoids and trioids are peculiar algebraic structures, with applications to dialgebra theory [2, 15] and trialgebra theory [1, 5, 16], respectively. For details, see, e.g., [23, 28, 34] and [20, 29, 36], respectively. It should be noted that doppelalgebras [18] are linear analogs of 2-tuple semigroups. The 2-tuple semigroups or, equivalently, doppelsemigroups were studied in [21, 24, 25, 27, 30, 31, 35]. The n-tuple semigroups also have relationships with duplexes [17], g-dimonoids [37], and restrictive bisemigroups [19]. These connections increase the motivation for studying n-tuple semigroups.

One of the fundamental problems in the variety theory of algebraic systems is the problem of constructing free algebras in a given variety. Some free systems in the variety of n-tuple semigroups were studied recently: the constructions of the free n-tuple semigroup, of the free commutative

n-tuple semigroup and of the free k-nilpotent n-tuple semigroup were presented in [22] and [33], respectively. The free product of arbitrary n-tuple semigroups was constructed in [32].

In this paper, we consider the variety of rectangular n-tuple semigroups which are analogs of rectangular semigroups. The main result of the paper is the construction of the free rectangular n-tuple semigroup of an arbitrary rank (Theorem 1). As a consequence, the free rectangular n-tuple semigroup of rank 1 is presented (Corollary 2). We also characterize the least rectangular congruence on the free n-tuple semigroup (Theorem 2), count the cardinality of the free rectangular n-tuple semigroup for a finite case, establish that the automorphism group of the free rectangular n-tuple semigroup is isomorphic to the symmetric group and the semigroups of the free rectangular n-tuple semigroup $(n > 1)$ are isomorphic.

The results obtained in the present paper extend some results in [35].

2. Preliminaries

A semigroup S is called rectangular [28] if $xyz = xz$ foll all $x, y, z \in S$. In [7], the lattice of subvarieties of the variety defined by the identity $xyz = xz$ was indicated. This variety is the union of the variety of left zero semigroups, the variety of right zero semigroups and the variety of zero semigroups, and the lattice of its subvarieties is an 8-element Boolean algebra. The variety of dimonoids with rectangular semigroups and the variety of rectangular doppelsemigroups were considered in [28] and [35], respectively.

For n-tuple semigroups, it is natural to introduce an analog of a rectangular semigroup. An n-tuple semigroup $(G, \boxed{1}, \boxed{2}, ..., \boxed{n})$ will be called rectangular if semigroups $(G, \boxed{1})$, $(G, \boxed{2})$,..., $(G, \boxed{n})$ are rectangular. The class of all rectangular n-tuple semigroups forms a subvariety of the variety of n-tuple semigroups. An n-tuple semigroup which is free in the variety of rectangular n-tuple semigroups will be called a free rectangular n-tuple semigroup. If ρ is a congruence on an n-tuple semigroup G' such that G'/ρ is a rectangular n-tuple semigroup, we say that ρ is a rectangular congruence. As usual, $\mathbb{N}$ denotes the set of all positive integers.

We will need the following two lemmas.

LEMMA 1. *([22], Lemma 1) In an n-tuple semigroup $(G, \boxed{1}, \boxed{2}, ..., \boxed{n})$, for any $1 < m \in \mathbb{N}$, and any $x_i \in G$, $1 \leqslant i \leqslant m+1$, and any $*_j \in \{\boxed{1}, \boxed{2}, ..., \boxed{n}\}$, $1 \leqslant j \leqslant m$, any parenthesizing of*

$$x_1 *_1 x_2 *_2 \ldots *_m x_{m+1}$$

gives the same element from G.

LEMMA 2. *In a rectangular n-tuple semigroup $(G, \boxed{1}, \boxed{2}, ..., \boxed{n})$, for any $a, b, x, y \in G$, and any $i, j \in \{1, 2, \ldots, n\}$ the following identity is satisfied:*

$$a \boxed{i} b \boxed{j} x = a \boxed{i} y \boxed{j} x.$$

PROOF. The proof follows from Lemma 2.2 of [35] and Lemma 1. □

Semigroups $(D, \dashv)$ and $(D, \vdash)$ are called $\mathcal{P}$-related [11] if $x \dashv y \dashv z = x \vdash y \vdash z$ for all $x, y, z \in D$.

Proposition 2.3 of [35] implies the following statement which establishes necessary and sufficient conditions under which the operations of a rectangular n-tuple semigroup $(n > 1)$ coincide.

PROPOSITION 1. *Let $1 < n \in \mathbb{N}$. The operations of a rectangular n-tuple semigroup $(G, \boxed{1}, \boxed{2}, ..., \boxed{n})$ coincide if and only if $(G, \boxed{1})$, $(G, \boxed{2})$,...,$(G, \boxed{n})$ are pairwise $\mathcal{P}$-related semigroups.*

An n-tuple semigroup which is free in the variety of n-tuple semigroups is called a free n-tuple semigroup [22]. The construction of the free n-tuple semigroup was first given in [22]. We recall it.

Let X be an arbitrary nonempty set, and let w be an arbitrary word in the alphabet X. The length of w will be denoted by l_w. Fix $n \in \mathbb{N}$ and let $Y = \{y_1, y_2, ..., y_n\}$ be an arbitrary set consisting of n elements. Let further $F[X]$ be the free semigroup on X, let $F^\theta[Y]$ be the free monoid on Y, and let $\theta \in F^\theta[Y]$ be the empty word. By definition, the length l_θ of θ is equal to 0. Define n binary operations $\boxed{1}, \boxed{2}, ..., \boxed{\text{n}}$ on

$$XY_n = \{(w, u) \in F[X] \times F^\theta[Y] \mid l_w - l_u = 1\}$$

by

$$(w_1, u_1)\boxed{i}(w_2, u_2) = (w_1 w_2, u_1 y_i u_2)$$

for all $(w_1, u_1), (w_2, u_2) \in XY_n$ and $i \in \{1, 2, \ldots, n\}$. The algebra $(XY_n, \boxed{1}, \boxed{2}, ..., \boxed{\text{n}})$ is denoted by $F_nTS(X)$. By Theorem 2 of [22], $F_nTS(X)$ is the free n-tuple semigroup.

If $f : S_1 \to S_2$ is a homomorphism of n-tuple semigroups, the kernel of f will be denoted by Δ_f.

3. Main results

In this section, we construct the free rectangular n-tuple semigroup of an arbitrary rank, consider separately singly generated free rectangular n-tuple semigroups and characterize the least rectangular congruence on the free n-tuple semigroup. We also count the cardinality of the free rectangular n-tuple semigroup for a finite case, establish that the automorphism group of the free rectangular n-tuple semigroup is isomorphic to the symmetric group and the semigroups of the free rectangular n-tuple semigroup $(n > 1)$ are isomorphic.

Let X be an arbitrary nonempty set, $n \in \mathbb{N}$ and Y as above. Define n binary operations $\boxed{1}, \boxed{2}, ..., \boxed{\text{n}}$ on $X \cup (Y \times X \times X \times Y)$ by

$$(a_1, b_1, c_1, d_1)\boxed{i}(a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2),$$

$$x\boxed{i}(a_1, b_1, c_1, d_1) = (y_i, x, c_1, d_1), \quad (a_1, b_1, c_1, d_1)\boxed{i}x = (a_1, b_1, x, y_i),$$

$$x\boxed{i}y = (y_i, x, y, y_i)$$

for all $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in Y \times X \times X \times Y$, $x, y \in X$ and $i \in \{1, 2, ..., n\}$. The obtained algebra will be denoted by $FR_nS(X)$.

The main result of the paper is the following theorem.

Theorem 1. *$FR_nS(X)$ is the free rectangular n-tuple semigroup.*

Proof. The proof that $FR_nS(X)$ is a rectangular n-tuple semigroup follows from the proof of Theorem 3.1 in [35]. Let us show that $FR_nS(X)$ is free rectangular.

Note that $FR_nS(X)$ is generated by X. Indeed,

$$(y_i, b_1, c_1, y_j) \in \{b_1\boxed{i}c_1, b_1\boxed{j}c_1, (b_1\boxed{i}c_1)\boxed{i}(b_1\boxed{j}c_1)\}$$

for $(y_i, b_1, c_1, y_j) \in Y \times X \times X \times Y$, and hence any element of $Y \times X \times X \times Y$ can be expressed by elements from X.

Let $(S, \boxed{1}', \boxed{2}', ..., \boxed{\text{n}}')$ be an arbitrary rectangular n-tuple semigroup, and let $\gamma : X \to S$ be an arbitrary map. Fix $\varepsilon \in S$ and define a map

$$\pi : FR_nS(X) \to (S, \boxed{1}', \boxed{2}', ..., \boxed{\text{n}}')$$

as follows:

$$(y_i, b_1, c_1, y_j)\pi = \begin{cases} b_1\gamma\boxed{i}' c_1\gamma, & \text{if} \quad i = j, \\ b_1\gamma\boxed{i}' \varepsilon\boxed{j}' c_1\gamma, & \text{if} \quad i \neq j, \end{cases}$$

$$x\pi = x\gamma$$

for $(y_i, b_1, c_1, y_j) \in Y \times X \times X \times Y$ and $x \in X$. By Lemmas 1 and 2, π is well-defined. In order to show that π is a homomorphism, we will use Lemmas 1, 2 and the identities of a rectangular n-tuple semigroup.

Let (y_i, b_1, c_1, y_j), $(y_s, b_2, c_2, y_k) \in Y \times X \times X \times Y$, $x, y \in X$ and $m \in \{1, 2, \ldots, n\}$. We have

$$((y_i, b_1, c_1, y_j)\boxed{m}(y_s, b_2, c_2, y_k))\pi$$

$$= (y_i, b_1, c_2, y_k)\pi = \begin{cases} b_1\gamma\boxed{i}' c_2\gamma, & \text{if} \quad i = k, \\ b_1\gamma\boxed{i}' \varepsilon\boxed{k}' c_2\gamma, & \text{if} \quad i \neq k, \end{cases}$$

$$(y_s, b_2, c_2, y_k)\pi = \begin{cases} b_2\gamma\boxed{s}' c_2\gamma, & \text{if} \quad s = k, \\ b_2\gamma\boxed{s}' \varepsilon\boxed{k}' c_2\gamma, & \text{if} \quad s \neq k. \end{cases}$$

Let $i = k$. Then

$$((y_i, b_1, c_1, y_j)\boxed{m}(y_s, b_2, c_2, y_k))\pi = b_1\gamma\boxed{i}' c_2\gamma$$

$$= (y_i, b_1, c_1, y_j)\pi\boxed{m}'(y_s, b_2, c_2, y_k)\pi.$$

In the case $i \neq k$ we get

$$((y_i, b_1, c_1, y_j)\boxed{m}(y_s, b_2, c_2, y_k))\pi = b_1\gamma\boxed{i}' \varepsilon\boxed{k}' c_2\gamma$$

$$= (y_i, b_1, c_1, y_j)\pi\boxed{m}'(y_s, b_2, c_2, y_k)\pi.$$

Moreover,

$$(x\boxed{m}y)\pi = (y_m, x, y, y_m)\pi = x\gamma\boxed{m}' y\gamma = x\pi\boxed{m}' y\pi.$$

Further,

$$((y_i, b_1, c_1, y_j)\boxed{m}x)\pi = (y_i, b_1, x, y_m)\pi = \begin{cases} b_1\gamma\boxed{i}' x\gamma, & \text{if} \quad i = m, \\ b_1\gamma\boxed{i}' \varepsilon\boxed{m}' x\gamma, & \text{if} \quad i \neq m. \end{cases}$$

Assume that $i = m$. In this case we obtain

$$((y_i, b_1, c_1, y_j)\boxed{m}x)\pi = b_1\gamma\boxed{i}' x\gamma = (y_i, b_1, c_1, y_j)\pi\boxed{m}' x\pi.$$

For $i \neq m$,

$$((y_i, b_1, c_1, y_j)\boxed{m}x)\pi = b_1\gamma\boxed{i}' \varepsilon\boxed{m}' x\gamma = (y_i, b_1, c_1, y_j)\pi\boxed{m}' x\pi.$$

Consider the remaining case:

$$(x\boxed{m}(y_i, b_1, c_1, y_j))\pi = (y_m, x, c_1, y_j)\pi = \begin{cases} x\gamma\boxed{m}' c_1\gamma, & \text{if} \quad m = j, \\ x\gamma\boxed{m}' \varepsilon\boxed{j}' c_1\gamma, & \text{if} \quad m \neq j. \end{cases}$$

If $m = j$, then

$$(x\boxed{m}(y_i, b_1, c_1, y_j))\pi = x\gamma\boxed{m}' c_1\gamma = x\pi\boxed{m}'(y_i, b_1, c_1, y_j)\pi.$$

For $m \neq j$,

$$(x\boxed{m}(y_i, b_1, c_1, y_j))\pi = x\gamma\boxed{m}'\varepsilon\boxed{j}'c_1\gamma = x\pi\boxed{m}'(y_i, b_1, c_1, y_j)\pi.$$

Consequently, π is a homomorphism of n-tuple semigroups.

Since $x\pi = x\gamma$ for all $x \in X$ and X generates $FR_nS(X)$, the uniqueness of the homomorphism π is obvious. Thus, $FR_nS(X)$ is free in the variety of rectangular n-tuple semigroups. □

Note that, for $n = 2$, Theorem 1 yields Theorem 3.1 in [35]. It is also worth noting that some facts that ψ is a homomorphism in the proof of Lemma 3.8 from [35] were left for independent reader's verification, unlike Theorem 1 for which we present the complete proof that π is a homomorphism.

Corollary 1. *The free rectangular n-tuple semigroup $FR_nS(X)$ generated by a finite set X is finite. Specifically, if $|X| = k$, then $|FR_nS(X)| = k(1 + n^2k)$.*

Now we construct an n-tuple semigroup which is isomorphic to the free rectangular n-tuple semigroup of rank 1.

Let e be an arbitrary symbol. Define n binary operations $\boxed{1}, \boxed{2}, ..., \boxed{\text{n}}$ on

$$(Y \times Y) \cup \{e\} \quad \text{by}$$

$$(a_1, d_1)\boxed{i}(a_2, d_2) = (a_1, d_2), \quad e\boxed{i}(a_1, d_1) = (y_i, d_1),$$

$$(a_1, d_1)\boxed{i}e = (a_1, y_i), \quad e\boxed{i}e = (y_i, y_i)$$

for all $(a_1, d_1), (a_2, d_2) \in Y \times Y$ and $i \in \{1, 2, ..., n\}$. The algebra

$$((Y \times Y) \cup \{e\}, \boxed{1}, \boxed{2}, ..., \boxed{\text{n}})$$

will be denoted by FR_nS_1. It is immediate to show that FR_nS_1 is an n-tuple semigroup.

Theorem 1 implies the following statement which describes singly generated free rectangular n-tuple semigroups.

Corollary 2. *If $|X| = 1$, then $FR_nS_1 \cong FR_nS(X)$.*

Proof. Let $X = \{e\}$. We define a map $\sigma : FR_nS_1 \to FR_nS(X)$ by the rule

$$e\sigma = e \text{ and } (a_1, d_1)\sigma = (a_1, e, e, d_1)$$

for all $(a_1, d_1) \in Y \times Y$. An immediate verification shows that σ is an isomorphism. □

The following statement establishes a relationship between the semigroups of the free rectangular n-tuple semigroup ($n > 1$).

Corollary 3. *Let $1 < n \in \mathbb{N}$ and $i, j \in \{1, 2, \ldots, n\}$. The semigroups*

$$(X \cup (Y \times X \times X \times Y), \boxed{i}) \text{ and } (X \cup (Y \times X \times X \times Y), \boxed{j})$$

of the free rectangular n-tuple semigroup $FR_nS(X)$ are isomorphic.

Proof. The proof is similar to the proof of Corollary 3.11 in [35]. □

It is not difficult to see that the free rectangular n-tuple semigroup $FR_nS(X)$ is determined uniquely up to isomorphism by cardinality of the set X. Hence the automorphism group of $FR_nS(X)$ is isomorphic to the symmetric group on X.

At the end of this section, we characterize the least rectangular congruence on the free n-tuple semigroup.

For every nonempty word w over an alphabet X, denote the first (respectively, last) letter of w by $w^{(0)}$ (respectively, $w^{(1)}$).

Theorem 2. *Let $F_nTS(X)$ be the free n-tuple semigroup, $(w_1,u_1),(w_2,u_2)\in F_nTS(X)$, and let $FR_nS(X)$ be the free rectangular n-tuple semigroup. Define a relation $\widetilde{\mu}$ on $F_nTS(X)$ by*

$$(w_1,u_1)\widetilde{\mu}(w_2,u_2)$$

if and only if

$$u_1\neq\theta,\ u_2\neq\theta\quad\text{and}\quad (u_1^{(0)},w_1^{(0)},w_1^{(1)},u_1^{(1)})=(u_2^{(0)},w_2^{(0)},w_2^{(1)},u_2^{(1)}),$$

$$\text{or}\quad (w_1,u_1)=(w_2,u_2).$$

Then $\widetilde{\mu}$ is the least rectangular congruence on $F_nTS(X)$.

Proof. Define a map $\mu: F_nTS(X)\to FR_nS(X)$ by

$$(w,u)\mapsto(w,u)\mu=\begin{cases}(u^{(0)},w^{(0)},w^{(1)},u^{(1)}), & \text{if}\quad u\neq\theta,\\ w, & \text{if}\quad u=\theta.\end{cases}$$

Using Theorem 1, similarly to the proof of Theorem 4.1 in [35], the facts that μ is an epimorphism and the least rectangular congruence Δ_μ on $F_nTS(X)$ coincides with $\widetilde{\mu}$ can be proved. □

Note that, for $n=2$, Theorem 2 implies Theorem 4.1 in [35].

4. Conclusions

In this paper, we consider n-tuple semigroups which are sets with n binary associative operations satisfying additional axioms. $n>1$ pairwise interassociative semigroups give rise to an n-tuple semigroup. The main result of this paper is the construction of the free rectangular n-tuple semigroup. We also present the least rectangular congruence on the free n-tuple semigroup.

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