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Расширения Любина — Тейта и модуль Карлица над проективной прямой: явная связь

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Аннотация

В данной статье рассматриваются различные подходы к построению максимальных абелевых расширений для локальных и глобальных геометрических полей. Теория Любина — Тейта играет ключевую роль в построении максимального Абелева расширения для локальных геометрических полей. В случае глобальных геометрических полей особый интерес представляют модули Дринфельда. В настоящей работе рассматривается самый простой частный случай модулей Дринфельда для проективной прямой, который называется модулем Карлица.

Во введении мы приводим мотивацию и краткую историческую справку по затронутым в работе темам.

В первом и втором разделах мы приводим краткую информацию о модулях Любина-Тейта и модуле Карлица.

В третьем разделе мы приводим два основных результата:

- установлена явная связь между теориями глобальных и локальных полей в геометрическом случае проективной прямой над конечным полем: доказано, что башня расширения модуля Карлица индуцирует башню расширений Любина-Тейта.
- установлена связь между отображениями Артина расширений функционального поля произвольной проективной гладкой неприводимой кривой и расширениями пополнений локальных колец в замкнутых точках этой кривой.

В последнем разделе мы формулируем различные открытые задачи и интересные направления для дальнейших исследований, которые включают обобщение первого результата для произвольной гладкой проективной кривой над конечным полем и рассмотрение модулей Дринфельда более высокого ранга.

Ключевые слова: теория полей классов, теория Любина — Тейта, модуль Карлица, модули Дринфельда, отображение Артина, максимальное абелево расширение, проективная прямая над конечным полем.

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Lubin–Tate extensions and Carlitz module over a projective line: an explicit connection

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Abstract

In this article we consider different approaches for constructing maximal abelian extensions for local and global geometric fields. The Lubin–Tate theory plays key role in the maximal abelian extension construction for local geometric fields. In the case of global geometric fields, Drinfeld modules are of particular interest. In this paper we consider the simpliest special case of Drinfeld modules for projective line which is called the Carlitz module.

In the introduction, we provide motivation and a brief historical background on the topics covered in the work.

In the first and second sections we provide brief information about Lubin–Tate modules and Carlitz module.

In the third section we present two main results:

- an explicit connection between the local and global field theory in the geometric case for projective line over finite field: it is proved that the extension tower of Carlitz module induces the tower of the Lubin–Tate extensions.
- a connection between Artin maps of extensions of a function field of an arbitrary projective smooth irreducible curve and extensions of completions of local rings at closed points of this curve.

In the last section we formulate different open problems and interesting directions for further research, which include generalization first result for an arbitrary smooth projective curve over a finite field and consideration Drinfeld modules of higher rank.

Keywords: class field theory, Lubin–Tate theory, Carlitz module, Drinfeld modules, Artin map, maximal abelian extension, projective line over a finite field.

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Introduction

The main motivation of this work is the study of Hilbert's 9th problem, in particular, an attempt to transfer the results of S. V. Vostokov [1] to the case of geometric fields. This problem is of high interest in modern algebraic number theory and has been discussed in numerous scientific papers [2] - [3].

In 1853, the famous *Kronecker–Weber theorem* was proved for the arithmetic global case. It says that an arbitrary finite abelian extension of the field of rational numbers lies in some cyclotomic extension of $\mathbb{Q}$.

Consider a geometric analogue of this statement. Let $\mathbb{F}_q(X)$ be a field of rational functions of the projective line $X = \mathbb{P}^1_{\mathbb{F}_q}$ over a finite field $\mathbb{F}_q$. Using the theory of *Carlitz module*, we can build explicitly cyclotomic extensions of $\mathbb{F}_q(X)$ and construct the maximum abelian extension of the *global field* $\mathbb{F}_q(X)$.

Teruyoshi Yoshida built the maximum abelian extension and the Artin map for an arbitrary *local field* of an arbitrary characteristic [6]. In the construction, the theory of formal Lubin–Tate modules was applied.

The first result of this paper is a construction of a connection between theories of building a maximal abelian extension for local and global fields. It is proved that the extension tower of *Carlitz module* of the global field induces the extension tower of *formal Lubin—Tate modules* over the completion of the local ring at the closed point of our curve *Theorem 1.*

The second result is a description of the connection between Artin mappings for an arbitrary projective irreducible smooth curve X and for completions of local rings at its closed points *Theorem 2.*

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Preliminaries and notation

Local fields

Throughout the work all the considered fields have characteristic other than 2.

Let p be a prime integer not equal to 2. $q = p^s$, where s is an arbitrary natural number. Then for $\mathbb{F}_q$ we denote a finite field consisting of q elements.

We call a field $\mathfrak{K}$ with non-Archimedean regular valuation $\nu_{\mathfrak{K}}$ *local* if:

- $\mathfrak{K}$ is complete with respect to $\nu_{\mathfrak{K}}$;
- residue field with respect to $\nu_{\mathfrak{K}}$ is finite.

We denote *ring of integers*, *simple ideal* and *residue field* by $\mathfrak{O}_{\mathfrak{K}}, \mathfrak{p}_{\mathfrak{K}}$ and $\mathfrak{k}_{\mathfrak{K}}$ respectively. If $\mathfrak{K}'$ is a finite extension of local field $\mathfrak{K}$ we denote ramification index and the degree of inertia by $e_{\mathfrak{K}'/\mathfrak{K}}$ and $f_{\mathfrak{K}'/\mathfrak{K}}$ respectively.

If $e_{\mathfrak{K}'/\mathfrak{K}} = 1$, then the extension is called *finite unramified.* If $f_{\mathfrak{K}'/\mathfrak{K}} = 1$, then we call such extension as *finite completely ramified.* Finally, if $e_{\mathfrak{K}'/\mathfrak{K}}$ is coprime with p, then the extension is called *finite weakly ramified*, and the ramification itself is called *finite tamely ramification.*

Now we need to take a closer look on infinite extensions.

A separable extension $E/\mathfrak{K}$ is said to be *unramified (completely ramified)* if the extension is obtained by the union of finite unramified (completely ramified) extensions over $\mathfrak{K}$. We assume that the *ring of integers* of an arbitrary separable extension $E/\mathfrak{K}$ is the complete closure of the ring of integers $\mathfrak{O}_{\mathfrak{K}}$ in E. In the case when all our extensions are finite, this definition coincides with the classical one. We also call separable extension $E/\mathfrak{K}$ a *finitely-ramified* if $E/\mathfrak{K}$ is a finite extension of some unramified extension.

It is well-known fact that for every positive integer n and arbitrary local field $\mathfrak{K}$ there is a unique finite unramified extension $\mathfrak{K}_n$ such that $[\mathfrak{K}_n : \mathfrak{K}]=n$. Moreover, $\mathrm{Gal}(\mathfrak{K}_n/\mathfrak{K})\cong\mathrm{Gal}(\mathfrak{t}_{\mathfrak{K}_n}/\mathfrak{t})\cong(\mathbb{Z}/n\mathbb{Z})^{\times}$. We denote the *Frobenius automorphism*, the generator of $\mathrm{Gal}(\mathfrak{K}_n/\mathfrak{K})$, by Δ.

Maximal unramified extension of the local field $\mathfrak{K}$ is $\bigcup\limits_{n\in\mathbb{N}} \mathfrak{K}_n$ and we denote it by $\mathfrak{K}^{ur}$. It is unramified by the very definition. What is more, $\mathrm{Gal}(\mathfrak{K}^{ur}/k)$ is a protective limit of $\mathrm{Gal}(\mathfrak{K}_n/\mathfrak{K})$:

$$\mathrm{Gal}(\mathfrak{K}^{ur}/\mathfrak{K}) \cong \varprojlim \mathrm{Gal}(\mathfrak{K}_n/\mathfrak{K}) \cong \varprojlim(\mathbb{Z}/n\mathbb{Z})^{\times} \cong \widehat{\mathbb{Z}}.$$

A Frobenius automorphism is the unique automorphism Δ in $\mathrm{Gal}(\mathfrak{K}^{ur}/\mathfrak{K})$ with the property $\Delta(x) \equiv x^q(\mathit{mod}\ \mathfrak{p}^{\mathfrak{K}^{ur}})$ for all $x \in \mathfrak{O}_{\mathfrak{K}^{ur}}$. The Frobenius element maps to identity under the isomorphism $\varprojlim(\mathbb{Z}/n\mathbb{Z})^{\times} \cong \widehat{\mathbb{Z}}$. Restriction of this automorphism on an arbitrary finite unramified field $\mathfrak{K}_n$ gives us the corresponding Frobenius automorphism in the finite extension. It explains the name.

Proposition 1. *Let $E/\mathfrak{K}$ be a finitely ramified separable extension.*

- *$\mathfrak{O}_E$ is a discrete valuation ring. Moreover, the valuation obtained from this ring is a continuation of the valuation $\nu_{\mathfrak{K}}$. According to this valuation, we can take the completion E. We denote it by $\widehat{E}$.*
- *If E'/E is a finite separable extension, then $E' \cdot \widehat{E} = \widehat{E'}$.*
- *$\widehat{E} \cap \mathfrak{K}^{sep} = E$. In particular, for finitely ramified extensions E, E' over $\mathfrak{K}$ if $\widehat{E} = \widehat{E'}$, then $E = E'$.*

We call the field L a *complete extension* of $\mathfrak{K}$ if L is the completion of some finitely ramified separable extension of $\mathfrak{K}$. L is a complete unramified extension if it is the completion of an unramified separable extension of $\mathfrak{K}$.

For convenience we denote the completion of $\mathfrak{K}^{ur}$ with repect to $\nu_{\mathfrak{K}}$ by $\mathfrak{F}$.

If $L/\mathfrak{K}$ is a complete unramified extension of $\mathfrak{K}$, and L' is a completely ramified extension of L, then we say that this is a *Galois extension* if any automorphism $\phi^i \in \mathrm{Aut}(L/\mathfrak{K})$ extends to $[L' : L]$ different automorphisms in $\mathrm{Aut}(L'/\mathfrak{K})$. We call the *Weil group* of such Galois extension a $W(L'/\mathfrak{K}) := \{\sigma \in \mathrm{Aut}(L'/\mathfrak{K}) \mid \sigma|_L \in \Delta^{\mathbb{Z}}\}$. If $L = \mathfrak{F}$, then we introduce the map:

$$v : W(L'/\mathfrak{K}) \to \mathbb{Z};$$

$$\sigma|_{\mathfrak{F}} \mapsto \Delta^{-v(\sigma)}.$$

Formal group laws

By a formal group law over an arbitrary ring R we mean a series $F(X,Y) \in R[[X,Y]]$ which satisfies:

- $F(X,Y) \equiv X + Y (\mathrm{mod\ deg\ } 2)$;
- $F(X, F(Y,Z)) \equiv F(F(X,Y),Z)$ (associativity);
- $F(X,Y) \equiv F(Y,X)$ (commutativity).

Lubin–Tate extensions and maximal abelian extensions

Definition 1. *We call the polynomial $f \in \mathfrak{O}_{\mathfrak{F}}[X]$ **Lubin–Tate polynomial** if it satisfies two properties:*

1. $f(X) \equiv \pi \cdot X \pmod{deg\, 2}$;
2. $f(X) \equiv X^q \pmod{\mathfrak{p}_{\mathfrak{F}}}$,

where by π we mean the arbitrary unifomizer of local field $\mathfrak{K}$. Let us also define the action of Δ on an arbitrary formal group law $F(X,Y) \in \mathfrak{O}_{\mathfrak{F}}[[X,Y]]$. If $F(X,Y) = \sum_{i,j\geqslant 1} a_{ij}X^iY^j$ then $F^{\Delta} := \sum_{i,j\geqslant 1} \Delta(a_{ij})X^iY^j$.

Then for arbitrary Lubin–Tate polynomial f it is known [6, Lemma 3.4] that there exists a unique formal group law $F(X,Y) \in \mathfrak{O}_{\mathfrak{F}}[[X,Y]]$:

$$f \circ F = F^{\Delta} \circ f.$$

We call such a law a formal group law which corresponds to f and denote it by F_f.

Lubin–Tate polynomials are very important in the explicit construction of the maximal abelian extension and Artin map for an arbitrary local field. In the general situation Lubin–Tate polynomials are generalized to Lubin–Tate series with coefficients not necessary in $\mathfrak{O}_{\mathfrak{F}}$ but in $\mathfrak{O}_L$, where L can be an arbitrary complete unramified extension of $\mathfrak{K}$. However, in the considered here cases the series belongs to $\mathfrak{O}_{\mathfrak{K}}[X]$. The next proposition partly explains it.

Proposition 2. *If $f \in \mathfrak{O}_{\mathfrak{K}}[X]$, then $F_f \in \mathfrak{O}_{\mathfrak{K}}[[X,Y]]$ and, therefore, $F_f^{\Delta} = F_f$.*

Proof. The idea of proof is taken from [6, Lemma 3.4]. It is enough to construct such a sequence of polynomials $\{F_m\}$ satisfying two properties:

1. $deg\, F_m \leqslant m$;
2. $f \circ F_m \equiv F_m^{\Delta} \circ f \pmod{deg\,(m+1)}$.

However, due to our needs we only prove that $F_m \in \mathfrak{O}_{\mathfrak{K}}[X,Y]$ by induction.

The base case is simple, because $F_1 = X + Y$ and required property is true. To understand inductive step we define $H_{m+1} = F_{m+1} - F_m$ and G_{m+1} as $f \circ F_m - F_m^{\Delta} \circ f$. To find F_{m+1} it is enough to find H_{m+1}.

From [6, Lemma 3.4] we obtain that if $\pi \cdot \beta_{ij}$ is the coefficient at X^iY^j in G_{m+1}, then the coefficient α_{ij} at the same monomial in H_{m+1} satisfies the equality:

$$\alpha_{ij} = -\beta_{ij} - \sum_{l=1}^{\infty}\prod_{i=0}^{l-1}\Delta^i(\pi^m)\cdot\Delta^l(\beta_{ij}).$$

According to the induction hypothesis, $\Delta(\beta_{ij}) = \beta_{ij}$. We note that since $\pi \in \mathfrak{O}_{\mathfrak{K}}$, then $\Delta(\pi) = \pi$. Thus, we get $\Delta(\alpha_{ij}) = \alpha_{ij}$. □

Now we briefly introduce Lubin–Tate modules. As we already mentioned, we work mostly with polynomials from $\mathfrak{O}_{\mathfrak{K}}[X]$. The technical details and proofs can be found in [6].

Proposition 3. *For any Lubin-–Tate polynomial $f \in \mathfrak{O}_{\mathfrak{F}}[[X]]$ and for any nonzero element $\theta \in \mathfrak{O}_{\mathfrak{K}}$ there exists a unique series $[\theta]_f \in \mathfrak{O}_{\mathfrak{F}}[[X]]$, satisfying two properties:*

1. $[\theta]_f \equiv \theta \cdot X \pmod{deg\, 2}$;

2. $f \circ [\theta]_f = [\theta]_f^{\Delta} \circ f$.

Moreover, for arbitrary non-zero θ_1 *and* θ_2 *of* $\mathfrak{O}_{\mathfrak{K}}$ *holds:*

- $[\theta_1]_f +_{F_f} [\theta_2]_f = [\theta_1 + \theta_2]_f$;
- $[\theta_1]_f \circ [\theta_2]_f = [\theta_1 \cdot \theta_2]_f$;
- $f = [\pi]_f$.

Thus $\{[\alpha]_f,\ \alpha \in \mathfrak{O}_{\mathfrak{K}}\}$ is isomorphic to $\mathfrak{O}_{\mathfrak{K}}$ as $\mathfrak{O}_{\mathfrak{K}}$-module with the addition in the form of a substitution in the F_f and multiplying by scalar in the form of taking composition with corresponding series.

For a Lubin–Tate polynomial $f \in \mathfrak{O}_{\mathfrak{K}}[X]$ we denote $\underbrace{f \circ f \circ \cdots \circ f}_{m \text{ times}}$ by f_m. Let $\mu_{f,m}$ be the set of roots f_m, $\mathfrak{F}_f^m$ be the decomposition field of f_m over $\mathfrak{F}$. Then $\mu_{f,m}$ is a $\mathfrak{O}_{\mathfrak{K}}$-module with addition in the form of a substitution in F_f and multiplication by scalars as a composition with $[\cdot]_f$. Moreover, $\mu_{f,m}^{\times} := \mu_{f,m} \backslash \mu_{f,m-1}$.

Before introducing the Artin map let us put all necessary facts in the next proposition.

Proposition 4. • *For any* $\alpha \in \mu_{f,m}^{\times}$ *the map:*

$$\mathfrak{O}_{\mathfrak{K}}/\mathfrak{p}_{\mathfrak{K}}^m \to \mu_{f,m}$$

$$a \to [a]_f(\alpha),$$

is an isomorphism of $\mathfrak{O}_{\mathfrak{K}}$*-modules. If* $\alpha \in \mu_{f,m}^{\times}$*, then* $\mathfrak{F}_f^m = \mathfrak{F}(\alpha)$*,* $N_{\mathfrak{F}_f^m/\mathfrak{F}}(-\alpha) = \Delta^{m-1}(\pi)$*.*

- α *is the uniformizer of* $\mathfrak{F}_f^m$ *and* $(\mathfrak{F}_f^m/\mathfrak{F})$ *is a completely ramified Galois extension of degree* $q^{m-1}(q-1)$*.*
- *The following isomorphism is defined:*

$$\rho_{f,m} : \mathrm{Gal}(\mathfrak{F}_f^m/L) \cong \mathrm{Aut}_{\mathfrak{O}_{\mathfrak{K}}}(\mu_{f,m}) \cong (\mathfrak{O}_{\mathfrak{K}}/\mathfrak{p}_{\mathfrak{K}}^m)^{\times}$$

$$(\alpha \mapsto [u](\alpha)\ \forall\, \alpha \in \mu_{f,m}) \mapsto u\ \ (mod\, \mathfrak{p}_{\mathfrak{K}}^m).$$

- $\mathfrak{F}_f^m$ *is a Galois extension over* $\mathfrak{K}$*. For any* $\alpha \in \mu_{f,m}^{\times}$*, the following map is bijective:*

$$\mathfrak{K}^{\times}/(1+\mathfrak{p}_{\mathfrak{K}}^m) \to \bigsqcup_{j\in\mathbb{Z}} \mu_{f,m}^{\times}$$

$$x\ \ (mod\,(1+\mathfrak{p}_{\mathfrak{K}}^m)) \mapsto [x\pi^j]_f(\alpha), \quad \nu_{\mathfrak{K}}(x) = -j.$$

- *The map* $\rho_{f,m}$ *mentioned above continues to isomorphism:*

$$W(\mathfrak{F}_f^m/\mathfrak{K}) \to \mathfrak{K}^{\times}/(1+\mathfrak{p}_{\mathfrak{K}}^m)$$

$$(\alpha \mapsto [x\pi^j](\alpha)) \mapsto x\ \ (mod\,(1+\mathfrak{p}_{\mathfrak{K}}^m)), \quad \nu_{\mathfrak{K}}(x) = -j.$$

Now if we define $\mathfrak{F}_f^{LT} = \bigcup\limits_{m\geqslant 1} \mathfrak{F}_f^m$, then passing to projective limit, we get:

$$\rho_f : W(\mathfrak{F}^{LT}/\mathfrak{K}) \cong \mathfrak{K}^{\times}.$$

And we immediately can define the Artin map as the inverse of the map ρ_f:

$$Art_{\mathfrak{K}} : \mathfrak{K}^{\times} \to W(\mathfrak{F}^{LT}/\mathfrak{K}).$$

Projective line over $\mathbb{F}_q$

Let X be an arbitrary projective smooth irreducible curve over $\mathbb{F}_q$. We denote its field of rational functions by $K(X)$. Let R be a discrete valuation ring containing $\mathbb{F}_q$ and whose field of quotients is isomorphic to $K(X)$. Then the unique maximal ideal $P \subset R$ is called the simple ideal of $K(X)$. The maximal ideal P defines the valuation on the field $K(X)$.

Then we say that:

- $f(X) \in K(X)$ has a pole in P, if $\nu_P(f) < 0$.
- $f(X) \in K(X)$ has zero in P, if $\nu_P(f) > 0$.
- The degree of a prime P is the dimension of the field of quotients R/P over $\mathbb{F}_q$.

Let L be a finite extension of the field $K(X)$. We say that the ideal $\mathfrak{B} \subset L$ lies above the ideal $P \subset K(X)$, if $\mathfrak{O}_P = K(X) \cap \mathfrak{O}_{\mathfrak{B}}$ and $P = \mathfrak{O}_P \cap \mathfrak{B}$. The concepts of ramification and inertia are introduced as in the classical case. We assume that an extension over $K(X)$ is unramified if it is unramified in all ideals.

Let us fix some simple ideal of $K(X)$ and denote it by ∞. Let $A := \{f(X) \in K(X) \mid f$ has no poles in ideals other than $\infty\}$, then A is the Dedekind integral domain [8]. It is well-known that all the remaining simple ideals of $K(X)$ are in one-to-one correspondence with the simple ideals $A[8, p.219]$.

From now we consider the projective line $\mathbb{P}^1_{\mathbb{F}_q}$ over $\mathbb{F}_q$. Consider any point of degree one and call such an ideal ∞. For example, the ideal $(\frac{1}{T})$ in the ring $\mathbb{F}_q[\frac{1}{T}]$. Then $\mathbb{F}_q[T]$ is such a subring of $\mathbb{F}_q(T)$, whose elements have poles only at the point ∞. All other points are in one-to-one correspondence with the unitary irreducible polynomials $\mathbb{F}_q[T]$.

Carlitz Module

Our goal is to find a connection between maximal abelian extenstion constructions for local and global geometric fields. As we saw, the Lubin–Tate modules are very useful for local geometric fields. Now we introduce the *Carlitz Module* the global analogue.

Let L be an arbitrary field. A polynomial $h \in L[X]$ is called additive if

$$h(X+Y) \equiv h(X) + h(Y).$$

A polynomial $h \in L[X]$ is called $\mathbb{F}_q$-linear if it is additive and for any $\alpha \in \mathbb{F}_q$

$$h(\alpha X) = \alpha h(X).$$

If the field L is of characteristic p and L contains $\mathbb{F}_q$, then the set of $\mathbb{F}_q$-linear polynomials coincides with the set of polynomials from X^q. On the set of $\mathbb{F}_q$-linear polynomials, can be introduced the ring structure with respect to addition and composition operations. Such a ring is denoted by $L\langle\tau\rangle$, where by τ we mean X^q.

For convenience we give an alternative ring construction of $L\langle\tau\rangle$: as a set, it is isomorphic to $L[X]$; the addition operation is determined by coefficient-wise addition at τ in equal degrees; the operation of multiplying two polynomials is a “twisted multiplication”: for any $\alpha \in L$ the following is true $\tau \cdot \alpha = \alpha^q \cdot \tau$.

It is easy to see that $L\langle\tau\rangle$ is a $\mathbb{F}_q$-algebra. If R is a subring of L, is closed under multiplication by elements of $\mathbb{F}_q$, then $R\langle\tau\rangle$ is a $\mathbb{F}_q$-subalgebra of $L\langle\tau\rangle$.

Definition 2. *Let* $L = \mathbb{F}_q(T)$ *and* $R = \mathbb{F}_q[T]$.

Consider the morphism of $\mathbb{F}_q$-algebras:

$$C : \mathbb{F}_q[T] \to \mathbb{F}_q(T)\langle\tau\rangle$$
$$C : T \mapsto T \cdot \tau^0 + \tau$$
$$C : 1 \mapsto \tau^0$$
$$C : 0 \mapsto 0.$$

*The image of $\mathbb{F}_q[T]$ in $\mathbb{F}_q(T)\langle\tau\rangle$ is called the **Carlitz module**. The image of an arbitrary element $a \in \mathbb{F}_q[T]$ under the action of C is denoted by C_a.*

As before we need to look through some statements to use them further. All technical details and proofs can be found in [8, 12 section]. For convenience we collected all necessary statements in one proposition.

We fix a unitary irreducible polynomial $P \in \mathbb{F}_q[T]$ of degree d_P.

Proposition 5. *1. C_P is the Eisenstein unitary polynomial in $\mathbb{F}_q[T]$ of degree q^{d_P} with respect to the ideal (P).*

2. *C_{P^e} is a separable polynomial.*

3. *Denote the set of zeros of the polynomial $C_{P^e}(X)$ in the algebraic closure[1] of $\mathbb{F}_q(T)$ by Λ_{P^e}. Then on Λ_{P^e} one can introduce the structure of a $\mathbb{F}_q[T]$-module with standard addition and multiplication by a scalar $a \in \mathbb{F}_q[T]$ as an application of C_a.*

4. *The set Λ_{P^e} is isomorphic, as $\mathbb{F}_q[T]$-module, to $\mathbb{F}_q[T]/(P^e \cdot \mathbb{F}_q[T])$. Moreover, for any $m \in \mathbb{F}_q[T]$ it is true that the set Λ_m is isomorphic, as $\mathbb{F}_q[T]$-module, to the set $\mathbb{F}_q[T]/(m \cdot \mathbb{F}_q[T])$, where Λ_m is the set of zeros of the polynomial $C_m[X]$.*

5. *Denote $\mathbb{F}_q(T)(\Lambda_m)$ by K_m. Then the extension $K_{P^e}/\mathbb{F}_q(T)$ is a Galois extension with a Galois group:*
$$\mathrm{Gal}(K_{P^e}/\mathbb{F}_q(T)) \cong (\mathbb{F}_q[T]/(P^e \cdot \mathbb{F}_q[T]))^\times.$$

6. *If $(Q) \neq (P)$ and $(Q) \neq \infty$, where ∞ is some simple ideal of $\mathbb{F}_q(T)$, (see the previous section) then K_{P^e} is unramified in (Q).*

7. *K_{P^e} is completely ramified in (P) and, if λ is an arbitrary generator of Λ_{P^e}, and $\mathfrak{O}_{K_{P^e}}$ is the integral closure of $\mathbb{F}_q[T]$ in K_{P^e}, then:*
$$(\lambda)^{(q^{d_P}-1)\cdot q^{d_P \cdot e}} \cdot \mathfrak{O}_{K_{P^e}} = (P) \cdot \mathfrak{O}_{K_{P^e}}.$$

8. *$\mathfrak{O}_{K_{P^e}} = \mathbb{F}_q[T](\lambda)$.*

Main results

Connection between the Lubin–Tate theory and the Carlitz module

In this section we consider projective line $\mathbb{P}_{\mathbb{F}_q}$. For this section we fix the notation:

- $K := \mathbb{F}_q(T)$ is a field of functions of $\mathbb{P}_{\mathbb{F}_q}$.
- $\infty := (\frac{1}{T})$ is an ideal in the subring $\mathbb{F}_q[\frac{1}{T}]$ that corresponds to one fixed point on our line.

[1] Throughout the paper, the algebraic closure is fixed $\mathbb{F}_q(T)$.

- $A := \mathbb{F}_q[T]$ is $\{f(X) \in K(X) \mid f$ has no poles in ideals other than $\infty\}$.
- $\mathfrak{K} := \mathbb{F}_q((T))$ is the completion of the K with respect to ideal (T).
- $\mathfrak{F} := \mathbb{F}_q((T))^{ur}$ is the maximal unramifed extension of $\mathfrak{K}$.

Лемма 1. *Consider the polynomial $f = T \cdot X + X^q$. Such a polynomial is the Lubin–Tate polynomial for the local field $\mathfrak{K} = \mathbb{F}_q((T))$. The Lubin–Tate law corresponding to it is the additive law $F_{add} = X + Y$.*

Since $\mathbb{F}_q[T] \subset \mathfrak{O}_{\mathfrak{K}}$, it means that for any $a \in \mathbb{F}_q[T]$ the series $[a]_f \in \mathfrak{O}_{\mathfrak{F}}$ is defined. It is claimed that for any $a \in \mathbb{F}_q[T]$:

$$[a]_f = C_a,$$

where C_a is considered as an element of $\mathbb{F}_q(T)[X]$.

Moreover, let $a \in \mathfrak{O}_{\mathfrak{K}}$, $\quad a = \sum\limits_{i=0}^{\infty} a_i \cdot T^i$, $a_i \in \mathbb{F}_q$. Then

$$[a]_f = \sum_{i=0}^{\infty} a_i \cdot C_{T^i}.$$

Proof. Note that T is uniformizer for $F_q((T))$. Also, the residue field of this local field is isomorphic to $\mathbb{F}_q$. From these two remarks it follows that C_T is the Lubin—Tate polynomial for $\mathbb{F}_q((T))$. According to proposition *2*, $F_{add} = F_{add}^{\Delta}$. Further, it is seen that:

$$f \circ F_{add} = T \cdot (X + Y) + (X + Y)^q,$$

which is equal to:

$$F_{add} \circ f = T \cdot X + X^q + T \cdot Y + Y^q.$$

The series $[a]_f$ is uniquely defined by two conditions:

1. $[a]_f \equiv aX \ \ (mod\, deg\, 2)$;
2. $f \circ [a]_f = [a]_f^{\Delta} \circ f$.

First, check the first condition. It is enough to note that when two polynomials are added, the coefficients at X are added, and during composition are multiplied.

For any $a \in \mathbb{F}_q[T]$, the polynomial C_a lies in $\mathbb{F}_q(T)[X]$, which means $C_a = C_a^{\Delta}$. The map C is a morphism of $\mathbb{F}_q$-algebras, and $\mathbb{F}_q[T]$ is a commutative ring, which means:

$$C_T \circ C_a = C_a \circ C_T.$$

Now we notice that $f = C_T$ and get:

$$f \circ C_a = C_a \circ f.$$

It is remaining to prove that if $a = \sum\limits_{i=0}^{\infty} a_i \cdot T^i$, $a_i \in \mathbb{F}_q$, then

$$[a]_f = \sum_{i=0}^{\infty} a_i \cdot C_{T^i}.$$

First, check that the series $\sum\limits_{i=0}^{\infty} a_i \cdot C_{T^i}$ is set correctly. Consider the coefficient b_{q^r} of this series at X^{q^r}. Let us prove that for T^k in b_{q^r} there is a finite sum of elements of the field $\mathbb{F}_q$.

C_{T^n} by definition is equal to $\prod_{j=1}^{n}(T\cdot\tau^0+\tau)$, where the multiplication is twisted. If $r<n$, then $\tau^r = X^{q^r}$ and X^{q^r} is in C_{T^n} with a coefficient divisible by at least T^{n-r}, since when opening the brackets, we must take $T\cdot\tau^0$ no later than in $(n-r)-$th bracket. Moreover, $T\cdot\tau^0$, and τ increase the degree of T by at least 1. Therefore, if $n>r+k$, then in C_n at X^{q^r} there are no terms whose valuation is less or equal to k. That is, for T^k in X^{q^r} there is a finite sum of elements from $\mathbb{F}_q$.

Note that $\sum_{i=0}^{\infty} a_i\cdot C_{T^i}\in\mathfrak{O}_{\mathfrak{K}}[[X]]$, which means:

$$\sum_{i=0}^{\infty} a_i\cdot C_{T^i} = (\sum_{i=0}^{\infty} a_i\cdot C_{T^i})^{\Delta}.$$

Now we are going to check two properties defining the series $[a]_f$.

The first property follows automatically from the fact that $C_{T^i}\equiv T^iX\ (mod\, deg\, 2)$.

For the second property, we should understand that:

- $f\circ(\sum_{i=0}^{\infty} a_i\cdot C_{T^i}) = \sum_{i=0}^{\infty} a_i\cdot(f\circ C_{T^i})$,
- $(\sum_{i=0}^{\infty} a_i\cdot C_{T^i})\circ f = \sum_{i=0}^{\infty} a_i\cdot(C_{T^i}\circ f)$.

Note that both properties are true if we consider expressions modulo $deg\, n$, since both series are polynomials of finite degree, and f is $\mathbb{F}_q-$linear. So both equities are true for these series.

Using the fact that f and C_{T_i} commute, then equating the right-hand sides of the last two expressions, we obtain the required property. □

We have already noticed that all points of our projective line except infinity are in one-to-one correspondence to unitary irreducible polynomials of $\mathbb{F}_q[T]$. The local field of the projective line at a point (T) is just $\mathbb{F}_q(T)$. The next proposition explores these fields for other points of $\mathbb{P}_{\mathbb{F}_q}$ except infinity.

Proposition 6. *Let $P\in\mathbb{F}_q[T]$ be an arbitrary unitary, irreducible polynomial from T, of degree d_P. Then the localization residue field of the ring $\mathbb{F}_q[T]$ by the simple ideal (P) is $\mathbb{F}_{q^{d_P}}$.*

Proof. Localization by the ideal (P) consists of rational functions of T whose denominators are coprime to P. The only maximum ideal of this ring is $P\cdot\mathbb{F}_q[T]_{(P)}$. By definition, the residue field is $\mathbb{F}_q[T]_{(P)}/(P\cdot\mathbb{F}_q[T]_{(P)})$. Let us prove that it is isomorphic to $\mathbb{F}_q[T]/(P\cdot\mathbb{F}_q[T])$.

(P) is the maximum ideal in $\mathbb{F}_q[T]$, then $\mathbb{F}_q[T]/(P\cdot\mathbb{F}_q[T])$ is a field. Consider an arbitrary element $\frac{h(T)}{g(T)}\in\mathbb{F}_q[T]_{(P)}/(P\cdot\mathbb{F}_q[T]_{(P)})$, where $g(T)$ is coprime with P. If:

$$h(T) = q_h(T)\cdot P + r_h(T),$$

$$g(T) = q_h(T)\cdot P + r_g(T),$$

then

$$\frac{h(T)}{g(T)}\equiv\frac{r_h(T)}{r_g(T)}\ (mod\,(P\cdot\mathbb{F}_q[T])).$$

Since $\mathbb{F}_q[T]/(P\cdot\mathbb{F}_q[T])$ is a field, then for any $r_g(T)$ there exists a polynomial $b(T)$, of degree less than d_P, such that $g(T)\cdot b(T)\equiv 1\ (mod\,(P\cdot\mathbb{F}_q[T]))$. Thus, $\mathbb{F}_q[T]_{(P)}/(P\cdot\mathbb{F}_q[T]_{(P)})$ is exactly the set of polynomials, which degree less than d_P. Let $\frac{h_1(T)}{g_1(T)}$ and $\frac{h_2(T)}{g_2(T)}$ be two arbitrary elements from $\mathbb{F}_q[T]_{(P)}/(P\cdot\mathbb{F}_q[T]_{(P)})$. Note that:

$$\frac{h_1(T)}{g_1(T)}+\frac{h_2(T)}{g_2(T)} = \frac{h_1(T)\cdot g_2(T)+h_2(T)\cdot g_1(T)}{g_1(T)\cdot g_2(T)}\equiv h_1(T)\cdot b_1(T)+h_2\cdot b_2(T).$$

Similarly:

$$\frac{h_1(T)}{g_1(T)} \cdot \frac{h_2(T)}{g_2(T)} \equiv (h_1(T) \cdot b_1(T)) \cdot (h_2(T) \cdot b_2(T)).$$

Therefore, the map $\frac{h(T)}{g(T)} \mapsto h(T) \cdot b(T)$ is an isomorphism. □

Note that when we take the completion of the field K with respect to the valuation associated with the ideal (P), the residue field and the uniformizer do not change with respect to this valuation. The proof of this statement can be found in [9, Claim 1.1.3].

Лемма 2. *C_P is the Lubin–Tate polynomial for the completion of K with respect to the valuation associated with the simple ideal (P).*

Proof. As already noted in *lemma 1*, $C_P \equiv P \cdot X \pmod{deg\, 2}$. Also according to propositions 5, C_P is the unitary polynomial of degree q^{d_P}, and all its coefficients, except the highest, are divisible by P. We write this in the form of two sequences:

1. $C_P \equiv P \cdot X \pmod{deg\, 2}$;
2. $C_P \equiv X^{q^{d_P}} \pmod{(P)}$.

Note that when the field $\mathbb{F}_q(T)$ is the completion with respect to the valuation associated with the ideal (P), the residue field and the uniformizer do not change with respect to this valuation. The proof of this statement can be found in [9, Claim 1.1.3].

Finally, according to the proposition 6 and remark above, the residue field of completion is isomorphic to $\mathbb{F}_{q^{d_P}}$, and the maximal ideal is the main ideal (P). □

We fix some irreducible polynomial $P \in \mathbb{F}_q[T]$. Let f_P be the Lubin–Tate polynomial of the field $K_{(P)}$, where $K_{(P)}$ is the completion of K with respect to the valuation, corresponding to the ideal (P). Let also $f_P \in \mathfrak{O}_{K_{(P)}}[X]$. Define $f_{P,m}$ as $\underbrace{f_P \circ f_P \circ \cdots \circ f_P}_{m \text{ times}}$, and $K_{(P),m}$ as the decomposition field of $f_{P,m}$ over the field $K_{(P)}$. Then the extension tower $K_{(P)} \subset K_{(P),1} \subset K_{(P),2} \subset \cdots$ we call the Lubin–Tate extensions tower with respect to the ideal (P), and $\bigcup_{m=1}^{\infty} K_{(P),m}$ by Lubin–Tate extension of the field K with respect to the ideal (P).

Теорема 1. *The tower of extensions $K \subset K_P \subset K_{P^2} \subset \cdots$ induces the Lubin–Tate extensions tower with respect to the ideal (P). The union of K_{P^e} over all positive integers e induces the Lubin–Tate extension with respect to the ideal (P).*

Proof. From the proposition 5, we know that for K_{P^e} there is only one ideal ramified, and it is defined as $(\lambda) \cdot \mathfrak{O}_{K_{P^e}}$, where $\mathfrak{O}_{K_{P^e}}$ is the integral closure of $\mathbb{F}_q[T]$ in K_{P^e}.

Consider the ring $(\mathfrak{O}_{K_{P^e}})_{(\lambda)}$. Since this is the localization of some ring by a simple ideal, $(\mathfrak{O}_{K_{P^e}})_{(\lambda)}$ is discrete valuation ring with maximum ideal (λ). The field of quotients of this ring is the field K_{P^e}, since all elements except λ and P are already invertible, and the addition of λ^{-1} automatically adds the inverse to P, due to the fact that:

$$(\lambda)^{(q^{d_P}-1)\cdot q^{d_P \cdot e}} = (P).$$

Therefore, localization by the ideal (λ) is the discrete valuation ring that lies above the localization $\mathbb{F}_q[T]$ by the ideal (P).

We denote the completion of K with respect to the valuation $\nu_{(P)}$ by $K_{(P)}$ and the completion of K_{P^e} with respect to the valuation $\nu_{(\lambda)}$ by $(K_{P^e})_{(\lambda)}$. Then, by the proposition *1*:

$$(K_{P^e})_{(\lambda)} = K_{(P)} \cdot K_{P^e} = K_{(P)}(\lambda),$$

where λ is the primitive root of C_{P^e}.

Since $C_P \in \mathbb{F}_q[T]\langle\tau\rangle$, it means that it lies in $\mathfrak{O}_{K_{(P)}}[X]$, and therefore for any m :

$$C_{P,m} = \underbrace{C_P \circ C_P \circ \cdots \circ C_P}_{m \text{ times}}.$$

Thus $C_{P^e} = C_{P,e}$. It remains to notice that according to the proposition *4*,

$$K_{(P)}(\lambda) = K_{(P)}(C_{P,e}).$$

Thus, the extension K_{P^e}/k induces on $K_{(P)}$ the Lubin–Tate extension of order e.

Passing to the projective limit with respect to e, we obtain that $\bigcup\limits_{e=1}^{\infty} K_{P^e}$ induces the maximum abelian extension of the local field $K_{(P)}$. □

Artin maps for global and local fields

In order to relate theories for the local and global cases, we need first to recall the Artin map in the global case.

Definition 3. *Let X be a projective smooth irreducible curve over $\mathbb{F}_q$ and $K = k(X)$ be its field of functions. Then:*

- *For an arbitrary closed point $P \in X$, we denote the completion of the local ring in P by $\widehat{\mathfrak{O}_P}$, and the field of quotients of $\widehat{\mathfrak{O}_P}$ by K_P.*
- *By the idel group $\mathbb{I}_K$ of the field K we call bounded product of groups of invertible elements K_p by groups of ring units $\widehat{\mathfrak{O}_P}$, by all closed points $P \in X$.*

Consider the maximal unramified extension $K^{u,ab}$. We call the Artin map [9]:

$$\Phi_K^u : \mathbb{I}_K / \prod_{P \in X} \widehat{\mathfrak{O}_P^{\times}} \to \mathrm{Gal}(K^{u,ab}/K)$$

$$(\cdots, a_P, \cdots) \mapsto Frob_P^{ord_P(a_P)}.$$

Remark 1. *If the extension of the field K is abelian, then for any point (ideal) P we can define the Frobenius automorphism[8, p. 136-137]. It can be described as a mapping $Frob_P \in \mathrm{Gal}(K^{u,ab}/K)$ such that for any ideal $\mathfrak{B}$ lying above P it holds:*

$$Frob_P(w) \equiv w^{N(P)} \pmod{\mathfrak{B}},$$

where $N(P)$ is the dimension of the residue field of the local ring at the point P over the field $\mathbb{F}_q$.

Consider the local case. Since the maximal unramified abelian extension K, is used, then in the local case it is natural to consider the completion of the maximal unramified extension for K_P, denoting it by K_P^{ur}. Then the map Art_{K_P}, (remark 9) restricted on $\mathrm{Gal}(K_P^{ur}/K)$ is defined as:

$$Art_{K_P} : K_P^{\times} \to \mathrm{Gal}(K_P^{ur}/K)$$

$$x \mapsto \phi_P^{-j}, \text{ if } \nu_P(x) = j,$$

where ϕ_P is the Frobenius map of the extension K_P^{ur}/K.

Теорема 2. *For a partially defined mapping, we introduce the notation* $\sigma_P : \mathrm{Gal}(K^{u,ab}/K) \to \mathrm{Gal}(K^{ur}/K)$, *which sends* $Frob_P$ *to* ϕ_P^{-1}. *Then the following commutative diagram holds:*

$$\begin{array}{ccc} K_P^{\times}/\widehat{\mathfrak{O}_P^{\times}} & \xrightarrow{\Phi_K^u|_{K_P^{\times}}} & \mathrm{Gal}(K^{u,ab}/K) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \sigma_P} \\ K_P^{\times}/\widehat{\mathfrak{O}_P^{\times}} & \xrightarrow{Art_{K_P}} & \mathrm{Gal}(K_P^{ur}/K_P) \end{array}$$

Proof. $Art_{K_P}|_{\mathrm{Gal}(K_P^{ur}/K)}$ converts any element from $\widehat{\mathfrak{O}_P^{\times}}$ to one, since for any x from $\widehat{\mathfrak{O}_P^{\times}}$ it is true that $\nu_P(x) = 0$.

Then a_P goes to $\sigma_P(Frob_P^{ord_P(a_P)})$, therefore it goes to $\phi_P^{-ord_P(a_P)}$, which by definition is the image of a_P under action of $Art_{K_P}|_{\mathrm{Gal}(K_P^{ur}/K)}$. □

Conclusion

In *Lemmas 1, 2* and in the *theorem 1* it is proved that the extension tower of *Carlitz module* induces the extension tower *of formal Lubin–Tate modules* over completion of the local ring at a closed point of the curve.

In the *theorem 2* the description was given of the connection between the Artin maps for an arbitrary projective smooth irreducible curve X and the completions of local rings at its closed points.

Open questions

- The Carlitz module is a special case of Drinfeld modules of rank 1, which can be considered for an arbitrary smooth projective curve over a finite field. One of the most interesting problems is the generalization of the *theorem 1* for an arbitrary smooth projective curve over a finite field.

- As a development of Theorem 1, we can consider a morphism that takes T to an arbitrary polynomial in $\mathbb{F}_q[T]\langle\tau\rangle$, which will be a Lubin–Tate polynomial of degree r. For example, for $r = 2$, one can send T to the polynomial $T \cdot X + X^q + T \cdot X^{q^2}$. Then, extending $\mathbb{F}_q(T)$ by the powers of this polynomial, we again get a tower of Lubin–Tate extensions for the ideal (T), but it remains unknown what will happen at the other points of our curve.

- Combining the previous two problems leads us to consider Drinfeld modules of rank greater than 1.

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