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ЭКСТРЕМУМЫ ВЕКТОРОЗНАЧНЫХ ФУНКЦИЙ НЕСКОЛЬКИХ ВЕЩЕСТВЕННЫХ ПЕРЕМЕННЫХ

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Аннотация

В данной работе мы попытались обобщить обычное понятие экстремума функции вещественного переменного на векторозначные функции нескольких вещественных переменных. Нашей задачей было построить такое обобщение, чтобы для него остались верными обычные свойства и соотношения для экстремума вещественнозначных функций. Рассматриваемое обобщение также характеризуется эквивалентным обобщением. Наши определения и связанные с ними результаты проиллюстрированы многочисленными примерами.

EXTREMUMS OF VECTOR-VALUED FUNCTIONS OF SEVERAL REAL VARIABLES

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Abstract

In this paper we try to give a generalization of the usual notion of extremum of real functions to the vector-valued functions of several real variables. Our aim is that in this generalization remain valid the usual properties and relations for extremum of real functions. A considered generalization is also characterized by an equivalent generalization. Our definitions and related results are illustrated by numerous examples.

1. The notion of extremums of vector-valued functions of several real variables

Let R be the set of all real numbers, and let R^n be the n-dimensional vector space with the usual Euclidean norm $\|\cdot\|$, that is, for $x = (x_1, \dots, x_n) \in R^n$, $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$.

Defintion 1. Let $f : U \to R^m$ be a function, where U is an open subset of R^n. The point $x_0 \in U$ is said to be *the extremum* of a function f in U if there holds

$$\|f(x_0)-f(a)\|^2+\|f(x_0)-f(b)\|^2 \geq \|f(a)-f(b)\|^2 \text{ for all } a,b\in U.$$

Although it seems that Definition 1 is abstract, it can be shown that it is a natural generalization of the notion of the usual local extremum of real functions (see e.g., [1, Lecture 19]). This is shown by the following result.

Theorem 1. *Let $f:U\to R$ be a function, where U is an open subset of R^n. Then the point $x_0\in U$ is an extremum of f U in accordance with Definition* 1 *if and only if x_0 the usual local extremum of f in U.*

Proof. Let x_0 be an extremum of f by Definition 1. Then there holds

$$|f(a)-f(x_0)|^2+|f(b)-f(x_0)|^2 \geq |f(a)-f(b)|^2 \text{ for all } a,b\in O(x_0)$$

where $O(x_0)$ is a neighbourhood of a point x_0. Suppose that x_0 is not the usual local extremum of a real function f. This means that there exist points $a,b\in U$ for which $\alpha=f(a)-f(x_0)$ and $\beta=f(x_0)-f(b)$ such that α and β are positive numbers. Substituting the previous equalities in above inequality, we obtain

$$\alpha^2+\beta^2\geq(\alpha+\beta)^2 \Leftrightarrow \alpha^2+\beta^2\geq\alpha^2+\beta^2+2\alpha\beta \Leftrightarrow 0\geq\alpha\beta.$$

This contradicts the fact that α and β are positive numbers, and hence, x_0 is the usual local extremal value of a real function f. Conversely, suppose that x_0 is a usual local extremum of a function f. Without loss of generality, we can suppose that a function f does not attain a local maximum at a point x_0. Then there exists a neighbourhood $O(x_0)$ of x_0 such that $f(a)\leq f(x_0)$ and $f(b)\leq f(x_0)$ for all $a,b\in O(x_0)$. Clearly, at least one of the following inequalities there holds: $f(a)\leq f(b)\leq f(x_0)$ or $f(a)\leq f(b)\leq f(x_0)$. If the first inequality is satisfied then $|f(a)-f(x_0)|\geq |f(a)-f(b)|$. If the second inequality is satisfied then $|f(b)-f(x_0)|\geq|f(a)-f(b)|$. In both cases we have $\max\{|f(a)-f(x_0)|,|f(b)-f(x_0)|\}\geq|f(a)-f(b)|$. This inequality yields $|f(a)-f(x_0)|^2+|f(b)-f(x_0)|^2\geq|f(a)-f(b)|^2$ for all $a,b\in O(x_0)$, i.e., x_0 is an extreme point of a function f by Definition 1. This completes the proof. ■

Hence, Definition 1 may be considered as a generalization of the notion of a usual local extremum of real functions to vector-valued functions of several variables (for more information on these functions see e.g., [2, Chapter XIV]).

The following result gives a necessary condition for a point to be an extremum of a vector-valued function of several variables, which is analogous to those of a real function.

Theorem 2 (*generalized Fermat's theorem*). *Let $f:U\to R^m$ be a function, where $U\subseteq R^n$ is an open subset of R^n. Suppose that $x_0\in U$ is an extreme point of f in accordance with Definition* 1. *If f is a differentiable function at a point x_0 then* $f'(x_0)=0$.

Proof. Let $x_0=(x_0^1,...,x_0^n)$ and $f(x)=(f_1(x),...,f_m(x))$ (f_i $i\in\{1,2,...,m\}$ are coordinate maps of f). Then there exists a neighbourhood $O(x_0)$ of x_0 such that for all $a=(a_0^1,...,a_0^n)\in O(x_0)$ and $b=(b_0^1,...,b_0^n)\in O(x_0)$ there holds

$$\textstyle\sum_{i=1}^m(f_i(a)-f_i(x_0))^2+\sum_{i=1}^m(f_i(x_0)-f_i(b))^2\geq\sum_{i=1}^m(f_i(a)-f_i(b))^2. \quad (1)$$

Let $a^{(\varepsilon)} = (x_0^1 + \varepsilon, ..., x_0^n)$ and $b^{(\varepsilon)} = (x_0^1 - \varepsilon, ..., x_0^n)$ where $\varepsilon \geq 0$ is chosen so that $a^\varepsilon, b^\varepsilon \in O(x_0))$. Replacing in (1) a and b by a^ε and b^ε, respectively, we find that

$$\sum_{i=1}^m((f_i(x_0^1+\varepsilon, ..., x_0^n) - f_i(x_0^1, ..., x_0^n))^2 + \sum_{i=1}^m((f_i(x_0^1, ..., x_0^n) - f_i(x_0^1-\varepsilon, ..., x_0^n))^2 \geq \sum_{i=1}^m((f_i(x_0^1 + \varepsilon, ..., x_0^n) - f_i(x_0^1 - \varepsilon, ..., x_0^n))^2. \quad (2)$$

By the differentiability of coordinate maps f_i with $i \in \{1, ..., m\}$ it follows that for sufficiently small ε we have

$$f_i(x_0^1 + \varepsilon, ..., x_0^n) - f_i(x_0^1, ..., x_0^n) = \frac{\partial f_i}{\partial x_1}(x_0)\varepsilon + o(\varepsilon).$$

Substituting the above equality in (2), we obtain

$$\sum_{i=1}^m(\frac{\partial f_i}{\partial x_1}(x_0)\varepsilon + o(\varepsilon))^2 + \sum_{i=1}^m(\frac{\partial f_i}{\partial x_1}(x_0)\varepsilon - o(-\varepsilon))^2 \geq \sum_{i=1}^m(\frac{\partial f_i}{\partial x_1}(x_0)\varepsilon + o(\varepsilon) + \frac{\partial f_i}{\partial x_1}(x_0)\varepsilon - o(-\varepsilon))^2$$

Dividing the above inequality by ε^2, we find that

$$\sum_{i=1}^m(\frac{\partial f_i}{\partial x_1}(x_0) + \frac{o(\varepsilon)}{\varepsilon})^2 + \sum_{i=1}^m(\frac{\partial f_i}{\partial x_1}(x_0) + \frac{o(-\varepsilon)}{-\varepsilon})^2 \geq \sum_{i=1}^m(2\frac{\partial f_i}{\partial x_1}(x_0) + \frac{o(\varepsilon)}{\varepsilon} + \frac{o(-\varepsilon)}{-\varepsilon})^2.$$

Letting $\varepsilon \to 0$ we get

$$2\sum_{i=1}^m(\frac{\partial f_i}{\partial x_1})^2 \geq 4\sum_{i=1}^m(\frac{\partial f_i}{\partial x_1})^2 \Leftrightarrow \sum_{i=1}^m(\frac{\partial f_i}{\partial x_1})^2 \leq 0 \Leftrightarrow \frac{\partial f_i}{\partial x_1} = 0 \text{ for all } i \in \{1, ..., m\}.$$

Assuming $a^{(\varepsilon)} = (x_0^1, x_0^2 + \varepsilon, ..., x_0^n)$ and $b^{(\varepsilon)} = (x_0^1, x_0^2 - \varepsilon, ..., x_0^n)$, and using the previous considerations, we obtain $\frac{\partial f_i}{\partial x_2} = 0$ for each $i \in \{1, ..., m\}$ ... etc. $a^{(\varepsilon)} = (x_0^1, x_0^2, ..., x_0^n + \varepsilon)$ and $b^{(\varepsilon)} = (x_0^1, x_0^2, ..., x_0^n - \varepsilon)$ ... $\Leftrightarrow \frac{\partial f_i}{\partial x_n} = 0$ for all $i \in \{1, ..., m\}$.

$$\Rightarrow [f'(x_0)] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_1}{\partial x_n} \\ . & ... & . \\ \frac{\partial f_m}{\partial x_1} & ... & \frac{\partial f_m}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 & ... & 0 \\ . & ... & . \\ 0 & ... & 0 \end{bmatrix}.$$

This shows that the Jacobian matrix $[f'(x_0)]_{n\times m}$ is a zero-matrix, and the proof is completed. ■

<u>EXAMPLES</u>

Example 1. The function $f : R^2 \to R^2$ defined as $f(x, y) = (x^2+y^3, x+y^2)$ does not attain an extreme value at the point $(0, 0)$ because of $[f'(0, 0)] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and so, $f'(0, 0) \neq O$. It is natural then to propose the question: is f has an extreme point?

Example 2. The function $f : R \to R^2$ defined as $f(x) = (x, x^2)$ does not attain an extreme value at the point 0 because of $f'(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e., $f'(0) \neq O$. It is natural to propose the question: whether f has an extreme point?

Example 3. The function $f : R \to R^2$ defined as $f(x) = (x^2, x^2)$ has an extreme point, namely, the point $0 \in R$. Let $I(0)$ be an arbitrary small interval which contains zero and let $\varepsilon, \delta \in I(0)$. Then

$$\|f(\varepsilon)-f(0)\|^2+\|f(\delta)-f(0)\|^2=\|(\varepsilon^2,\varepsilon^2)\|+\|(\delta^2,\delta^2)\|=2\varepsilon^4+2\delta^4$$

$$\|f(\varepsilon)-f(\delta)\|^2=\|(\varepsilon^2-\delta^2,\varepsilon^2-\delta^2\|^2=2(\varepsilon^2-\delta^2)^2=2\varepsilon^4+2\delta^4-4\varepsilon^2\delta^2$$

From the above equalities it follows that

$$\|f(\varepsilon)-f(0)\|^2+\|f(\delta)-f(0)\|^2\geq\|f(\varepsilon)-f(\delta)\|^2,$$

i.e., $x_0=0$ is an extreme point of the function f. This is the only extreme point of the function f because of if $x_0\neq 0$ then $f'(x_0)=\begin{pmatrix}2x_0\\2x_0\end{pmatrix}\neq\begin{pmatrix}0\\0\end{pmatrix}$, and hence x_0 is not an extreme point of the function f.

Example 4. The function $f:R\to R^2$ defined as $f(\varphi)=(\cos\varphi,\sin\varphi)$ has no extreme point because of $f'(\varphi)=\begin{pmatrix}\cos\varphi\\\sin\varphi\end{pmatrix}$ for all $\varphi\in R$.

However, there exist different examples of functions $f:U\to R^m$ where U is an open subset of R^n, such that there exists a point $x_0\in U$ for which $f'(x_0)=O$. In order to verify whether some point is an extreme point of a function f we have only Definition 1, but such a verification is very complicated and non-practical in general. The following considerations solve this problem.

Definition 2. Let $f:U\to R^m$ be a function, where $U\subseteq R^n$ is an open set, $f=(f_1,...,f_m)$ and let $x_0\in U$. A point x_0 is said to be an extreme point of a function f if the function $\Gamma:U\times U\in R^{2n}\to R$ defined as

$$\Gamma(a_1,...,a_n,b_1,...,b_n)=$$
$$\sum_{i=1}^m f_i(a_1,...,a_n)f_i(b_1,...,b_n)-\sum_{i=1}^m(f_i(a_1,...,a_n)+f_i(b_1,...,b_n))f_i(x_0)$$

attains a local minimum at a point $(x_0,x_0)=(x_0^1,x_0^2,...,x_0^n,x_0^1,x_0^2,...,x_0^n)\in R^{2n}$.

It seems that Definition 2 is abstract, but the following result gives its complete characterization.

Theorem 3. *Let $f:U\to R^m$ be a function, where $U\subseteq R^n$ is an open set and let $x_0\in U$. Then x_0 is an extreme point in accordance with Definition* 1 *if and only if x_0 is an extreme point of a function f in accordance with Definition* 2.

Proof. Let x_0 be an extreme point of a function f by Definition 1. Then there exists a neighbourhood $O(x_0)$ of a point $x_0\in U$ such that for all $a,b\in O(x_0)$ holds

$$\|f(x_0)-f(a)\|^2+\|f(x_0)-f(b)\|^2\geq\|f(a)-f(b)\|^2\quad(1)$$

For fixed $a,b\in O(x_0)$ consider the vectors $f(x_0)-f(a)$ and $f(x_0)-f(b)$ in the space R^m. Then we have

$$2(f(x_0)-f(a),f(x_0)-f(b))=$$
$$\|f(x_0)-f(a)\|^2+\|f(x_0)-f(b)\|^2-\|f(x_0)-f(a)-f(x_0)+f(b)\|^2\ (\text{iz }(1))$$
$$\Rightarrow(f(x_0)-f(a),f(x_0)-f(b))\geq 0\quad(2)$$

Hence, the inequality (2) is equivalent with the inequality (1). Multiplying scalary in (2), we get $\|f(x_0)\|^2-(f(a)+f(b),f(x_0))+(f(a),f(b))\geq 0$, i.e.,

$\sum_{i=1}^{m} f_i(a_1,...,a_n)f_i(b_1,...,b_n) - \sum_{i=1}^{m}(f_i(a_1,...,a_n)+f_i(b_1,...,b_n))f_i(x_0)+\|f(x_0)\|^2 \geq 0,$

i.e.,

$$\Gamma(a_1,...,a_n,b_1,...,b_n)+\|f(x_0)\|^2 \geq 0 \Leftrightarrow \Gamma(a_1,...,a_n,b_1,...,b_n) \geq -\|f(x_0)\|^2 \quad (3)$$

The above relation is satisfied for all $(a_1,...,a_n,b_1,...,b_n) \in O(x_0) \times O(x_0)$ while for

$$(a_1,...,a_n,b_1,...,b_n) = (x_0^1, x_0^2,...,x_0^n, x_0^1, x_0^2,...,x_0^n)$$

we have

$$\Gamma(a_1,...,a_n,b_1,...,b_n) = -\|f(x_0)\|^2. \quad (4)$$

Conversely, if x_0 is an extreme point by Definition 2, then the relations (3) and (4) are satisfied, where in (3) a neighbourhood $O(x_0) \times O(x_0)$ of a point x_0 is not explicitly given, while this is the case for a neighbourhood $O(x_0, x_0) \in R^{2n}$ (but each neighbourhood of type $O(x_0, x_0)$ contains a neighbourhood of type $O(x_0) \times O(x_0)$). The relations (3) and (4) are equivalent with the relation (2) , while the relation (2) is equivalent with the inequality (1), that is, x_0 is an extreme point by Definition 1. The proof is completed.■

Remark. We have used the fact that the scalar product $(\cdot,\cdot)$ in a real Hilbert space H may be expressed in terms of related norm by the following identity:

$$(x,y) = \frac{\|x\|^2+\|y\|^2-\|x-y\|^2}{2} \quad \text{for all } x,y \in H.$$

.

Remark. As noticed above, by using Definition 2 it is very complicated and non-practical to verify whether a point $x_0 \in U \subseteq R^n$ is an extreme point of a function $f: U \subseteq R^n \Rightarrow R^m$ where U is an open set.

By Definition 2, it is sufficient to examine whether $(x_0, x_0) \in R^{2n}$ is a local extreme point of a real function $\Gamma : U \times U \in R^{2n} \to R$, and the function Γ can be easily consctructed from a function f. In dependence of the behaviour of a function Γ in some neighbourhood of the point (x_0, x_0), we have different investigations related to the question whether this point is a local minimum of the function Γ. The following considerations gives analytic solution of this problem.

2. The investigation of a function Γ in a neighbourhood of the point $x_0^2 (= (x_0, x_0))$

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Theorem 4 (*Necessary conditions for optimality*). *Let U be an open subset of R^n and let $f : U \Rightarrow R^m$ be a function of class C^1 in some neighbourhood of a point*

$x_0 \in U$ such that f is a twice differentiable function at a point x_0. If x_0 is an extreme point of a function f, then the partial derivatives of the first and the second order of f at a point x_0 are equal to zero.

***Proof*.** By the assumptions for a function f, we conclude that there exist the partial derivatives of the function Γ up to second order at a point x_0^2. Clearly, by Fermat's theorem, the partial derivatives of the first order of Γ at a point x_0^2 are equal to zero. For the partial derivatives of the second order we have

$$\frac{\partial^2\Gamma}{\partial a_k \partial a_s}(x_0^2) = \sum_{i=1}^m \frac{\partial^2 f_i}{\partial a_k \partial a_s}(x_0) f_i(x_0) - \sum_{i=1}^m \frac{\partial^2 f_i}{\partial a_k \partial a_s}(x_0) f_i(x_0) = 0$$

$$\frac{\partial^2\Gamma}{\partial b_k \partial b_s}(x_0^2) = \sum_{i=1}^m \frac{\partial^2 f_i}{\partial b_k \partial b_s}(x_0) f_i(x_0) - \sum_{i=1}^m \frac{\partial^2 fi}{\partial b_k \partial b_s}(x_0) f_i(x_0) = 0$$

$$\frac{\partial^2\Gamma}{\partial a_k \partial b_s}(x_0^2) = \sum_{i=1}^m \frac{\partial f_i}{\partial a_k}(x_0) \frac{\partial f_i}{\partial b_s}(x_0) = 0$$

for all $k, s \in \{1, ..., n\}$ because of all partial derivatives of the first order of the function f at a point x_0 are equal to zero. The proof is completed$\blacksquare$

Remark. Theorem 4 shows that the second form $\Gamma''(x_0^2)$ of the function Γ cannot be applied for examining whether a point $x_0^2 = (x_0, x_0)$ is a local minimum of the function Γ, i.e., whether x_0 is an extreme point of the function f. However, we have the following result which gives sufficient conditions of optimality.

Theorem 5 (*Sufficient conditions of optimality*). *Let $U \subseteq R^n$ be an open set and let $f : U \to R^m$ be a $2p$ times differentiable function at a point $x_0 \in U$ ($p \in N$ and $p \geq 2$). If the forms $\Gamma^{(k)}(x_0^2)(h, h, ..., h) \equiv 0$ for $k \in \{3, ..., 2p-1\}$ and $\Gamma^{(2p)}(x_0^2)(h, h, ..., h) > 0$ then x_0 an extreme point of the function f.*

***Proof*.** By the assumptions, we find that

$$\Gamma(x_0^2 + h) - \Gamma(x_0^2) = \frac{\Gamma'(x_0^2)h}{1!} + \frac{\Gamma''(x_0^2)(h,h)}{2!} + ... + \frac{\Gamma^{(2p)}(x_0^2)(h,h,...,h)}{(2p)!} + r(h)$$

where $\frac{r(h)}{\|h\|^{2p}} \to 0$ as $h \to 0$. As the forms $\Gamma'(x_0^2)$ $\Gamma''(x_0^2)$ are trivial by Theorem 4, we obtain

$$\Gamma(x_0^2 + h) - \Gamma(x_0^2) = \frac{\Gamma^{(2p)}(x_0^2)(h,h,...,h)}{(2p)!} + r(h). \quad (1)$$

Suppose that Γ does not have a local minimum at a point x_0. Then there exists a sequence $\{x_k^2\} \subset R^{2n}$ such that $x_k^2 \to x_0^2$ as $k \to \infty$ and $\Gamma(x_k^2) < \Gamma(x_0^2)$ for all $k \in N$. x_k^2 can be written as $x_k^2 = x_0^2 + \|x_k^2 - x_0^2\| \frac{x_k^2 - x_0^2}{\|x_k^2 - x_0^2\|}$. Denoting $\alpha_k = \|x_k^2 - x_0^2\|$ i $h_k = \frac{x_k^2 - x_0^2}{\|x_k^2 - x_0^2\|}$ the previous equality becomes $x_k^2 = x_0^2 + \alpha_k h_k$ where $\alpha_k \to 0$ as $k \to \infty$ and $\|h_k\| = 1$ for all $k \in N$. In view of the fact that $\{h_k\}$ is a subset of a compact sphere in the space R^{2n} it follows that it contains a convergent subsequence h_{kl}, so that $h_{kl} \to \overline{h}$ with $\|\overline{h}\| = 1$. In view of (1) for each l we have

$$0 > \Gamma(x_{kl}^2) - \Gamma(x_0^2) = \Gamma(x_0^2 + \alpha_{k_l} h_{k_l}) - \Gamma(x_0^2) = \frac{\Gamma^{(2p)}(x_0^2)(\alpha_{k_l} h_{kl}, \alpha_{k_l} h_{kl}, ..., \alpha_{k_l} h_{kl})}{(2p)!} + r(\alpha_{k_l} h_{k_l})$$

whence it follows that
$\alpha_{k_l}^{2p}(\frac{\Gamma^{(2p)}(x_0^2)(h_{kl},h_{kl},...,h_{kl})}{(2p)!}+\frac{r(\alpha_{k_l}h_{kl})}{\alpha_{k_l}^{2p}})<0 \Rightarrow \frac{\Gamma^{(2p)}(x_0^2)(h_{kl},h_{kl},...,h_{kl})}{(2p)!}+\frac{r(\alpha_{k_l}h_{kl})}{\|\alpha_{k_l}h_{kl}\|^{2p}}<0.$

Letting $l\to\infty$ in above inequality, and after this multiplying by $(2p)!$ we obtain $\Gamma^{(2p)}(x_0^2)(\overline{h},\overline{h},...,\overline{h})\le 0$ where $\overline{h}$ is a non-zero vector (because of $\|\overline{h}\|=1$).

This is a contradiction with the assumption of Theorem 5 that $\Gamma^{(2p)}(x_0)$ is a positive definite form. It follows that x_0^2 is a local minimum of the function Γ, i.e., x_0 is an extreme point of the function f. This completes the proof. ■

The following theorem gives a good classification of extreme points on an open set $U\subseteq R^n$ (i.e., it immediately excludes some points).

Theorem 6. *Let U be an open subset of R^n and let $f:U\to R^m$ be a $2p+1$ times differentiable function at a point $x_0\in U$ ($p\in N$ and $p\ge 1$) and $\Gamma^{(2p+1)}(x_0^2)$ is a nontrivial form where the forms $\Gamma^{(k)}(x_0^2)$ are trivial for all $k\in\{1,...,2p\}$. Then x_0 is not an extreme point of a function f.*

Proof. By the assumptions of Theorem 6 it follows that for all $h\in R^{2n}\setminus\{0\}$

$$\Gamma(x_0^2+h)-\Gamma(x_0^2)=\frac{\Gamma^{(2p+1)}(x_0^2)(h,h,...,h)}{(2p+1)!}+r(h)$$

where $\frac{r(h)}{\|h\|^{2p+1}}\to 0$ as $h\to 0$. Since $\Gamma^{(2p+1)}(x_0^2)$ is a nontrivial form, it follows that there exists a vector $\overline{h}\ne 0$ such that $\Gamma^{(2p+1)}(x_0^2)(\overline{h},\overline{h},...,\overline{h})\ne 0$. Furthermore, we have

$$\Gamma(x_0^2+\alpha\overline{h})-\Gamma(x_0^2)=\frac{\Gamma^{(2p+1)}(x_0^2)(\alpha\overline{h},\alpha\overline{h},...,\alpha\overline{h})}{(2p+1)!}+r(\alpha\overline{h})\Rightarrow\frac{\Gamma(x_0^2+\alpha\overline{h})-\Gamma(x_0^2)}{\alpha^{2p+1}}=$$
$$\frac{\Gamma^{(2p+1)}(x_0^2)(\overline{h},\overline{h},...,\overline{h})}{(2p+1)!}+\frac{r(\alpha\overline{h})}{sgn(\alpha)\|\alpha\overline{h}\|^{2p+1}}\|\overline{h}\|^{2p+1}\quad(1)$$

Now suppose that x_0 is an extreme point of a function f.

Letting $\alpha\to 0+$, we find that the left hand side of (1) is positivna in this case (because Γ attains a minimum at a point x_0^2 i $\alpha>0$) and it converges as $\alpha\to 0+$ because of the right hand side of the equality converges to $\frac{\Gamma^{(2p+1)}(x_0^2)(\overline{h},\overline{h},...,\overline{h})}{(2p+1)!}$. Consequently, we have $\frac{\Gamma^{(2p+1)}(x_0^2)(\overline{h},\overline{h},...,\overline{h})}{(2p+1)!}\ge 0$. Letting $\alpha\to 0-$ and proceeding in a a similar manner as previously, we arrive to the inequality $\frac{\Gamma^{(2p+1)}(x_0^2)(\overline{h},\overline{h},...,\overline{h})}{(2p+1)!}\le 0$.

The last two inequalities yields $\frac{\Gamma^{(2p+1)}(x_0^2)(\overline{h},\overline{h},...,\overline{h})}{(2p+1)!}=0$, which contradicts the assertion that $\Gamma^{(2p+1)}(x_0^2)$ is a nontrivial form. Hence, x_0 is not an extreme point of a function f. This completes the proof.■

Corollary. If $x_0\in R^n$ is a point such that the third form $\Gamma'''(x_0^2)$ is nontrivial, then x_0 is not an extreme point of a function f.

EXAMPLES

Example 1. The function $f:R^2\to R^2$ defined as $f(x,y)=(x^2+y^2,x^2+y^2)$ has the extreme point $x_0=(0,0)$ because of for the function $\Gamma:R^4\to R$ we have $\Gamma(a_1,a_2,b_1,b_2)=2(a_1^2+a_2^2)(b_1^2+b_2^2)$. Obviously, Γ atttains a local minimum at a point $x_0^2=(0,0,0,0)$, and it is easy to verify that $\Gamma'''(x_0^2)$ is a trivial form ($\Gamma(x_0^2)$ and $\Gamma'(x_0^2)$ are trivial forms by Theorem 4) and

$$\Gamma^{(4)}(x_0^2)((h_1,h_2,h_3,h_4)(h_1,h_2,h_3,h_4)(h_1,h_2,h_3,h_4)(h_1,h_2,h_3,h_4)) = 8(h_1^2h_3^2+h_1^2h_4^2+h_2^2h_3^2+h_2^2h_4^2)\geq 0,$$

and therefore, $\Gamma(x_0^2+h)-\Gamma(x_0^2)=\frac{\Gamma^{(4)}(x_0^2)(h,h,h,h)}{4!}\geq 0$. This means that x_0^2 is a local minimum of the function Γ, and hence, $x_0=(0,0)$ is an extreme point of a function f.

Example 2. The function $f:R^2\to R^2$ defined as $f(x,y)=(x^2+y,x+y^2)$ does not have none extreme point in R^2 because of
$\Gamma(a_1,a_2,b_1,b_2)=(a_1^2+a_2)(b_1^2+b_2)+(a_2^2+a_1)(b_2^2+b_1)-(a_1^2+a_2+b_1^2+b_2)(x_0^2+y_0)-(a_2^2+a_1+b_2^2+b_1)(y_0^2+x_0)$. This shows that $\frac{\partial^3\Gamma}{\partial a_1^2\partial b_2}(x_0,y_0,x_0,y_0)=2$, i.e., $\Gamma'''(x_0^2)$ is a nontrivial form for all $x_0^2\in R^4$, and by the previous consequence of Theorem 6 it folows that f does not have none extreme point.

Example 3. The function $f:R\to R^3$ defined as $f(x)=(x^2,x^3,x^5)$ has the extreme point $x_0=0$ because of
$\Gamma(a,b)=a^2b^2+a^3b^3+a^5b^5=a^2b^2(1+ab+a^3b^3)\geq 0=\Gamma(0,0)$ for small a and b, i.e., Γ attains a local minimum at the point $(0,0)$, and therefore, x_0 is an extreme point of a function f.

Example 4. The function $f:R\to R^2$ defined as $f(\varphi)=(\cos\varphi,\sin\varphi)$ does not have none extreme point because of

$$\Gamma(a,b)=\cos a\cos b+\sin a\sin b-(\sin a+\sin b)\sin x_0-(\cos a+cosb)\cos x_0 = \cos(a-b)-\cos(a-x_0)-\cos(b-x_0).$$

For $a=b=x_0$ we have $\Gamma(x_0,x_0)=1-1-1=-1$.

For sequences $a_n=x_0+\frac{\pi}{n}$ and $b_n=x_0-\frac{\pi}{n}$ holds $(a_n,b_n)\to(x_0,x_0)$ as $n\to\infty$, and it folows that each neighbourhood of a point $x_0^2=(x_0,x_0)$ contains points of the form (a_n,b_n). We have

$$\Gamma(a_n,b_n)=\cos\frac{2\pi}{n}-2\cos\frac{\pi}{n}=2\cos^2\frac{\pi}{n}-2\cos\frac{\pi}{n}-1.$$

As $z(t)=t^2-2t-1$ is a decreasing function in a neighbourhood of the point $t=1$ and $\cos\frac{\pi}{n}<1$ for all n, it follows that $\Gamma(a_n,b_n)<-1$ for sufficiently large n, whence it follows that (x_0,x_0) is not a local minimum of the function Γ. This shows that x_0 is not an extreme point of a function f.

Example 5. Let $f:R^n\to R^m$ be a function defined as $f(x_1,...,x_n)=c=(c_1,...,c_m)\in R^m$ (c is a constant). Then $\Gamma(a_1,...,a_n,b_1,...,bn)=\sum c_i^2-\sum 2c_ic_i=-\|c\|^2$ $(=const)$. It follows that Γ attains a local minimum at every point $x_0^2=(x_0,x_0)\in R^{2n}$, that is, each $x_0\in R^n$ is an extreme point of the function f.

Example 6. (The function with countably many extreme points). Let $f:U\subset R^n\to R^m$ where $U=\bigcup U_i$ (U_i are open disjoint subsets of the set U). Clearly, such a set U exists. For example, about every point in $Z\times Z\times...\times Z\subset R^n$ we describe a ball with the radius $\frac{1}{2}$. Define
$f(u_1,...,u_n)=((u_1-\alpha_1^i)^2+...+(u_n-\alpha_n^i)^2,...,(u_1-\alpha_1^i)^2+...+(u_n-\alpha_n^i)^2)$ na U_i, $(\alpha_1^i,...,\alpha_n^i)\in U_i$, $i\in\{1,...,n\}$.

Before we give another definition of extremum of a function $f : U \subseteq R^n \to R^m$, we will prove the following two lemmas.

Lemma 1. *Let $\Re : R^m \to R^m$ be a mapping of the space R^m into itself and let $\{e_1, ..., e_m\}$ be the orthonormal base of this space. Then the following assertions are equivalent.*

1. *$\Re$ preserves the scalar product.*
2. *$\Re$ is a linear isometry, i.e., holds*
   (a) *$\Re$ is a linear mapping.*
   (b) *For each $x \in R^m$ $\|\Re x\| = \|x\|$ holds.*
3. *$\Re$ is a linear orthogonal operator, i.e., it holds*
   (a) *$\Re$ is a linear mapping.*
   (b) *For all $i, j \in \{1, ..., m\}$ holds $(\Re e_i, \Re e_j) = \delta_i^j$, where δ_i^j is the Kronecker delta.*

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Proof. (1) $\Rightarrow$ (2). We have $(\Re 0, \Re 0) = (0, 0) = 0$, i.e., $\|\Re 0\|^2 = 0 \Rightarrow \|\Re 0\| = 0 \Rightarrow \Re 0 = 0$. Let $x \in R^m$ be an arbitrary vector. Then

$$\|\Re x\| = (\Re x, \Re x)^{\frac{1}{2}} = (x, x)^{\frac{1}{2}} = \|x\|.$$

Prove that $\Re$ is a linear mapping. Let $x, y \in R^m$. Then we have

$$\begin{gathered}\|\Re(x+y) - \Re x - \Re y\|^2 = (\Re(x+y) - \Re x - \Re y, \Re(x+y) - \Re x - \Re y) = \\ (\Re(x+y), \Re(x+y)) - 2(\Re(x+y), \Re x) - 2(\Re(x+y), \Re y) + 2(\Re x, \Re y) + (\Re x, \Re x) + \\ (\Re y, \Re y) = (x+y, x+y) - 2(x+y, x) - 2(x+y, y) + 2(x, y) + (x, x) + (y, y) = \\ (x+y, x+y) - 2(x+y, x+y) = -(x+y, x+y) + 2(x, y) + (x, x) + (y, y) = \\ -(x, x) - 2(x, y) - (y, y) + (x, x) + 2(x, y) + (y, y) = 0,\end{gathered}$$

i.e.,

$$\|\Re(x+y) - \Re x - \Re y\|^2 = 0 \Rightarrow \Re(x+y) = \Re x + \Re y.$$

By the additivity of the operator $\Re$ and the fact that $\Re 0 = 0$, it can be easily seen that $\Re(qx) = q\Re x$ for each rational number q and for each vector $x \in R^m$.

Now let α be an irrational number. Then there exists a sequence of rational numbers q_n that converge to the irrational number α. We will now prove that $\Re$:

is a continuous operator.

$$\|\Re x - \Re y\| = (\Re x - \Re y, \Re x - \Re y)^{\frac{1}{2}} = (\Re(x-y), \Re(x-y))^{\frac{1}{2}} = (x-y, x-y)^{\frac{1}{2}} = \|x-y\|,$$

i.e., $\Re$ satisfies the Lipschitz condititition, and thus, $\Re$ is continuous. It follows that $\Re(\alpha_n x) = \alpha_n \Re x$. Letting $n \to \infty$ and since $\alpha_n x \to \alpha x$ it follows that $\Re(\alpha x) = \alpha \Re x$.

(2) $\Rightarrow$ (1). Let $x, y \in R^m$. Then

$$(\Re x, \Re y) = \frac{\|\Re x\|^2+\|\Re y\|^2-\|\Re x-\Re y\|^2}{2} = \frac{\|x\|^2+\|y\|^2-\|\Re(x-y)\|^2}{2} = \frac{\|x\|^2+\|y\|^2-\|x-y\|^2}{2} = (x,y).$$

(2) $\Rightarrow$ (3). We also see from (2) that $\Re$ is a linear map. Now assume that $\Re(e_1) = (a_{11},...,a_{m1})$.... etc. $\Re(e_m) = (a_{1m},...,a_{mm})$ and $R = \begin{bmatrix} 11 & ... & a_{1m} \\ . & ... & . \\ a_{m1} & ... & a_{mm} \end{bmatrix}$. We have

$$R^T R = \begin{bmatrix} (\Re e_1, \Re e_1) & ... & (\Re e_1, \Re e_m) \\ . & ... & . \\ (\Re e_m, \Re e_1) & ... & (\Re e_m, \Re e_m) \end{bmatrix}.$$

It follows (2) from that $(\Re e_i, \Re e_j) = (e_i, e_j) = \delta_i^j$, and hence, $R^T R = E_m$.

Besides (3) it is satisfied $1 = \det(R^T R) = \det R^T \det R = \det R \det R = \det R^2$ whence it follows that $\det R = 1$ ili $\det R = 1$.

(3) $\Rightarrow$ (2). Again we have that $\Re$ is a linear map, and as it is showed previously, from (2) $\Rightarrow$ (3) we have $R^T R = E_m$. It follows that $(\Re x, \Re x) = (Rx, Rx) = (R^T Rx, x) = (E_m x, x) = (x, x)$, i.e., $(\Re x, \Re x) = (x, x) \Rightarrow \|\Re x\|^2 = \|x\|^2 \Rightarrow \|\Re x\| = \|x\|$. ($R$ denotes the matrix of the linear operator $\Re$). The proof is completed.■

Definition. The mapping which satisfies any of the conditions (1), (2) or (3) of Lemma 1 is said to be the rotation of the space R^m.

Corollary (of Lemma 1). Let$\Re$ be a rotation of the space $\Re^2$, and let R be a matrix of the operator $\Re$, i.e., $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$. By Lemma 1, we have $R^T R = E_2$, $\begin{cases} \alpha^2 + \beta^2 = 1 \\ \alpha\gamma + \beta\delta = 0 \\ \gamma^2 + \delta^2 = 1 \end{cases}$.

It follows that there exist real numbers $\varphi, \theta \in [0, \pi)$ such that $\alpha = \cos\varphi$, $\beta = \sin\varphi$, $\gamma = \cos\theta$, $\delta = \sin\theta$ and from $\alpha\gamma + \beta\delta = 0$ it follows that $\cos\varphi\cos\theta + \sin\varphi\sin\theta = 0$, i.e., $\cos(\varphi - \theta) = 0 \Rightarrow \varphi - \theta = \frac{\pi}{2} \Rightarrow \theta = \varphi - \frac{\pi}{2}$. Therefore, $R = \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}$ for some $\varphi \in [0, \pi)$.

Lemma 2. *Let S be a set of vectors in the space R^m where $m \in \{1, 2\}$. If S is a set such that $(a, b) \geq 0$ for all $a, b \in S$, then there exists an orthonormal base $(v_1, ..., v_m)$ of the space R^m such that $a = \sum_{i=1}^m \alpha_i^a v_i$ for all $a \in S$, where $\alpha_i^a \geq 0$ for all $i \in \{1, ..., m\}$.*

Conversely, let m be any positive integer and let S be a set of vectors of R^m such that $a = \sum_{i=1}^m \alpha_i^a v_i$ for all $a \in S$, where $\{v_i\}_{i=1}^m$ is a certain fixed orthonormal base of the space R^m. Then $(a, b) \geq 0$ for all $a, b \in S$.

Proof. Let $m = 1$ and S be a subset of R such that $(a, b) \geq 0$ for all $a, b \in S$. Let $\{v_1\}$ be a base of the space R. Without loss of generality, we can suppose that $\|v_1\| = 1$ (otherwise, we assume $\frac{v_1}{\|v_1\|}$). If S contains only zero vector, then $(0, v_1) = 0$ and clearly, the assertion is true.

Now we suppose that for $a \in S$ it holds $(a, v_1) \neq 0$. Then either $(a, v_1) > 0$ or $(a, v_1) < 0$. If $(a, v_1) < 0$ then instead of v_1 we assume $-v_1$ ($\{-v_1\}$ is also an orthonormal base of R). It follows that $(a, v_1) > 0$. We will prove the last inequality for all $a \in S \setminus \{0\}$. Suppose contrary, i.e., that there exists $b \in S \setminus \{0\}$ such that $(b, v_1) < 0$. Therefore, $0 \leq (a, b) = ((a, v_1)v_1, (b, v_1)v_1)) = (a, v_1)(b, v_1) < 0$. This contradiction implies that the assertion is true for $m = 1$.

Next suppose that $m = 2$. If S contains only zero vector, then proceeding as in the previous case, we find that the assertion is true for any orthonormal base. Further, we assume that there exists $a \in S$ different from zero. Consider the function $F(x) = (\frac{a}{\|a\|}, x)$ for $x \in \overline{S}$. We can assume that each element a in $\overline{S}$ of length 1 (if this is not true, then instead of a we consider the vector $\frac{a}{\|a\|}$). ($\overline{S}$ is a closure of the set S).

The function F is continuous and $\overline{S}$ is a compact set as a closed subset of a compact central sphere with the radius 1. By Weierstrass theorem, there exists $v_1 \in \overline{S}$ such that $\inf_{x\in\overline{S}} F(x) = (\frac{a}{\|a\|}, v_1)$. Since $F(x) \geq 0$ for all $x \in \overline{S}$ it follows that $F(v_1) = (\frac{a}{\|a\|}, v_1)$. Consider the orthogonal subspace $L^{\perp}(v_1)$ to the vector v_1. This is one-dimensional vector space, and hence, its base is $\{v_2\}$, so that $\|v_2\| = 1$. Clearly, we have $(v_1, v_2) = 0$ and $\|v_1\| = 1$ because of $v_1 \in \overline{S}$. If $(a, v_2) \geq 0$ then for such a v_2 we have an orthonormal base $\{v_1, v_2\}$, while if this is not true then instead of v_2 we assume $-v_2$ such that $(a, -v_2) > 0$. In both cases we have an orthonormal base $\{v_1, v_2\}$ of R^2 for which $(a, v_2) \geq 0$ and $(a, v_1) \leq (\frac{a}{\|a\|}, b)$ for all $b \in \overline{S}$. Now let $b \in S$ be a non-zero vector. Then we have $b = (b, v_1)v_1 + (b, v_2)v_2$. It follows that

$(b, v_1) = \|b\|(\frac{b}{\|b\|}, v_1) \geq \|b\|(\frac{a}{\|a\|}, v_1) = \frac{\|b\|}{\|a\|}(a, v_1) \geq 0$. It remains to prove that $(b, v_2) \geq 0$. Suppose conversely, i.e., that $(b, v_2) < 0$. Then we have $(\frac{b}{\|b\|}, \frac{a}{\|a\|}) \geq (v_1, \frac{a}{\|a\|})$, or equivalently,
$(\frac{b}{\|b\|}, v_1)(\frac{a}{\|a\|}, v_1) + (\frac{b}{\|b\|}, v_2)(\frac{a}{\|a\|}, v_2) \geq (v_1, \frac{a}{\|a\|})$. >From this we find that
$(\frac{b}{\|b\|}, v_2)(\frac{a}{\|a\|}, v_2) \geq (\frac{a}{\|a\|}, v_1)(1 - (\frac{b}{\|b\|}, v_1)$. Multiplying this by $\|a\|\|b\|$ we obtain

$(b, v_2)(a, v_2) \geq (a, v_1)(\|b\| - (b, v_1))$. (*)

Now consider the following three cases.

Case 1. $(a, v_1) = 0$. Then since $a \neq 0$, $a = (a, v_1)v_1 + (a, v_2)v_2$ and $(a, v_2) \geq 0$ it follows that $(a, v_2) > 0$. Then in (*) we obtain $(b, v_2)(a, v_2) \geq 0$ which is impossible.

Case 2. $\|b\| = (b, v_1) \Rightarrow \|(b, v_1)v_1 + (b, v_2)v_2\| = (b, v_1) \Rightarrow (b, v_1)^2 + (b, v_2)^2 = (b, v_1)^2 \Rightarrow (b, v_2) = 0$, which contradicts the assumption that $(b, v_2) < 0$. From the previous consideration and since by the Cauchy-Schwarz inequality, $(b, v_1) \geq \|b\|\|v_1\| = \|b\|$ we find that $(b, v_1) < \|b\|$, i.e., $\|b\| - (b, v_1) > 0$.

Case 3. $(a, v_2) = 0$. Since $a \neq 0$, $a = (a, v_1)v_1 + (a, v_2)v_2$ and $(a, v_1) \geq 0$ it follows that $(a, v_1) > 0$. In this case the left hand side in the relation (*) is equal to 0 while its right hand side is greater that 0e. A contradiction.

Thus, from the previous cases we conclude that $(a, v_1) > 0, (a, v_2) > 0$ and $\|b\| - (b, v_1) > 0$, while from (*) we have $(b, v_2) \geq \frac{(a,v_1)(\|b\|-(b,v_1)}{(a,v_2)} > 0$, we have $(b, v_2) > 0$, which contradicts the assumption $(b, v_2) < 0$. Hence, we must have $(b, v_2) \geq 0$. Therefore, $b = (b, v_1)v_1 + (b, v_2)v_2$ where (b, v_1) and (b, v_2) are nonnegative numbers, and so, the assertion is proved for the case $m = 2$.

It remains to prove the converse assertion of the lemma. Let $m \in N$ and let S be a set of vectors in R^m for which there exists an orthonormal base $\{v_1, ..., v_m\} \subset R^m$ such that $a = \sum_{i=1}^{m} \alpha_i^a v_i$ for all $a \in S$, where $\alpha_i^a \geq 0$ for each $i \in \{1, ..., n\}$. Now assume that $a, b \in S$. We have $a = \sum_{i=1}^{m} \alpha_i^a v_i$ and $b = \sum_{i=1}^{m} \alpha_i^b v_i$ where $\alpha_i^a, \alpha_i^b \geq 0$ for all $i \in \{1, ..., n\}$. It follows that $(a, b) = \sum_{i=1}^{m} \alpha_i^a \alpha_i^b \geq 0$. The proof of lemma is completed. ■

Example (which shows that the first part of the assertion of Lemma 2 is not true for $m \geq 3$).

Firstly, we will show the following auxiliary assertion.

The assertion. *There exists a countable set $\{\xi_1, \xi_2, ..., \xi_i, ..\}$ of vectors in the space $R^n (n \geq 2)$ such that every its subset consisting of n elements is a linearly independent set.*

Proof. Assume that $\xi_1, \xi_2, ..., \xi_n$ is a standard base of the space R^n. Choose a vector ξ_{n+1} such that it does not belong to the set $L(\{f_1, f_2, ..., f_{n-1}\})$ ($L(A)$ is a vector space generated by the set A) where $\{f_1, f_2, ..., f_{n-1}\}$ are arbitrary subsets of the set $\{\xi_1, \xi_2, ..., \xi_n\}$ consisting of $n-1$ elements. It follows that the set $\{\xi_1, \xi_2, ..., \xi_{n+1}\}$ possesses the above property. Choose ξ_{n+2} such that it does not belong to the subsets $L(\{f_1, f_2, ..., f_{n-1}\})$ where $\{f_1, f_2, ..., f_{n-1}\}$ is an arbitrary subset of $\{\xi_1, \xi_2, ..., \xi_{n+1}\}$ consisting of $n-1$ elements, ,... etc, we procced inductively.

In other words, ξ_{n+k} is chosen so that it does not belong to the the subsets $L(\{f_1, f_2, ..., f_{n-1}\})$ where $\{f_1, f_2, ..., f_{n-1}\}$ is any subset of $\{\xi_1, \xi_2, ..., \xi_{n+k-1}\} (k \in N)$ consisting of $n-1$ elements.

Every subset of the set $\{\xi_1, \xi_2, ..., \xi_i, ..\}$ consisting of n elements is a linearly independent set. Clearly, if there would be be exist a subset consisting of n linearly dependent vectors, then one of thes vectors should be a linear combination of other vectors (their maximal number is $n-1$), which contradicts a construction of this vector. This completes the proof. ■

Now we return to the example. Consider the following subset of $R^m (m \geq 3)$:
$K_m = \{(x_1, x_2, ..., x_m) | \; x_m = \sqrt{\sum_{i=1}^{m-1} x_i^2} \;\}$ (K_m is a conus).
Assume that $x, y \in K_m$, i.e.,

$$x = (x_1, x_2, ..., x_{m-1}, \sqrt{\sum_{i=1}^{m-1} x_i^2}) \text{ and } y = (y_1, y_2, ..., y_{m-1}, \sqrt{\sum_{i=1}^{m-1} y_i^2}).$$

Using Cauchy-Schwarz inequality, we have
$(x, y) = \sum_{i=1}^{m-1} x_i y_i + \sqrt{\sum_{i=1}^{m-1} x_i^2} \sqrt{\sum_{i=1}^{m-1} y_i^2} \geq 0.$

Suppose that

$$K \subset A = \left\{ \sum_{i=1}^{m} \alpha_i v_i \;\middle|\; \begin{array}{c} \alpha_i \geq 0 \text{ and} \\ \{v_1, v_2, ..., v_m\} \text{ is an orthonormal base of the space } R^m \end{array} \right\}.$$

Assuming that $a, b \in A$ and $(a, b) = 0$ then $(\sum_{i=1}^{m} a_i v_i, \sum_{i=1}^{m} b_i v_i) = 0$, whence it follows that $\sum_{i=1}^{m} a_i b_i = 0$. The last equality shows that at least half of coordinates of the vector a or b is equal to zero. It is necessary that the set K also has this property i.e., that for every pair of vectors in K that are mutually orthogonal, and one of these vectors must lie in a vector subspace of $\leq \lfloor \frac{m}{2} \rfloor$ of the space R^m. The number of such subspaces is finite, because of they are subspaces of the space R^m

and generated by subsets of the set $\{v_1, v_2, ..., v_m\}$ (v_i belongs to the subspace if for all $x, y \in K$ with $(x, y) = 0$, assuming that x is such a vector which has greater than half of its coordinates in the base $\{v_1, v_2, ..., v_m\}$ više that are equal to zero, then ith coordinate of the vector x is $\neq 0$).

Now we claim that there exists a countable subset E of K satisfying the property: $E = \{a_1, b_1, a_2, b_2, ..., a_r, b_r, ...\}$ with $(a_i, b_i) = 0$ for all $i \in N$, and every subset of the set $\{a_1, a_2, ..., a_i, ..\}$ consisting of $m - 1$ elements is a linearly independent set, and every subset of the set $\{b_1, b_2, ..., b_i, ..\}$ consisting of $m - 1$ elements is also linearly independent set.

Firstly, we will construct the vectors $(a_i)_{i \in N}$. Let $\{\xi_i\}_{i \in N}$ be a set constructed in the previous assertion, regarded as a subset of the space R^{m-1}. If we define $a_i = (\xi_i, \|\xi_i\|)$, then $a_i \in K_m$ for all $i \in N$ and it is easily seen that the set $\{a_1, a_2, ..., a_i, ...\}$ has the property that every its subset consisting of $m-1$ elements is linearly independent set. Take $b_i = (-\xi_i, \|\xi_i\|)$. We also have $b_i \in K_m$ for all $i \in N$ and $(a_i, b_i) = -(\xi_i, \xi_i) + \|\xi_i\|^2 = 0$ holds. Hence, we have constructed the set E with the mentioned properties.

As the pairs $\{a_i, b_i\}$ are mutually orthogonal and they lie in K_m, it follows that at least one of vectors a_i or b_i must belong to the previous constructed vector spaces of dimensions $\leq \lfloor \frac{m}{2} \rfloor$ for all $i \in N$. It follows that there exists a subsequence of a sequence $(a_i)_{i \in N}$ or $(b_i)_{i \in N}$ (we assume that $(a_{i_j})_{j \in N} \subset (a_i)_{i \in N}$) that lie in subspaces of dimensions $\leq \lfloor \frac{m}{2} \rfloor$. Since the number of these subspaces is finite, it follows that there exists infinitely many terms of a sequence $(a_{i_j})_{j \in N}$, namely, its subsequence $(a_{i_{j_k}})_{k \in N}$ which lies in one of these subspaces of dimension $\leq \lfloor \frac{m}{2} \rfloor$. Assume that $a_{i_{j_1}}, a_{i_{j_2}}, ..., a_{i_{j_{m-1}}} \in (a_{i_{j_k}})_{k \in N} \subset (a_i)_{i \in N}$. It follows that these vectors are linearly independent, their number is $m-1$ and they lie in a subspace of the space R^m whose dimension is $\leq \lfloor \frac{m}{2} \rfloor < m-1$ for $m \geq 3$. A contradiction! Therefore, it is not possible to lie the conus K_m in the set $A = \{\sum_{i=1}^{m} \alpha_i v_i \mid \alpha_i \geq 0 \text{ for all } i \in \{1, 2, .., m\}\}$ where $\{v_1, v_2, ..., v_m\}$ is a orthonormal base of the space R^m.

Remark. The answer to the question why it is not possible to apply the previous proof in the cases when $m < 3$ is as follows: it is not possible to construct E with the mentioned properties.

Definition 3. Let $f : U \to R^m$ be a function where $U \subseteq R^n$ is an open set, and let $x_0 \in U$. A point x_0 is said to be a strong extreme point of a function f if there exist a rotation $\Re$ of the space R^m such that the mapping $F : U \to R^m$ defined as $F(x) = f(x_0) + \Re(f(x) - f(x_0))$ has the property that all its coordinate maps $F_i : U \to R$ with $i \in \{1, ..., m\}$ attain a local minimum.

Theorem 7. *Let $f : U \to R^m$ be a function where $U \subseteq R^n$ is an open set, and let $x_0 \in U$. If x_0 is a strong extreme point of a function f, then x_0 is an extreme point of a function f. Conversely, if $m \leq 2$ ($m \in N$), then all extreme points are also strong extreme points of a function f.*

Proof. Let x_0 be a strong extreme point of a function f. Then there exist a rotation $\Re$ prostora R^m satisfying the following property: There exists a neighbourhood $O(x_0)$ of x_0 such that the funcction $F(x) = f(x_0) + \Re(f(x) - f(x_0))$ in this neighbourhood has the property that the coordinate maps attain a local minimum.

Let $e_1, ..., e_m$ be the standard orthonormal base of the space R^m. Then for each $x \in O(x_0)$ we have $F(x) - f(x_0) = \Re(f(x) - f(x_0)) \in \{\sum_{i=1}^m \alpha_i e_i \mid \alpha_i \geq 0\}$, i.e., $\Re(f(O(x_0)) - f(x_0)) \subset \{\sum_{i=1}^m \alpha_i e_i \mid \alpha_i \geq 0\}$. It follows from Lemma 2 that $(\Re(f(x) - f(x_0)), \Re(f(y) - f(y_0))) \geq 0$ for each $(x, y) \in O(x_0, x_0)$. It follows from Lemma 1 that $(f(x)-f(x_0), f(y)-f(y_0)) \geq 0$. This shows that $\Gamma(x, y)+\|f(x_0)\|^2 \geq 0 \Rightarrow \Gamma(x, y) \geq -\|f(x_0)\|^2$ i $\Gamma(x_0, x_0) = -\|f(x_0)\|^2$, i.e., Γ attains a local minimum at the point $x_0^2 = (x_0, x_0)$ and hence, x_0 is an extreme point of the mapping f.

Conversely, suppose that $m \leq 2$ and $x_0 \in U \subseteq R^n$ is an extreme point of the function f. Then the function $\Gamma(x, y) = (f(x) - f(x_0), f(y) - f(y_0)) - \|f(x_0)\|^2$ attains a local minimum at a point $x_0^2 = (x_0, x_0)$. Since $\Gamma(x_0)^2 = -\|f(x_0)\|^2$, this is equivalent with $(f(x) - f(x_0), f(y) - f(y_0)) \geq 0$ for all $x, y \in O(x_0^2)$. Hence, there exists a neighbourhood $O(x_0)$ of a point x_0 which lies in $U \subset R^n$ such that the above relation is valid for all $x, y \in O(x_0)$.

It follows by Lemma 2 that there exists an orthonormal base $\{v_1, ..., v_m\}$ of the space R^m such that $f(x) - f(x_0) \in \{\sum_{i=1}^m \alpha_i e_i \mid \alpha_i \geq 0\}$ for all $x \in O(x_0)$. Hence, $f(x) - f(x_0) = \sum_{i=1}^m \alpha_i^x v_i$ where $\alpha_i^x \geq 0$ for each $i \in \{1, ..., m\}$. By using Lemma 1, there exists a rotation $\Re$ of the space R^m such that $\Re(v_i) = e_i$ for all $i \in \{1, ..., m\}$, where $(e_i)_{i=1}^m-$ is a standard orthonormal base of the space R^m. It follows that $\Re(f(x) - f(x_0)) = \Re(\sum_{i=1}^m \alpha_i^x v_i) = \sum_{i=1}^m \alpha_i^x \Re(v_i) = \sum_{i=1}^m \alpha_i^x e_i$ for all $x \in O(x_0)$.

Therefore, the coordinate maps of the function $\Re(f(x) - f(x_0))$ attain local minimums at a point x_0 which are equal to zero. It follows that this is also true for the function $F(x) = f(x_0)+\Re(f(x)-f(x_0))$ where the local minimums of coordinate maps F_i of this function at a point x_0 are equal to $f_i(x_0)$ (f_i are the coordinate maps of the function f). The proof is completed. ■

Corollary. *If $f : U \to R^m$ is a function where $U \subseteq R^n$ is an open set, and if $x_0 \in U$ is a strong extreme point of a function f, then* $f'(x_0) = 0$.

Proof. By Theorem 7, x_0 is an extreme point of a function f, and thus the assertion follows from Theorem 1.

We give here another proof of the corollary. Namely, $F'(x) \equiv 0$ holds (because of $F'(x_0)(h_1, ..., h_n) = \sum_{i=1}^m f_i'(x_0)h_i = \sum_{i=1}^m 0$), and therefore, $(\Re(f(x) - f(x_0)))'_{x_0} = 0$, i.e., $(\Re f(x_0))' = 0 \Rightarrow \Re f'(x_0) = 0 \Rightarrow f'(x_0) \equiv 0$.

Remark. In the case when $m = 2$, i.e., for a function $f : U \subseteq R^n \to R^2$ we have that $x_0 (x_0 \in U)$ is an extreme point of a function f, if a function f maps points that are “near"to a point x_0 in a rectangular part of the plane R^2 with a vertex at a point $f(x_0)$.

In order to verify extreme points of these functions, or to determine all extreme points of f there are numerous criteria (necessary and sufficient conditions).

<u>EXAMPLES</u>

Example 1. The function $f : R \to R^2$ defined as $f(x) = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} (1 - \cos x, x^2)$ has a strong extreme point or an extreme point, namely, the point $x = 0$, because of $R = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & -\cos\frac{\pi}{6} \end{bmatrix}$ is a matrix of rotation for the angle $\frac{\pi}{6}$, and hence,

$$f(x) = \Re_{\frac{\pi}{6}}(1-\cos x, x^2) \Rightarrow$$
$$\Rightarrow (1-\cos x, x^2) = \Re_{\frac{\pi}{6}}{}^{-1} f(x) = \Re_{-\frac{\pi}{6}}(f(x)-f(0)) + f(0) = F(x).$$

Since the coordinate maps F attain local minimums at the point $x_0 = 0$, it follows that $x_0 = 0$ is a strong extreme point of a function f and hence, it is an extreme point of a function f.

Example 2. We have previously determined the extreme points and strong extreme points of a given function f. For a given point $x_0 \in R^n$ we will now determine the function $f : R^n \to R^2$ such that a point x_0 is an extreme point of f. We proceed as follows.

There exists a $\Re-$rotation of the space R^2 for which
$F(x) = f(x_0) + \Re(f(x) - f(x_0)) \Leftrightarrow f(x) = \Re^{-1}(F(x) - f(x_0)) + f(x_0)$, i.e., $f(x) = \Re_1(F(x) - f(x_0)) + f(x_0)(R_1 = \Re^{-1}-$ is also a rotation).

Now we proceed as follows. Choose an arbitrary function $F = (F_1, ..., F_m)$ such that the coordinate maps $F_i : R^n \to R$ $(i \in \{1, ..., m\})$ attain a local minimum at a point x_0. Let $f_i(x_0) = F_i(x_0)$ za $i \in \{1, ..., m\}$, and define $f(x) = \Re(F(x) - f(x_0)) + f(x_0)$ for an arbitrary rotation $\Re$. Clearly, x_0 is a strong extreme point, and hence, it is an extreme point of a function f.

Example 3. Consider the mapping $f : R \to R^2$ defined as $f(x) = (\cos x, \sin x)$. If there would be exist a point x_0 which is a strong extreme point of f, then would be exist a rotation $\Re$ of the space R^2 such that

$$\Re(f(x)) = \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix} (\cos x, \sin x) =$$
$$(\cos x \cos\varphi + \sin x \sin\varphi, \cos x \sin\varphi - \sin x \cos\varphi) = (\cos(x-\varphi), \sin(x-\varphi)) \Rightarrow$$

$$F(x) = \Re(f(x) - f(x_0)) + f(x_0)) =$$
$$(\cos(x-\varphi) - \cos(x_0-\varphi) + \cos x_0, \sin(x-\varphi) - \sin(x_0-\varphi) + \sin x_0).$$

Then must be exist φ such that $F_1(x_0)$ and $F_2(x_0)$ are local minimums of the functions F_1 and F_2. As F_1 and F_2 are differentiable functions, it follows that $F_1'(x_0) = 0$ and $F_2'(x_0) = 0$, i.e., $\sin(x_0 - \varphi) = 0$ and $\cos(x_0 - \varphi) = 0$, which is impossible because of the functions sin i cos does not vanish at the same point.

Example 4. Consider the function $f : R^2 \to R^3$ defined as $f(x, y) = (x, y, \sqrt{x^2 + y^2}$. The point $(0, 0)$ is an extreme point of the function f because of $\Gamma((a_1, a_2, b_1, b_2)) = a_1b_1 + a_2b_2 + \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2} \geq 0$ by Cauchy-Schwarz inequality, and $\Gamma(0, 0, 0, 0) = 0$, i.e., $(0, 0, 0, 0)$ is a local minimum of the mapping Γ, and so, $(0, 0)$ is an extreme point of the function f. Now we will prove that the point $x_0 = (0, 0)$ is not a strong extreme point of the function f.

Suppose contrary, i.e., that there exists a rotation $\Re$ of the space R^3 such that

$$F(x,y) = f(0,0) + \Re(f(x,y) - f(0,0)) = \Re(x, y, \sqrt{x^2+y^2}) =$$
$$= (F_1(x,y), F_2(x,y), F_3(x,y)) \geq (f_1(0,0), f_2(0,0), f_3(0,0)) = (0,0,0)$$

for each $(x,y) \in O((0,0)) \subset R^2$, i.e., that there exists a rotation $\Re$ of the space R^3 such that the set $\Re(x, y, \sqrt{x^2+y^2})$ for (x,y) in this neighbourhood $O((0,0))$, is a subset of the set $A = \{\sum_{i=1}^3 \alpha_i e_i \mid \alpha_i \geq 0, i \in \{1,2,3\}\}$, or equivalently, $(x, y, \sqrt{x^2+y^2}) \subset B = \{\sum_{i=1}^3 \alpha_i \Re^{-1} e_i \mid \alpha_i \geq 0, i \in \{1,2,3\}\}$. By Lemma 1 it follows that $\{\Re^{-1}e_1, \Re^{-1}e_2, \Re^{-1}e_3\}$ is an orthonormal base of the space R^3.

In the same manner as in example for conus K_m we can prove that the the previously mentioned is not possible. The only change consists in the fact that the set $E = \{a_1, b_1, ..., a_i, b_i, ..\}$ (see the example for conus K_m) must belong to $f(O(0,0))$. This can be made in the manner that instead of a_i $(i \in N)$ we assume $\frac{a_i}{n_{O(0,0)}}$ where $n_{O(0,0)} \in N$ with $\frac{a_i}{n_{O(0,0)}} \in f(O(0,0))$. In a similar manner we proceed for vectors b_i $(i \in N)$. Hence, the point $(0,0)$ is not a strong extreme point of the mapping f.

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