Совместная дискретная универсальность дзета-функций Лерха

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Аннотация

После 1975 г. работы Воронина известно, что некоторые дзета и $L$-функции универсальны в том смысле, что их сдвигами приближается широкий класс аналитических функций. Рассматриваются два типа сдвигов: непрерывный и дискретный.

В работе изучается универсальность дзета-функций Лерха $L(\lambda, \alpha, s), s = \sigma + it$, которые в полуплоскости $\sigma > 1$ определяются рядами Дирихле с членами $e^{2\pi i \lambda m (m + \alpha)^{-s}}$ с фиксированными параметрами $\lambda \in \mathbb{R}$ и $\alpha$, $0 < \alpha \leq 1$, и мероморфно продолжаются на всю комплексную плоскость. Получены совместные дискретные теоремы универсальности для дзета-функций Лерха. Именно, набор аналитических функций $f_1(s), \ldots, f_r(s)$ одновременно приближаются сдвигами $L(\lambda_1, \alpha_1, s + ikh), \ldots, L(\lambda_r, \alpha_r, s + ikh), k = 0, 1, 2, \ldots$, где $h > 0$ - фиксированное число. При этом требуется линейная независимость над полем рациональных чисел множества $\{\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \ldots, r\}$. Доказательство теорем универсальности использует вероятностные предельные теоремы о слабой сходимости вероятностных мер в пространстве аналитических функций.

Ключевые слова: дзета-функция Лерха, пространство аналитических функций, слабая сходимость, теорема Мергеля, универсальность.

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Joint discrete universality for Lerch zeta-functions

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Abstract

After Voronin’s work of 1975, it is known that some of zeta and $L$-functions are universal in the sense that their shifts approximate a wide class of analytic functions. Two cases of shifts, continuous and discrete, are considered.

The present paper is devoted to the universality of Lerch zeta-functions $L(\lambda, \alpha, s)$, $s = \sigma + it$, which are defined, for $\sigma > 1$, by the Dirichlet series with terms $e^{2\pi i \lambda m} (m+\alpha)^{-s}$ with parameters $\lambda \in \mathbb{R}$ and $\alpha$, $0 < \alpha \leq 1$, and by analytic continuation elsewhere. We obtain joint discrete universality theorems for Lerch zeta-functions. More precisely, a collection of analytic functions $f_1(s), \ldots, f_r(s)$ simultaneously is approximated by shifts $L(\lambda_1, \alpha_1, s+ikh), \ldots, L(\lambda_r, \alpha_r, s+ikh)$, $k = 0, 1, 2, \ldots$, where $h > 0$ is a fixed number. For this, the linear independence over the field of rational numbers for the set $\{(\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \ldots, r), \frac{2\pi}{h}\}$ is required. For the proof, probabilistic limit theorems on the weak convergence of probability measures in the space of analytic function are applied.

Keywords: Lerch zeta-function, Mergelyan theorem, space of analytic functions, universality, weak convergence.

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1. Introduction

In [18], see also [4], S.M. Voronin discovered the universality of the Riemann zeta-function \( \zeta(s), s = \sigma + it \), that a wide class of analytic functions can be approximated by shifts \( \zeta(s + i\tau), \tau \in \mathbb{R} \). After Voronin’s work, various authors extended his universality theorem for some other zeta- and \( L \)-functions, and classes of Dirichlet series. One of universal zeta-functions is the Lerch zeta-function \( L(\lambda, \alpha, s) \) with parameters \( \lambda \in \mathbb{R} \) and \( \alpha, 0 < \alpha \leq 1 \), which is defined, for \( \sigma > 1 \), by the Dirichlet series

\[
L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s}.
\]

The function \( L(\lambda, \alpha, s) \) was introduced and studied independently by R. Lipschitz [14] and M. Lerch [13]. The analytic properties of \( L(\lambda, \alpha, s) \) depend on the parameters \( \lambda \) and \( \alpha \), and in particular case, this is true for the analytic continuation to the whole complex plane. If \( \lambda \notin \mathbb{Z} \), then \( L(\lambda, \alpha, s) \) is an entire function, while, for \( \lambda \in \mathbb{Z} \), \( L(\lambda, \alpha, s) \) reduces to the Hurwitz zeta-function

\[
\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}, \quad \sigma > 1,
\]

which is analytically continued to the whole complex plane, except for a simple pole at the point \( s = 1 \) with residue 1. In virtue of the periodicity of \( e^{2\pi i \lambda m} \), it suffices to suppose that \( 0 < \lambda \leq 1 \).

The theory of the Lerch zeta-function is given in [7].

The first universality result for the function \( L(\lambda, \alpha, s) \) was obtained in [5]. Let

\[
D = \left\{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \right\},
\]

\( \mathcal{K} \) be the class of compact subsets of the strip \( D \) with connected complements, and let \( H(K) \) with \( K \in \mathcal{K} \) denote the class of continuous functions on \( K \) that are analytic in the interior of \( K \). Let \( \text{meas} A \) denote the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \). Then it was obtained in [5] that if \( \alpha \) is transcendental, then for \( K \in \mathcal{K}, f(s) \in H(K) \), \( 0 < \lambda \leq 1 \) and every \( \varepsilon > 0 \),

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |L(\lambda, \alpha, s + i\tau) - f(s)| < \varepsilon \right\} > 0.
\]

The case of rational \( \alpha \) is more complicated. Some conditional result in this direction has been obtained in [7]. If both \( \alpha \) and \( \lambda \) are rational, then the function \( L(\alpha, \lambda, s) \) becomes the periodic Hurwitz zeta-function, and, for it, an universality theorem of type of [9] is true. In this case, a certain condition connecting \( \alpha \) and \( \lambda \) is involved.

The universality of \( L(\alpha, \lambda, s) \) with algebraic irrational \( \alpha \) is an open problem. The case of \( \alpha \) with linearly independent set \( L(\alpha) = \{ \log(m + \alpha) : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \} \) over the field of rational numbers \( \mathbb{Q} \) can be viewed as a certain approximation to that problem, see [17] and [6].

For the function \( L(\alpha, \lambda, s) \), also a discrete universality when \( \tau \) in \( L(\lambda, \alpha, s + i\tau) \) takes values from a certain discrete set is considered. One of the simplest discrete sets is the arithmetic progression \( \{ kh : k \in \mathbb{N}_0 \} \) with \( h > 0 \). Denote by \( \# A \) the cardinality of the set \( A \). If \( \alpha \) is transcendental and the number \( \exp \left( \frac{2\pi i \lambda}{k} \right) \) is rational, then it is known [3], [8] that, for \( K \in \mathcal{K}, f(s) \in H(K), 0 < \lambda \leq 1 \) and every \( \varepsilon > 0 \),

\[
\liminf_{N \to \infty} \frac{1}{N + 1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K} |L(\lambda, \alpha, s + ikh) - f(s)| < \varepsilon \right\} > 0.
\]
Let, for $h > 0$,

$$L(\alpha, h, \pi) = \left\{ \left( \log(m + \alpha) : m \in \mathbb{N}_0 \right), \frac{2\pi}{h} \right\}.$$  

Then, in [12], the transcendence of $\alpha$ and rationality of $\exp\left\{ \frac{2\pi}{h} \right\}$ were replaced by the linear independence over $\mathbb{Q}$ of the set $L(\alpha, h, \pi)$.

The aim of this paper is joint discrete universality theorems for Lerch zeta-functions. We note that the joint universality for Lerch zeta-functions is an interesting problem connecting algebraic properties of the parameters $\alpha_1, \ldots, \alpha_r$ and $\lambda_1, \ldots, \lambda_r$ with analytic properties of a collection $L(\lambda_1, \alpha_1, s), \ldots, L(\lambda_r, \alpha_r, s)$, therefore, there are many results of such a kind. The first joint universality theorem for Lerch zeta-functions was proved in [10], [11].

**Theorem 1.** Suppose that the parameters $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, $\lambda_1 = \frac{a_1}{q_1}, \ldots, \lambda_r = \frac{a_r}{q_r}$, $(a_1, q_1) = 1, \ldots, (a_r, q_r) = 1$, are rational numbers, $k$ is the least common multiple of $q_1, \ldots, q_r$, and that the rank of the matrix

$$\begin{pmatrix}
e^{2\pi i \lambda_1} & e^{2\pi i \lambda_2} & \cdots & e^{2\pi i \lambda_r} \\
e^{4\pi i \lambda_1} & e^{4\pi i \lambda_2} & \cdots & e^{4\pi i \lambda_r} \\
\vdots & \vdots & \ddots & \vdots \\
e^{2k\pi i \lambda_1} & e^{2k\pi i \lambda_2} & \cdots & e^{2k\pi i \lambda_r}
\end{pmatrix}$$

is equal to $r$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j \in H(K_j)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \lim_{N \to \infty} \frac{1}{N + 1} \# \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} \left| L(\lambda_j, \alpha_j, \tau) - f_j(s) \right| < \varepsilon \right\} > 0.$$

Let

$$L(\alpha_1, \ldots, \alpha_r) = \left\{ (\log(m + \alpha_1) : m \in \mathbb{N}_0), \ldots, (\log(m + \alpha_r) : m \in \mathbb{N}_0) \right\}.$$

Then in [16], under the hypothesis that the set $L(\alpha_1, \ldots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, it was obtained that the inequality of Theorem 1 is true for all $0 < \lambda \leq 1$, $j = 1, \ldots, r$.

We will focus on joint discrete analogues of the above results. For $h > 0$, define the set

$$L(\alpha_1, \ldots, \alpha_r; h, \pi) = \left\{ (\log(m + \alpha_1) : m \in \mathbb{N}_0), \ldots, (\log(m + \alpha_r) : m \in \mathbb{N}_0), \frac{2\pi}{h} \right\}.$$

Then we have

**Theorem 2.** Suppose that the set $L(\alpha_1, \ldots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$, $f_j \in H(K_j)$ and $0 < \lambda_j \leq 1$. Then, for every $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N + 1} \# \left\{ 0 \leq k \leq N : \sup_{1 \leq j \leq r} \sup_{s \in K_j} \left| L(\lambda_j, \alpha_j, k + i\lambda_j) - f_j(s) \right| < \varepsilon \right\} > 0.$$

Theorem 2 has the following modification.

**Theorem 3.** Suppose that the set $L(\alpha_1, \ldots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$, $f_j \in H(K_j)$ and $0 < \lambda_j \leq 1$. Then the limit

$$\lim_{N \to \infty} \frac{1}{N + 1} \# \left\{ 0 \leq k \leq N : \sup_{1 \leq j \leq r} \sup_{s \in K_j} \left| L(\lambda_j, \alpha_j, k + i\lambda_j) - f_j(s) \right| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The proofs of Theorems 2 and 3 are based on statistical properties of Lerch zeta-functions, more precisely, on limit theorems of weakly convergent probability measures in the space of analytic functions.
2. Discrete limit theorems

Denote by $B(X)$ the Borel $\sigma$-field of the space $X$. We recall that $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Denote by $H(D)$ the space of analytic functions on $D$ endowed with the topology of uniform convergence on compacta. In this section, we consider the weak convergence of probability measures defined on $(H(D), B(H(D)))$.

We use the notation $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and define

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. Then, by the famous Tikhonov theorem, the torus $\Omega$ with the product topology and pointwise multiplication is a compact topological Abelian group. Putting

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \ldots, r$, by the Tikhonov theorem again, we have that $\Omega^r$ is a compact topological Abelian group. Therefore, on $(\Omega^r, B(\Omega^r))$, the probability Haar measure $m_H$ can be defined. This gives the probability space $(\Omega^r, B(\Omega^r), m_H)$. Denote by $m_{jH}$ the probability Haar measure on $(\Omega_j, B(\Omega^j))$, $j = 1, \ldots, r$. Then we have that

$$m_H = m_{1H} \times \cdots \times m_{rH}.$$

Let $\omega_j$ be the elements of $\Omega_j$, $j = 1, \ldots, r$, and $\omega = (\omega_1, \ldots, \omega_r)$ denote the elements of $\Omega^r$. Moreover, denote by $\omega_j(m)$ the projection of an element $\omega_j \in \Omega_j$ to the circle $\gamma_m$, $m \in \mathbb{N}_0$, $j = 1, \ldots, r$. Now, on the probability space $(\Omega^r, B(\Omega^r), m_H)$, define the $H^r(D)$-valued random element $L(\lambda, \alpha, s, \omega)$, where $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\alpha = (\alpha_1, \ldots, \alpha_r)$, by

$$L(\lambda, \alpha, s, \omega) = (L_1(\lambda_1, \alpha_1, s, \omega_1), \ldots, L_r(\lambda_r, \alpha_r, s, \omega_r)),$$

where

$$L_j(\lambda_j, \alpha_j, s, \omega_j) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m}}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r.$$

We note that the latter series are uniformly convergent on compact subsets of the strip $D$ [7], thus, they define the $H(D)$-valued random elements.

Having the above definitions, we state a joint discrete limit theorem for Lerch zeta-functions.

**Theorem 4.** Suppose that the set $L(\alpha_1, \ldots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. Then

$$P_N(A) \overset{\text{def}}{=} \frac{1}{N+1} \# \{0 \leq k \leq N : L(\lambda, \alpha, s + ikh) \in A\}, \quad A \in B(H^r(D)),$$

converges weakly to the distribution $P_L$ of the random element $L(\lambda, \alpha, s, \omega)$ as $N \to \infty$.

We remind that, for $A \in B(H^r(D))$,

$$P_L(A) = m_H \{\omega \in \Omega^r : L(\lambda, \alpha, s, \omega) \in A\}.$$

We divide the proof of Theorem 4 into lemmas. The first of them deals with the weak convergence of probability measures on $(\Omega^r, B(\Omega^r))$, and for that the linear independence of the set $L(\alpha_1, \ldots, \alpha_r; h, \pi)$ is essentially applied.

Let, for $A \in B(\Omega^r)$,

$$Q_N(A) = \frac{1}{N+1} \# \{0 \leq k \leq N : ((m + \alpha_1)^{-ikh} : m \in \mathbb{N}_0), \ldots, ((m + \alpha_r)^{-ikh} : m \in \mathbb{N}_0) \in A\}.$$
**Lemma 1.** Suppose that the set $L(\alpha_1, \ldots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. Then $Q_N$ converges weakly to the Haar measure $m_H$ as $N \to \infty$.

**Proof.**

We consider the Fourier transform of $Q_N$. Since characters of the group $\Omega^r$ are of the form

$$\prod_{j=1}^{r} \prod_{m=0}^{\infty} \omega_j^{k_{jm}}(m),$$

where only a finite number of integers $k_{jm}$ are distinct from zero, we have that the Fourier transform $g_N(\overline{k_1}, \ldots, \overline{k_r})$, $\overline{k_j} = (k_{jm} : k_{jm} \in \mathbb{Z}, m \in \mathbb{N}_0)$, $j = 1, \ldots, r$, of $Q_N$ is

$$g_N(\overline{k_1}, \ldots, \overline{k_r}) = \int_{\Omega^r} \prod_{j=1}^{r} \prod_{m=0}^{\infty} \omega_j^{k_{jm}}(m) dQ_N = \frac{1}{N+1} \sum_{k=0}^{N} \prod_{j=1}^{r} \prod_{m=0}^{\infty} (m + \alpha_j)^{-ikhk_{jm}},$$

where $\sum'$ means that only a finite number of integers $k_{jm}$ are distinct from zero. Clearly, $g_N(\overline{0}, \ldots, \overline{0}) = 1$.

Since the set $L(\alpha_1, \ldots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$,

$$\exp \left\{ -ih \sum_{j=1}^{r} \sum_{m=0}^{\infty'} k_{jm} \log(m + \alpha_j) \right\} \neq 1$$

for $(\overline{k_1}, \ldots, \overline{k_r}) \neq (\overline{0}, \ldots, \overline{0})$. Actually, if this inequality is not true, the

$$h \sum_{j=1}^{r} \sum_{m=0}^{\infty'} k_{jm} \log(m + \alpha_j) - \frac{2\pi l}{h} = 0$$

with $l \in \mathbb{Z}$, and this contradicts the linear independence of the set $L(\alpha_1, \ldots, \alpha_r; h, \pi)$. Thus, in this case, we find by (1) that

$$g_N(\overline{k_1}, \ldots, \overline{k_r}) = \frac{1 - \exp \left\{ -(N+1)ih \sum_{j=1}^{r} \sum_{m=0}^{\infty} k_{jm} \log(m + \alpha_j) \right\}}{(N+1) \left( 1 - \exp \left\{ -ih \sum_{j=1}^{r} \sum_{m=0}^{\infty} k_{jm} \log(m + \alpha_j) \right\} \right)}.$$

This and (2) show that

$$\lim_{N \to \infty} g_N(\overline{k_1}, \ldots, \overline{k_r}) = \begin{cases} 1 & \text{if } (\overline{k_1}, \ldots, \overline{k_r}) = (\overline{0}, \ldots, \overline{0}), \\ 0 & \text{if } (\overline{k_1}, \ldots, \overline{k_r}) \neq (\overline{0}, \ldots, \overline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure $m_H$, the lemma is proved. $\square$

Now, we will apply Lemma 1 to obtain a joint limit theorem in the space of analytic functions for functions given by absolutely convergent Dirichlet series connected to Lerch zeta-functions. Let $\sigma > \frac{1}{2}$ be a fixed number, and, for $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$,

$$v_n(m, \alpha_j) = \exp \left\{ -\left( \frac{m + \alpha_j}{n + \alpha_j} \right)^{\sigma} \right\}, \quad j = 1, \ldots, r.$$
Define
\[ L_n(\Lambda, \alpha, s) = (L_n(\lambda_1, \alpha_1, s), \ldots, L_n(\lambda_r, \alpha_r, s)) \]
and
\[ L_n(\lambda, \alpha, s, \omega) = (L_n(\lambda_1, \alpha_1, s, \omega_1), \ldots, L_n(\lambda_r, \alpha_r, s, \omega_r)), \]
where
\[ L_n(\lambda_j, \alpha_j, s) = \sum_{m=0}^{\infty} e^{2\pi i m} \frac{v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r, \]
and
\[ L_n(\lambda_j, \alpha_j, s, \omega) = \sum_{m=0}^{\infty} e^{2\pi i m} \frac{\omega_j(m) v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r, \]
It is known \[7\] that the series for \( L_n(\lambda_j, \alpha_j, s) \) and \( L_n(\lambda_j, \alpha_j, s, \omega_j) \) are absolutely convergent for \( \sigma > \frac{1}{2} \).

The next lemma deals with weak convergence for
\[ P_{N,n}(A) \overset{\text{def}}{=} \frac{1}{N+1} \# \{ 0 \leq k \leq N : L_n(\Lambda, \alpha, s + ikh) \in A \}, \quad A \in \mathcal{B}(H^r(D)). \]
Define the function \( u_n : \Omega^r \to H^r(D) \) by the formula
\[ u_n(\omega) = L_n(\Lambda, \alpha, s, \omega), \quad \omega \in \Omega. \]
Since the series for \( L_n(\lambda_j, \alpha_j, s, \omega_j) \), \( j = 1, \ldots, r \), are absolutely convergent for \( \sigma > \frac{1}{2} \), the function \( u_n \) is continuous, hence it is \((\mathcal{B}(\Omega^r), \mathcal{B}(H^r(D)))\)-measurable. Therefore, the measure \( m_H \) induces \[1\] on \((H^r(D), \mathcal{B}(H^r(D)))\) the unique probability measure \( \hat{P}_n \overset{\text{def}}{=} m_H u_n^{-1} \), where, for \( A \in \mathcal{B}(H^r(D)) \),
\[ \hat{P}_n(A) = m_H u_n^{-1}(A) = m_H(u_n^{-1}A). \]

**Lemma 2.** Suppose that the set \( L(\alpha_1, \ldots, \alpha_r; h, \pi) \) is linearly independent over \( \mathbb{Q} \). Then \( P_{N,n} \) converges weakly to \( \hat{P}_n \) as \( N \to \infty \).

**Proof.**
Let \( Q_N \) be defined in Lemma 1. Then the definitions of \( P_{N,n} \), \( Q_N \) and \( u_n \) show that, for every \( A \in \mathcal{B}(H^r(D)) \),
\[ P_{N,n}(A) = \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \left( (m + \alpha_1)^{ikh} : m \in \mathbb{N}_0 \right), \ldots, \left( (m + \alpha_r)^{ikh} : m \in \mathbb{N}_0 \right) \in u_n^{-1}A \right\} = Q_N(u_n^{-1}A), \]
i.e., \( P_{N,n} = Q_N u_n^{-1} \). This, Lemma 1, the continuity of \( u_n \) and Theorem 5.1 from \[1\] show that \( P_{N,n} \) converges weakly to the measure \( m_H u_n^{-1} \) as \( N \to \infty \).

Now, we will approximate \( L(\Lambda, \alpha, s) \) by \( L_n(\Lambda, \alpha, s) \). For \( g_1, g_2 \in H(D) \), let
\[ \rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}, \]
where \( \{K_l : l \in \mathbb{N}\} \) is a sequence of compact subsets of the strip \( D \) such that
\[ D = \bigcup_{l=1}^{\infty} K_l, \]
$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact subset, then $K \subset K_l$ for some $l$. The proof of the existence of the sequence $\{K_l : l \in \mathbb{N}\}$ can be found, for example, in [2]. The metric $\rho$ induces the topology of the space $H(D)$ of uniform convergence on compacta. The metric $\rho$ in $H^r(D)$ inducing the product topology is defined by

$$\rho(\eta_1, \eta_2) = \max_{1 \leq j \leq r} \rho(\eta_{1j}, \eta_{2j}),$$

where $\eta_1 = (g_{11}, \ldots, g_{1r})$, $\eta_2 = (g_{21}, \ldots, g_{2r}) \in H^r(D)$. \Box

**Lemma 3.** For all $\lambda, \alpha$ and $h > 0$,

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho(L(\lambda, \alpha, s + ikh), L_n(\lambda, \alpha, s + ikh)) = 0,$$

for all $j = 1, \ldots, r$, that were obtained in Lemma 3 of [12]. \Box

We recall that the measure $\hat{P}_n$ was defined in Lemma 2.

**Lemma 4.** Suppose that the set $L(\alpha_1, \ldots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. Then the sequence $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight, i.e., for every $\epsilon > 0$, there exists a compact subset $K = K(\epsilon) \subset H^r(D)$ such that

$$\hat{P}_n(K) > 1 - \epsilon$$

for all $n \in \mathbb{N}$.

**Proof.**

Consider the marginal measures of $\hat{P}_n$, i.e., the measures

$$\hat{P}_{n,j}(A) = \hat{P}_n \left( \frac{H(D) \times \cdots \times H(D)}{j-1} \times A \times H(D) \times \cdots \times H(D) \right), \quad A \in \mathcal{B}(H(D)),$$

where $j = 1, \ldots, r$. The linear independence of the set $L(\alpha_1, \ldots, \alpha_r; h, \pi)$ implies that for $L(\lambda, h, \pi)$, $j = 1, \ldots, r$. Therefore, in view of the proof of Lemma 5 from [12], we have that $\hat{P}_{n,j}$ converges weakly to the distribution $P_{L_j}$ of the random element $L_j(\lambda, \alpha_j, s, \omega_j)$ as $n \to \infty$, $j = 1, \ldots, r$. Hence, the sequence $\{\hat{P}_{n,j} : n \in \mathbb{N}\}$ is relatively compact, $j = 1, \ldots, r$. Since the set $H(D)$ is complete and separable, by the inverse Prokhorov Theorem [1, Theorem 6.2], the sequence $\{\hat{P}_{n,j} : n \in \mathbb{N}\}$ is tight, $j = 1, \ldots, r$. Thus, for every $\epsilon > 0$, there exists a compact subset $K_j \subset H(D)$ such that

$$\hat{P}_n(K_j) > 1 - \frac{\epsilon}{r}, \quad j = 1, \ldots, r,$$

for all $n \in \mathbb{N}$. The set $K = K_1 \times \cdots \times K_r$ is compact in $H^r(D)$. Moreover,

$$\hat{P}_n(H^r(D) \setminus K) = \hat{P}_n \left( \bigcup_{j=1}^{r} (H(D) \setminus K_j) \right) \leq \sum_{j=1}^{r} \hat{P}_{n,j}(H(D) \setminus K_j) < \epsilon$$

for all $n \in \mathbb{N}$, i.e., the sequence $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight. \Box

For convenience, we recall one result from [1]. Suppose that $(S, \rho)$-valued random elements $Y_n, X_{1n}, X_{2n}, \ldots$ are defined on the same probability space with measure $\mathbb{P}$, and that the space $S$ is separable.
Lemma 5. Suppose that, for every $k$,
\[ X_{kn} \xrightarrow{D_{n \to \infty}} X_k \]
and
\[ X_k \xrightarrow{D_{k \to \infty}} X. \]
Moreover, for every $\epsilon > 0$, let
\[ \lim_{k \to \infty} \limsup_{n \to \infty} \mathbb{P}\{\rho(X_{kn}, Y_n) \geq \epsilon\} = 0. \]
Then $Y_n \xrightarrow{D_{n \to \infty}} X$.

The lemma is Theorem 4.2 from [1].

Proof of Theorem 3. By Lemma 4 and the Prokhorov theorem [1, Theorem 6.1], the sequence
\[ \{\hat{P}_n : n \in \mathbb{N}\} \]
is relatively compact. Hence, every subsequence of $\hat{P}_n$ contains a subsequence
\[ \{\hat{P}_{nk}\} \]
such that $\hat{P}_{nk}$ converges weakly to a certain probability measure $P$ on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \to \infty$. Therefore, denoting by $\hat{X}_n = \hat{X}_n(s)$ the $H^r(D)$-valued random element having the distribution $\hat{P}_n$, we have that
\[ \hat{X}_{nk} \xrightarrow{D_{k \to \infty}} P. \]  
(3)
Moreover, by Lemma 2,
\[ X_{N,n} \xrightarrow{D_{N \to \infty}} \hat{X}_n, \]  
(4)
where the $H^r(D)$-valued random element $X_{N,n} = X_{N,n}(s)$ is defined by
\[ X_{N,n}(s) = L_n(\Lambda, \alpha, s + i\theta_N), \]
and $\theta_N$ is a random variable defined on a certain probability space $(\hat{\Omega}, \mathcal{F}, \mathbb{P})$ by the formula
\[ \mathbb{P}(\theta_N = kh) = \frac{1}{N + 1}, \quad k = 0, 1, \ldots, N. \]
Define one more $H^r(D)$-valued random element
\[ Y_N = Y_N(s) = L(\Lambda, \alpha, s + i\theta_N). \]
Then, in view of Lemma 3, for every $\epsilon > 0$,
\[ \lim_{n \to \infty} \limsup_{N \to \infty} \mathbb{P}(\rho(X_{N,n}, Y_N) \geq \epsilon) \]
\[ = \lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N + 1} \sum_{0 \leq k \leq N} \mathbb{P}(\rho(L(\Lambda, \alpha, s + ikh), L_n(\Lambda, \alpha, s + ikh)) \geq \epsilon) \]
\[ \leq \lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{(N + 1)\varepsilon} \sum_{k=0}^{N} \mathbb{P}(\rho(L(\Lambda, \alpha, s + ikh), L_n(\Lambda, \alpha, s + ikh)) = 0. \]
This equality together with relations (3) and (4) shows that all hypotheses of Lemma 5 are satisfied. Therefore, we obtain the relation
\[ Y_N \xrightarrow{D_{N \to \infty}} P. \]  
(5)
Thus, we have that $P_N$ converges weakly to $P$ as $N \to \infty$. Moreover, the relation (5) shows that the measure $P$ is independent of the choice of the subsequence $\hat{P}_{nk}$. Since the sequence $\hat{P}_n$ is relatively compact, hence we obtain that
\[ \hat{X}_n \xrightarrow{D_{n \to \infty}} P. \]
This means that \( \hat{X}_n \) converges weakly to \( P \) as \( n \to \infty \). The latter remark allows easily to identify the measure \( P \). Actually, in [16], it was obtained that, under hypothesis that the set \( L(\alpha_1, \ldots, \alpha_r) \) is linearly independent over \( \mathbb{Q} \),
\[
\frac{1}{T} \text{meas} \{ \tau \in [0, T] : L(\alpha, s + i\tau) \in A \}, \quad A \in \mathcal{B}(H^r(D)),
\]
also converges weakly to the limit measure \( P \) of \( \hat{P}_n \) as \( n \to \infty \), and that \( P \) coincides with \( P_L \). Obviously, the linear independence of the set \( L(\alpha_1, \ldots, \alpha_r; h, \pi) \) implies that of the set \( L(\alpha_1, \ldots, \alpha_r) \).

Therefore, \( P_N \) also converges weakly to \( P_L \) which is the limit measure of \( \hat{P}_n \). The theorem is proved.

\[ \square \]

3. Proofs of universality

We remind the Mergelyan theorem on approximation of analytic functions by polynomials [15].

**Lemma 6.** Let \( K \) be a compact subset on the complex plane with connected complement, and let \( f(s) \) be a function continuous on \( K \) and analytic in the interior of \( K \). Then, for every \( \varepsilon > 0 \), there exists a polynomial \( p(s) \) such that
\[
\sup_{s \in K} |f(s) - p(s)| < \varepsilon.
\]

We also need the explicit form of the support of the measure \( P_L \). We recall that the support of \( P_L \) is a closed minimal set \( S_L \) such that \( P_L(S_L) = 1 \). The set \( S_L \) consists of all \( g \in H^r(D) \) such that, for every open neighbourhood \( G \) of \( g \), the inequality \( P_L(G) > 0 \) is true.

**Lemma 7.** The support of the measure \( P_L \) is the whole of \( H^r(D) \).

**Proof.**
It was observed above that \( P_L \) is the limit measure of (6). Thus, the lemma follows from [16], see the proof of Theorem 2.1. \( \square \)

We also recall two equivalents of the weak convergence of probability measures. Let \( P_n, n \in \mathbb{N}, \) and \( P \) be probability measures on \( (X, \mathcal{B}(X)) \). The set \( A \in \mathcal{B}(X) \) is called a continuity set of \( P \) if \( P(\partial A) = 0 \), where \( \partial A \) is the boundary of \( A \).

**Lemma 8.** The following statements are equivalent:
1° \( P_n \) converges weakly to \( P \);
2° for every open set \( G \subset X \),
\[
\liminf_{n \to \infty} P_n(G) \geq P(G),
\]
3° for every continuity set \( A \) of the measure \( P \),
\[
\lim_{n \to \infty} P_n(A) = P(A).
\]

The lemma is a part of Theorem 2.1 from [1].

**Proof of Theorem 2.**
In view of Lemma 6, there exist polynomials \( p_1(s), \ldots, p_r(s) \) such that
\[
\sup_{1 \leq j \leq r} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{2}.
\]

Consider the set
\[
G_\varepsilon = \left\{ (g_1, \ldots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - p_j(s)| < \frac{\varepsilon}{2} \right\}.
\]
Then the set $G_\varepsilon$ is open, and, by Lemma 7, is a neighborhood of the collection $(p_1(s), \ldots, p_r(s))$ which is an element of the support of the measure $P_L$. Therefore, the inequality

$$P_L(G_\varepsilon) > 0$$

is satisfied. Hence, by Theorem 4 and 2° of Lemma 8,

$$\liminf_{N \to \infty} P_N(G_\varepsilon) \geq P_L(G_\varepsilon) > 0.$$  \hspace{1cm} (9)

This, and the definitions of $P_N$ and $G_\varepsilon$ show that

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + ikh) - p_j(s)| < \frac{\varepsilon}{2} \right\} > 0.  \hspace{1cm} (10)$$

Let $k \in \mathbb{N}$ satisfy the inequality

$$\sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + ikh) - p_j(s)| < \frac{\varepsilon}{2}.  \hspace{1cm}$$

Then, for such $k$, (7) implies the inequality

$$\sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + ikh) - f_j(s)| < \varepsilon.  \hspace{1cm}$$

Therefore, (10) gives the assertion of the theorem. □

**Proof of Theorem 3.**

Consider the set

$$\hat{G}_\varepsilon = \left\{ (g_1, \ldots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$  

Then the set $\hat{G}_\varepsilon$ is open. Moreover, the boundary $\partial G_\varepsilon$ lies in the set

$$\left\{ (g_1, \ldots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| = \varepsilon \right\}.  \hspace{1cm}$$

Therefore, $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \emptyset$ for positive $\varepsilon_1 \neq \varepsilon_2$. From this, it follows that $P_L(\hat{G}_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$, i.e., the set $\hat{G}_\varepsilon$ is a continuity set of $P_L$ for all but at most countably many $\varepsilon > 0$. Hence, by Theorem 4, and 1° and 3° of Lemma 8, the limit

$$\lim_{N \to \infty} P_N(\hat{G}_\varepsilon) = P_L(\hat{G}_\varepsilon)  \hspace{1cm} (11)$$

exists for all but at most countably many $\varepsilon > 0$. Moreover, it is not difficult to see that if $(g_1, \ldots, g_r) \in G_\varepsilon$, where $G_\varepsilon$ is defined in the proof of Theorem 2, then, taking into account (7), we find that

$$\sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| \leq \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - p_j(s)| + \sup_{1 \leq j \leq r} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \varepsilon.$$  

This shows that $G_\varepsilon \subset \hat{G}_\varepsilon$. Since, by (9), $P_L(G_\varepsilon) > 0$, the monotonicity of the measure gives the inequality $P_L(\hat{G}_\varepsilon) > 0$. This inequality and (11) prove the theorem. □
4. Conclusions

The Lerch zeta-function $L(\lambda, \alpha, s) = \sigma + it$, with parameters $\lambda \in \mathbb{R}$ and $0 < \alpha \leq 1$ is defined, for $\sigma > 1$, by the series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s},$$

and by analytic continuation elsewhere. In the paper, it is obtained that a collection of Lerch zeta-functions $(L(\lambda_1, \alpha_1, s), \ldots, L(\lambda_r, \alpha_r, s))$ has a discrete universality property, i.e., a wide class of analytic functions can be approximated by shifts $L(\lambda_1, \alpha_1, s + ikh), \ldots, L(\lambda_r, \alpha_r, s + ikh)$, $h > 0$, $k = 0, 1, 2, \ldots$. For this, the linear independence over $\mathbb{Q}$ of the set

$$\left\{ (\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \ldots, r), \frac{2\pi}{h} \right\}$$

is required. More precisely, if $K_1, \ldots, K_r$ are compact subsets of the strip $\{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \}$ with connected complements, and $f_1(s), \ldots, f_r(s)$ are functions continuous on $K_1, \ldots, K_r$ and analytic in the interior of $K_1, \ldots, K_r$, respectively, then, for every $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N + 1} \# \left\{ 0 \leq k \leq N : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(\lambda_1, \alpha_1, s + ikh) - f_j(s)| < \varepsilon \right\} > 0.$$

It is possible to consider a more general situation, i.e., to consider the approximation of $f_1(s), \ldots, f_r(s)$ by different shifts $L(\lambda_1, \alpha_1, s + ikh_1), \ldots, L(\lambda_r, \alpha_r, s + ikh_r)$ with $h_1 > 0, \ldots, h_r > 0$. For this case, a new more general method than that of the paper is required, and it will be developed in a subsequent paper.

**СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ**


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