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О совместном распределении значений дзета-функций Гурвица[1]

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Аннотация

Хорошо известно, что некоторые дзета и L-функции универсальны в смысле Воронина, т.е., ими приближается широкий класс аналитических функций. Некоторые из этих функций также совместно универсальны. В этом случае, набор аналитических функций одновременно приближается набором дзета-функций. В статье рассматривается проблема, связанная со совместной универсальностью дзета-функций Гурвица. Известно, что дзета-функции Гурвица $\zeta(s,\alpha_1),\dots,\zeta(s,\alpha_r)$ совместно универсальны, если параметры $\alpha_1,\dots,\alpha_r$ алгебраически независимы над полем рациональных чисел $\mathbb{Q}$, или в более общем случае, если множество $\{\log(m+\alpha_j) : m\in\mathbb{N}_0,\ j=1,\dots,r\}$ линейно независимо над $\mathbb{Q}$. Мы рассматриваем случай произвольных параметров $\alpha_1,\dots,\alpha_r$ и получаем, что существует непустое замкнутое множество функций $F_{\alpha_1,\dots,\alpha_r}$ пространства $H^r(D)$ аналитических в полосе $D=\left\{s\in\mathbb{C} : \frac{1}{2}<\sigma<1\right\}$ такое, что для любых компактных множеств $K_1,\dots,K_r\subset D$, функций $(f_1,\dots,f_r)\in F_{\alpha_1,\dots,\alpha_r}$ и всякого $\varepsilon>0$ множество $\left\{\tau\in\mathbb{R} : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<\varepsilon\right\}$ имеет положительную нижнюю плотность. Также рассматривается случай положительной плотности этого множества.

Ключевые слова: вероятностная мера, дзета-функция Гурвица, пространство аналитических функций, слабая сходимость, универсальность.

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On joint value distribution of Hurwitz zeta-functions[2]

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Abstract

It is well known that some zeta and L-functions are universal in the Voronin sense, i.e., they approximate a wide class of analytic functions. Also, some of them are jointly universal. In this case, a collection of analytic functions is simultaneously approximated by a collection of zeta-functions. In the paper, a problem related to joint universality of Hurwitz zeta-functions is discussed. It is known that the Hurwitz zeta-functions $\zeta(s,\alpha_1),\dots,\zeta(s,\alpha_r)$ are jointly universal if the parameters $\alpha_1,\dots,\alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$, or, more generally, if the set $\{\log(m+\alpha_j) : m \in \mathbb{N}_0,\ j=1,\dots,r\}$ is linearly independent over $\mathbb{Q}$. We consider the case of arbitrary parameters $\alpha_1,\dots,\alpha_r$ and obtain that there exists a non-empty closed set $F_{\alpha_1,\dots,\alpha_r}$ of the space $H^r(D)$ of analytic functions on the strip $D=\left\{s\in\mathbb{C} : \frac{1}{2}<\sigma<1\right\}$ such that, for every compact sets $K_1,\dots,K_r\subset D$, $f_1,\dots,f_r\in F_{\alpha_1,\dots,\alpha_r}$ and $\varepsilon>0$, the set $\left\{\tau\in\mathbb{R} : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<\varepsilon\right\}$ has a positive lower density. Also, the case of positive density of the latter set is discussed.

Keywords: Hurwitz zeta-function, probability measure, space of analytic functions, universality, weak convergence.

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1. Introduction

The Hurwitz zeta-function $\zeta(s,\alpha)$, $s=\sigma+it$, with parameter α, $0<\alpha\leqslant 1$, is defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},$$

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and has the analytic continuation to the whole complex plane, except for a simple pole at the point $s=1$ with residue 1. For $\alpha=1$, the Hurwitz zeta-function reduces to the Riemann zeta-function

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}, \quad \sigma>1,$$

and

$$\zeta\left(s,\frac{1}{2}\right)=(2^s-1)\,\zeta(s).$$

Thus, $\zeta(s,\alpha)$ is a generalization of the Riemann zeta-function. The function $\zeta(s)$ has the Euler product over primes

$$\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1},$$

while the function $\zeta(s,\alpha)$, except for the values $\alpha=1$ and $\alpha=\frac{1}{2}$, has no such a product. This fact reflects in value distribution differences of the functions $\zeta(s)$ and $\zeta(s,\alpha)$. For example, it is well known that $\zeta(s)\neq 0$, while the function $\zeta(s,\alpha)$ has infinitely many zeros for all $\alpha\neq 1,\frac{1}{2}$ in the half plane $\sigma>1$. On the other hand, the functions $\zeta(s)$ and $\zeta(s,\alpha)$ for some classes of the parameter α have a common property of the approximation of a wide class of analytic functions. This interesting property is called universality, and for the function $\zeta(s)$ was obtained by S. M. Voronin [12]. For modern statements of universality theorems it is convenient to use the following notation. Let $D=\left\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\right\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K\in\mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then the modern Voronin universality theorem, see, for example, [7], says that for every $K\in\mathcal{K}$, $f\in H_0(K)$ and $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right\}>0.$$

The later inequality shows that there are infinitely many shifts $\zeta(s+i\tau,\alpha)$ approximating with accuracy ε a given function $f(s)\in H_0(K)$. Yuri Vladimirovich Linnik knew the Voronin theorem and highly valued it. Moreover, Il'dar Abdulovich Ibragimov imformed the second author that Yu. V. Linnik had a conjecture that all Dirichlet series satisfying some natural growth conditions are universal in the Voronin sense. Now this conjecture is called the Linnik-Ibragimov conjecture (or problem), see, for example, [11].

The universality of the Hurwitz zeta-function differs slightly from that of the function $\zeta(s)$. Denote by $H(K)$ with $K\in\mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Thus, $H_0(K)\subset H(K)$ for all $K\in\mathcal{K}$. Then the following universality theorem for the function $\zeta(s,\alpha)$ is known.

Theorem 1. *Suppose that the parameter α is transcendental or rational $\neq 1,\frac{1}{2}$. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon\right\}>0. \tag{1}$$

The theorem in the case of rational α was already known to Voronin [14]. In a slightly different form, the theorem was obtained independently by S. M. Gonek and B. Bagchi in their theses [5], [1].

Unfortunately, the universality of $\zeta(s,\alpha)$ with algebraic irrational parameter α is an open problem. This problem is closely connected to linear independence over the field of rational numbers $\mathbb{Q}$ of the set $L(\alpha)=\{\log(m+\alpha):m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\}$. Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. Then, in [2], the following result towards to universality problem of $\zeta(s,\alpha)$ with algebraic irrational α was obtained.

THEOREM 2. *Suppose that the parameter α is algebraic irrational. Then there exists a closed non-empty set $F_\alpha \subset H(D)$ such that, for every compact set $K \subset D$, $f(s) \in F_\alpha$ and $\varepsilon > 0$, the inequality* (1) *is true.*

Some of zeta-functions are also jointly universal. In this case, a collection of analytic functions are simultaneously approximated by a collection of zeta-functions. The first joint universality results belong to S. M. Voronin. In [13], he considered the joint functional independence of Dirichlet L-functions, and, for this, he applied their joint universality. It is clear, that in the case of joint universality, the approximating zeta-functions must be in some sense independent. For Hurwitz zeta-functions this independence in [10] was described by the algebraic independence over $\mathbb{Q}$ of the parameters $\alpha_1, \dots, \alpha_r$. In [8], the algebraic independence was replaced by the linear independence over $\mathbb{Q}$ for the set

$$L(\alpha_1, \dots, \alpha_r) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0,\ j = 1, \dots, r\}.$$

Thus, the following theorem is known [8].

THEOREM 3. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\} > 0.$$

The aim of this paper is to prove a joint generalization of Theorem 2, i.e., to prove a certain theorem on joint approximation by the functions $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$ without using any independence condition.

THEOREM 4. *Suppose that the numbers α_j, $0 < \alpha_j < 1$, $\alpha_j \neq \frac{1}{2}$, $j = 1, \dots, r$, are arbitrary. Then there exists a closed non-empty set $F_{\alpha_1,\dots,\alpha_r} \subset H^r(D)$ such that, for every compact sets $K_1, \dots, K_r \subset D$, $(f_1, \dots, f_r) \in F_{\alpha_1,\dots,\alpha_r}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\} > 0.$$

Theorem 4 has the following modification.

THEOREM 5. *Suppose that the numbers α_j, $0 < \alpha_j < 1$, $\alpha_j \neq \frac{1}{2}$, $j = 1, \dots, r$, are arbitrary. Then there exists a closed non-empty set $F_{\alpha_1,\dots,\alpha_r} \subset H^r(D)$ such that, for every compact sets $K_1, \dots, K_r \subset D$ and $(f_1, \dots, f_r) \in F_{\alpha_1,\dots,\alpha_r}$, the limit*

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For the proof of above theorems we will apply the probabilistic approach. This is influenced in a certain sense by Yu. V. Linnik who was an expert not only in number theory but also in probability theory and mathematical statistics.

2. Auxiliary results

In this section, we will prove a joint limit theorem for the functions $\zeta(s,\alpha_1),\dots,\zeta(s,\alpha_r)$ in the space of analytic functions. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and, for $A\subset\mathcal{B}(H^r(D))$, define

$$P_{T,\underline{\alpha}}(A)=\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\underline{\zeta}(s+i\tau,\underline{\alpha})\in A\right\},$$

where $\underline{\alpha}=(\alpha_1,\dots,\alpha_r)$ and

$$\underline{\zeta}(s,\underline{\alpha})=\left(\zeta(s,\alpha_1),\dots,\zeta(s,\alpha_r)\right).$$

Theorem 6. *Suppose that the numbers α_j, $0<\alpha_j<1$, $\alpha_j\neq\frac{1}{2}$, $j=1,\dots,r$, are arbitrary. Then, on $(H^r(D),\mathcal{B}(H^r(D)))$, there exists a probability measure $P_{\underline{\alpha}}$ such that $P_{T,\underline{\alpha}}$ converges weakly to $P_{\underline{\alpha}}$ as $T\to\infty$.*

We divide the proof of Theorem 6 into lemmas.

Denote by γ the unit circle on the complex plane, and define the set

$$\Omega=\prod_{m\in\mathbb{N}_0}\gamma_m,$$

where $\gamma_m=\gamma$ for all $m\in\mathbb{N}_0$. By the classical Tikhonov theorem, the infinite-dimensional torus Ω with the product topology and pointwise multiplication is a compact topological Abelian group. Define one more set

$$\Omega^r=\prod_{j=1}^{r}\Omega_j,$$

where $\Omega_j=\Omega$ for all $j=1,\dots,r$. Then again by the Tikhonov theorem, Ω^r is a compact topological Abelian group. Denote by $\underline{\omega}=(\omega_1,\dots,\omega_r)$, $\omega_1\in\Omega_1,\dots,\omega_r\in\Omega_r$, the elements of Ω^r Moreover, let $\omega_j(m)$ be the m-th component of the element $\omega_j\in\Omega$, $j=1,\dots,r$, $m\in\mathbb{N}_0$.

For $A\in\mathcal{B}(\Omega^r)$, define

$$Q_{T,\underline{\alpha}}(A)=\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\left(\left((m+\alpha_1)^{-i\tau}:m\in\mathbb{N}_0\right),\dots,\left((m+\alpha_r)^{-i\tau}:m\in\mathbb{N}_0\right)\right)\in A\right\}.$$

Lemma 1. *On $(\Omega^r,\mathcal{B}(\Omega^r))$, there exists a probability measure $Q_{\underline{\alpha}}$ such that $Q_{T,\underline{\alpha}}$ converges weakly to $Q_{\underline{\alpha}}$ as $T\to\infty$.*

Proof. We apply the Fourier transform method. The dual group of Ω^r is isomorphic to

$$\mathcal{G}=\bigoplus_{j=1}^{r}\bigoplus_{m\in\mathbb{N}_0}\mathbb{Z}_{mj},$$

where $\mathbb{Z}_{mj}=\mathbb{Z}$ for all $j=1,\dots,r$, $m\in\mathbb{N}_0$. The element $\underline{k}=(k_{mj}:k_{mj}\in\mathbb{Z},\ j=1,\dots,r,\ \ m\in\mathbb{N}_0)$ $\in\mathcal{G}$, where only a finite number of integers k_{mj} are distinct from zero, acts on Ω^r by

$$\omega\to\omega^{\underline{k}}=\prod_{j=1}^{r}\prod_{m\in\mathbb{N}_0}\omega_j^{k_{mj}}(m).$$

Therefore, the Fourier transform $g_T(\underline{k})$ of $Q_{T,\underline{\alpha}}$ is of the form

$$g_T(\underline{k})=\int_{\Omega^r}\left(\prod_{j=1}^{r}{\prod_{m\in\mathbb{N}_0}}'\omega_j^{k_{jm}}(m)\right)\mathrm{d}Q_{T,\underline{\alpha}},$$

where the sign "$'$" shows that only a finite number of integers k_{mj} are distinct from zero. Thus, by the definition of $Q_{T,\underline{\alpha}}$,

$$g_T(\underline{k}) = \frac{1}{T}\int_0^T \prod_{j=1}^{r} {\prod_{m\in\mathbb{N}_0}}' (m+\alpha_j)^{-i\tau k_{mj}}\,\mathrm{d}\tau = \frac{1}{T}\int_0^T \exp\left\{-i\tau\sum_{j=1}^{r}{\sum_{m\in\mathbb{N}_0}}' k_{mj}\log(m+\alpha_j)\right\}\mathrm{d}\tau. \quad (2)$$

Define two collections of integers

$$\{\underline{k}'\} = \left\{k_{mj} : \sum_{j=1}^{r}{\sum_{m\in\mathbb{N}_0}}' k_{mj}\log(m+\alpha_j) = 0\right\}$$

and

$$\{\underline{k}''\} = \left\{k_{mj} : \sum_{j=1}^{r}{\sum_{m\in\mathbb{N}_0}}' k_{mj}\log(m+\alpha_j) \neq 0\right\}.$$

Obviously, in view of (2),

$$g_T(\underline{k}) = 1 \quad (3)$$

for $\underline{k}\in\{\underline{k}'\}$. If $\underline{k}\in\{\underline{k}''\}$, then integrating in (2), we find that

$$g_T(\underline{k}) = \frac{1-\exp\left\{-iT\sum_{j=1}^{r}{\sum}'_{m\in\mathbb{N}_0} k_{mj}\log(m+\alpha_j)\right\}}{iT\sum_{j=1}^{r}{\sum}'_{m\in\mathbb{N}_0} k_{mj}\log(m+\alpha_j)}.$$

This and (3) show that

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k}\in\{\underline{k}'\},\\ 0 & \text{if } \underline{k}\in\{\underline{k}''\}.\end{cases}$$

The right-hand side of the later equality is continuous in the discrete topology. Therefore, by a continuity theorem for probability measures on compact groups, we obtain that $Q_{T,\underline{\alpha}}$, as $T\to\infty$, converges weakly to a probability measure $Q_{\underline{\alpha}}$ on $(\Omega^r,\mathcal{B}(\Omega^r))$ defined by the Fourier transform

$$g(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k}\in\{\underline{k}'\},\\ 0 & \text{if } \underline{k}\in\{\underline{k}''\}.\end{cases}$$

The lemma is proved. □

Unfortunately, the limit measure $Q_{\underline{\alpha}}$ in Lemma 1 is given by its Fourier transform, we do not know the explicit form of $Q_{\underline{\alpha}}$, and this reflects in Theorems 4 and 5 with non-effective set $F_{\alpha_1,\dots,\alpha_r}$. For example, if the set $L(\alpha_1,\dots,\alpha_r)$ is linearly independent over $\mathbb{Q}$, then

$$g(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k}=\underline{0},\\ 0 & \text{if } \underline{k}\neq\underline{0},\end{cases}$$

and we have that the limit measure $Q_{\underline{\alpha}}$ coincides with the Haar measure on $(\Omega^r,\mathcal{B}(\Omega^r))$.

The next lemma is a joint limit theorem in the space $H^r(D)$ for absolutely convergent Dirichlet series.

Let σ_0 be a fixed number. For $m\in\mathbb{N}_0$ and $n\in\mathbb{N}$, set

$$v_n(m,\alpha_j) = \exp\left\{-\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\sigma_0}\right\}, \quad j=1,\dots,r,$$

and define the functions

$$\zeta_n(s,\alpha_j) = \sum_{m=0}^{\infty}\frac{v_n(m,\alpha_j)}{(m+\alpha_j)^s}, \quad j=1,\dots,r.$$

It is known [9] that the series for $\zeta_n(s,\alpha_j)$ are absolutely convergent for $\sigma > \frac{1}{2}$. For brevity, let

$$\underline{\zeta}_n(s,\underline{\alpha}) = \left(\zeta_n(s,\alpha_1),\dots,\zeta_n(s,\alpha_r)\right),$$

and

$$P_{T,n,\underline{\alpha}}(A) = \frac{1}{T}\text{meas}\left\{\tau\in[0,T] : \underline{\zeta}_n(s+i\tau,\underline{\alpha})\in A\right\},\quad A\in\mathcal{B}(H^r(D)).$$

Lemma 2. *On $(H^r(D),\mathcal{B}(H^r(D)))$, there exists a probability measure $P_{n,\underline{\alpha}}$ such that $P_{T,n,\underline{\alpha}}$ converges weakly to $P_{n,\underline{\alpha}}$ as $T\to\infty$.*

Proof. For $\omega_j\in\Omega_j$, define the functions

$$\zeta_n(s,\omega_j,\alpha_j) = \sum_{m=0}^{\infty}\frac{\omega_j(m)v_n(m,\alpha_j)}{(m+\alpha_j)^s},\quad j=1,\dots,r.$$

Since $|\omega_j(m)|=1$, the series for $\zeta_n(s,\omega_j,\alpha_j)$ is also absolutely convergent for $\sigma>\frac{1}{2}$. Let

$$\underline{\zeta}_n(s,\omega,\underline{\alpha}) = \left(\zeta_n(s,\omega_1,\alpha_1),\dots,\zeta_n(s,\omega_r,\alpha_r)\right).$$

Consider the function $u_{n,\underline{\alpha}}:\Omega^r\to H^r(D)$ given by the formula

$$u_{n,\underline{\alpha}}(\omega) = \underline{\zeta}_n(s,\omega,\underline{\alpha}).$$

In virtue of the absolute convergence of the series for $\zeta_n(s,\omega_j,\alpha_j)$, $j=1,\dots,r$, the function $u_{n,\underline{\alpha}}$ is continuous. Moreover,

$$u_{n,\underline{\alpha}}\left(\left((m+\alpha_1)^{-i\tau}:m\in\mathbb{N}_0\right),\dots,\left((m+\alpha_r)^{-i\tau}:m\in\mathbb{N}_0\right)\right) = \underline{\zeta}_n(s+i\tau,\underline{\alpha}).$$

Therefore, for every $A\in\mathcal{B}(H^r(D))$,

$$P_{T,n,\underline{\alpha}}(A) = \\ = \frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\left\{\left((m+\alpha_1)^{-i\tau}:m\in\mathbb{N}_0\right),\dots,\left((m+\alpha_r)^{-i\tau}:m\in\mathbb{N}_0\right)\right\}\in u_{n,\underline{\alpha}}^{-1}A\right\} = \\ = Q_{T,\underline{\alpha}}(u_{n,\underline{\alpha}}^{-1}A).$$

Hence, $P_{T,n,\underline{\alpha}} = Q_{T,\underline{\alpha}}u_{n,\underline{\alpha}}^{-1}$. Therefore, Theorem 5.1 of [3], Lemma 1 and the continuity of the function $u_{n,\underline{\alpha}}$ imply that $P_{T,n,\underline{\alpha}}$ converges weakly to the measure $P_{n,\underline{\alpha}} = Q_{\underline{\alpha}}u_{n,\underline{\alpha}}^{-1}$ as $T\to\infty$, where $Q_{\underline{\alpha}}$ is the limit measure in Lemma 1. □

The next step of the proof of Theorem 6 consists of the approximation of $\underline{\zeta}(s,\underline{\alpha})$ by $\underline{\zeta}_n(s,\underline{\alpha})$. For this, we recall the metric in the space $H^r(D)$. It is known, see, for example, [4], that there exists a sequence of compact sets $\{K_l : l\in\mathbb{N}\}\subset D$ such that

$$D=\bigcup_{l=1}^{\infty}K_l,$$

$K_l\subset K_{l+1}$ for all $l\in\mathbb{N}$, and, for every compact set $K\subset D$, there exists K_l such that $K\subset K_l$. Let, for $g_1,g_2\in H(D)$,

$$\rho(g_1,g_2) = \sum_{l=1}^{\infty}2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|}.$$

Then ρ is a metric in the space $H(D)$ inducing the topology of uniform convergence on compacta. Now, setting, for $\underline{g}_1=(g_{11},\dots,g_{1r}),\underline{g}_2=(g_{21},\dots,g_{2r})\in H^r(D)$,

$$\underline{\rho}(\underline{g}_1,\underline{g}_2) = \max_{1\leqslant j\leqslant r}\rho(g_{1j},g_{2j})$$

gives a metric in the space $H^r(D)$ inducing its product topology.

LEMMA 3. *The equality*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \underline{\rho}\Big(\underline{\zeta}(s+i\tau,\underline{\alpha}),\underline{\zeta}_n(s+i\tau,\underline{\alpha})\Big)\,\mathrm{d}\tau=0$$

holds.

PROOF. The proof of the lemma does not depend on the arithmetic of the numbers $\alpha_1,\dots,\alpha_r$, and can be found in [8], Lemma 7. □

Now, we consider the sequence $\{P_{n,\underline{\alpha}} : n\in\mathbb{N}\}$, where $P_{n,\underline{\alpha}}$ is the limit measure in Lemma 2.

LEMMA 4. *The sequence $P_{n,\underline{\alpha}}$ is tight, i.e., for every $\varepsilon>0$, there exists a compact set $K=K_\varepsilon\subset H^r(D)$ such that*

$$P_{n,\underline{\alpha}}(K)>1-\varepsilon$$

for all $n\in\mathbb{N}$.

PROOF. For an arbitrary α, $0<\alpha<1$, define

$$P_{T,n,\alpha}(A)=\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\zeta_n(s+i\tau,\alpha)\in A\right\},\quad A\in\mathcal{B}(H(D)),$$

and denote by $P_{n,\alpha}$ the limit measure of $P_{T,n,\alpha}$ as $T\to\infty$. Then, in [2], it was obtained that the sequences $\{P_{n,\alpha}:n\in\mathbb{N}\}$ is tight. Hence, the sequences

$$\{P_{n,\alpha_j}:n\in\mathbb{N}\},\quad j=1,\dots,r,$$

are tight. Clearly, P_{n,α_j} are the marginal measures of the measure $P_{n,\underline{\alpha}}$, i.e.,

$$P_{n,\alpha_j}(A)=P_{n,\underline{\alpha}}\Bigg(\underbrace{H(D)\times\dots\times H(D)}_{j-1}\times A\times H(D)\times\dots\times H(D)\Bigg),\quad A\in\mathcal{B}(H(D)),\tag{4}$$

$j=1,\dots,r$. Since the sequence $\{P_{n,\alpha_j}\}$ is tight, for every $\varepsilon>0$, there exists a compact set $K_j=K_j(\varepsilon)\subset H(D)$ such that

$$P_{n,\alpha_j}(K_j)>1-\frac{\varepsilon}{r},\quad j=1,\dots,r,\tag{5}$$

for all $n\in\mathbb{N}$. We put $K=K_1\times\dots\times K_r$. Then the set K is compact in the space $H^r(D)$. Moreover, in view of (4) and (5),

$$\begin{aligned}P_{n,\underline{\alpha}}(H^r(D)\setminus K)&=P_{n\underline{\alpha}}\Bigg(\bigcup_{j=1}^{r}\Bigg(\underbrace{H(D)\times\dots\times H(D)}_{j-1}\Bigg)\times(H(D)\setminus K_j)\times H(D)\times\dots\times H(D)\Bigg)\\&\leqslant\sum_{j=1}^{r}P_{n,\underline{\alpha}}\Bigg(\underbrace{H(D)\times\dots\times H(D)}_{j-1}\times(H(D)\setminus K_j)\times H(D)\times\dots\times H(D)\Bigg)\\&=\sum_{j=1}^{r}P_{n,\alpha_j}(H(D)\setminus K_j)\leqslant\sum_{j=1}^{r}\frac{\varepsilon}{r}=\varepsilon\end{aligned}$$

for all $n\in\mathbb{N}$. Therefore,

$$P_{n,\underline{\alpha}}(K)\geqslant 1-\varepsilon$$

for all $n \in \mathbb{N}$. The lemma is proved. □

PROOF.[Proof of Theorem 6] We will use the language of convergence in distribution ($\xrightarrow{\mathcal{D}}$). Let the random variable θ be defined on a certain probability space with measure μ, and be uniformly distributed on $[0,1]$. Define the $H^r(D)$-valued random element by the formula

$$X_{T,n,\underline{\alpha}} = X_{T,n,\underline{\alpha}}(s) = \underline{\zeta}_n(s + i\theta T, \underline{\alpha}).$$

Moreover, let $X_{n,\underline{\alpha}} = X_{n,\underline{\alpha}}(s)$ be the $H^r(D)$-valued random element having the distribution $P_{n,\underline{\alpha}}$. Then the assertion of Lemma 2 can be written in the form

$$X_{T,n,\underline{\alpha}} \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,\underline{\alpha}}. \tag{6}$$

Since the sequence $\{P_{n,\underline{\alpha}} : n \in \mathbb{N}\}$ is tight, by the Prokhorov theorem ([3, Theorem 6.1]), it is relatively compact. Therefore, there is a subsequence $\{P_{n_k,\underline{\alpha}}\} \subset \{P_{n,\underline{\alpha}}\}$ such that $P_{n_k,\underline{\alpha}}$ converges weakly to a certain probability measure $P_{\underline{\alpha}}$ on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \to \infty$. In other words, we have the relation

$$X_{n_k,\underline{\alpha}} \xrightarrow[k\to\infty]{\mathcal{D}} P_{\underline{\alpha}}. \tag{7}$$

Define one more $H^r(D)$-valued random element $X_{T,\underline{\alpha}}$ by the formula

$$X_{T,\underline{\alpha}} = X_{T,\underline{\alpha}}(s) = \underline{\zeta}(s + i\theta T, \underline{\alpha}).$$

Then, the application of Lemma 3 shows that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mu\left\{\underline{\rho}\left(X_{T,\underline{\alpha}}, X_{T,n,\underline{\alpha}}\right) \geqslant \varepsilon\right\} \\
&\quad = \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\rho}\left(\underline{\zeta}(s+i\tau,\underline{\alpha}), \underline{\zeta}_n(s+i\tau,\underline{\alpha})\right) \geqslant \varepsilon\right\} \\
&\quad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_0^T \underline{\rho}\left(\underline{\zeta}(s+i\tau,\underline{\alpha}), \underline{\zeta}_n(s+i\tau,\underline{\alpha})\right) \mathrm{d}\tau = 0.
\end{aligned}$$

The latter equality together with relations (6) and (7) shows that all hypotheses of Theorem 4.2 of [3] are satisfied. Therefore, we obtain the relation

$$X_{T,\underline{\alpha}} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\underline{\alpha}},$$

which is equivalent to the weak convergence of $P_{T,\underline{\alpha}}$ to $P_{\underline{\alpha}}$ as $T \to \infty$. The theorem is proved. □

3. Proof of Theorems 4 and 5

Theorems 4 and 5 follow easily from Theorem 6. For this, the notion of the support of a probability measure is applied. Denote by $F_{\alpha_1,\dots,\alpha_r}$ the support of the limit measure $P_{\underline{\alpha}}$ in Theorem 6. We remind that $F_{\alpha_1,\dots,\alpha_r} \subset H^r(D)$ is a minimal closed set such that $P_{\underline{\alpha}}(F_{\alpha_1,\dots,\alpha_r}) = 1$. The set $F_{\alpha_1,\dots,\alpha_r}$ consists of all elements $\underline{g} \in H^r(D)$ such that, for every open neighborhood G of $\underline{g}$, the inequality $P_{\underline{\alpha}}(G) > 0$ is satisfied.

Also, we will use two equivalents of the weak convergence of probability measures. We recall that a set A is a continuity set of the probability measure P if $P(\partial A) = 0$, where ∂A is the boundary of the set A.

LEMMA 5. *Let P_n, $n \in \mathbb{N}$, and P be the probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then the following statements are equivalent:*

1° *P_n converges weakly to P as $n \to \infty$;*
2° *For every open set $G \subset \mathbb{X}$,*

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$

3° *For every continuity set A of the measure P,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

The lemma is Theorem 2.1 of [3].

PROOF.[Proof of Theorem 4] Suppose that $F_{\alpha_1,\dots,\alpha_r}$ is the support of the measure $P_{\underline{\alpha}}$. Then $F_{\alpha_1,\dots,\alpha_r}$ is non-empty closed set of the space $H^r(D)$.

Let $(f_1,\dots,f_r) \in F_{\alpha_1,\dots,\alpha_r}$, $K_1,\dots,K_r$ are compact sets of the strip D and $\varepsilon > 0$. Define

$$G_\varepsilon = \left\{ (g_1,\dots,g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$

Then the set G_ε is an open neighborhood of the element $(f_1,\dots,f_r)$ which belongs to the support of the measure $P_{\underline{\alpha}}$. Therefore,

$$P_{\underline{\alpha}}(G_\varepsilon) > 0. \tag{8}$$

Moreover, in view of Theorem 6, and 1° and 2° of Lemma 5, we have that

$$\liminf_{T\to\infty} P_{T,\underline{\alpha}}(G_\varepsilon) \geqslant P_{\underline{\alpha}}(G_\varepsilon).$$

This, the definitions of $P_{T,\underline{\alpha}}$ and G_ε, and (7) show that

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{ \tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau,\alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

□

PROOF. [Proof of Theorem 5] We use the same notation as in the proof of Theorem 4. We observe that the boundaries $\partial G_{\varepsilon_1}$ and $\partial G_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. Therefore, $P_{\underline{\alpha}}(G_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$. This shows that that the set G_ε is a continuity set of the measure $P_{\underline{\alpha}}$ for all but at most countably many $\varepsilon > 0$. Therefore, using Theorem 6, 1° and 3° of Lemma 5, and inequality (7), we obtain that the limit

$$\lim_{T\to\infty} P_{T,\underline{\alpha}}(G_\varepsilon) = P_{\underline{\alpha}}(G_\varepsilon) > 0$$

exists for all but at most countably many $\varepsilon > 0$. Thus, the definitions of $P_{T,\underline{\alpha}}$ and G_ε prove the theorem. □

4. Conclusions

The Hurwitz zeta-function $\zeta(s,\alpha)$ depends on the parameter α whose arithmetic properties influence the analytic behavior of $\zeta(s,\alpha)$, including the universality. The universality problem is related to the linear independence over $\mathbb{Q}$ of the set

$$L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}.$$

If the parameter α is algebraic irrational, them we have not much information on the set $L(\alpha)$, it is only known by the Cassels theorem that at least 51 percent of elements $L(\alpha)$ in the sense of density are linearly independent over $\mathbb{Q}$. However, there is not any idea how to use the Cassels theorem for the proof of universality.

A similar situation arises in the investigation of the joint universality for Hurwitz zeta-functions. The linear independence of the set

$$L(\alpha_1, \dots, \alpha_r) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0,\ j = 1, \dots, r\}$$

leads to joint universality for the functions $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$. In the paper, we search a way how to avoid involving of the set $L(\alpha_1, \dots, \alpha_r)$. Without using any information about the set $L(\alpha_1, \dots, \alpha_r)$, we prove that there exists a closed non-empty set of analytic functions such that the collections of those functions can be approximated by shifts $(\zeta(s + i\tau, \alpha_1), \dots, \zeta(s + i\tau, \alpha_r))$. It remains a very difficult problem to describe the mentioned set of analytic functions.

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