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# Слабые алгебры Фаддеева — Тахтаджана — Волкова. Решеточные $W_n$ алгебры[1]

Ф. Разавиниа

**Фаррох Разавиниа** — Московский физико-технический институт (г. Москва).
*e-mail: f.razavinia@phystech.edu*

**Аннотация**

В этой статье мы начнем с обсуждения исторического общего вида нашего проекта, а затем попытаемся построить новую скобку Пуассона на нашем простейшем примере $sl_2$, а затем попытаемся дать универсальную конструкцию на основе наших универсальных переменных, а затем постараемся построить решеточные $W_2$-алгебры, которые будут играть ключевую роль в других наших конструкциях на решетчатых $W_3$-алгебрах, и, наконец, мы попытаемся найти единственный нетривиальный зависимый генератор наших решеточных $W_4$-алгебр и т. д. для решетки $W_n$-алгебры.

*Ключевые слова:* Решетки, $W$ алгебры, квантовые группы, гомоморфизмы Фейгина.

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# Weak Faddeev–Takhtajan–Volkov algebras. Lattice $W_n$ algebras

F. Razavinia

**Farrokh Razavinia** — Moscow Institute of Physics and Technology (Moscow).
*e-mail: f.razavinia@phystech.edu*

**Abstract**

In this paper, we will start by deliberating at our project's historical general view and then we will try to construct a new Poisson bracket on our simplest example $sl_2$ and then we will try to give a universal construction based on our universal variables and then will try to construct lattice $W_2$ algebras which will play a key role in our other constructions on lattice $W_3$ algebras and finally we will try to find the only nontrivial dependent generator of our lattice $W_4$ algebras and so on for lattice $W_n$ algebras.

*Keywords:* Lattice $W$ algebras, quantum groups, Feigin's homomorphisms.

*Bibliography:* 11 titles.

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Dedicated to the 80-th anniversary of A.V.Michalev and 70-th aniversary of A.L.Semenov

# 1. Introduction

There is an old problem which has been considered and introduced by Boris Feigin in 1992. It's been born in its new formulation on quantum Gelfand-Kirillov conjecture in a public talk at RIMS in 1992 based on the nilpotent part of $U_q(g)$ i.e. $U_q(\mathfrak{n})$ for $g$ a simple Lie algebra.
Now, this problem is known as ″Feigin's Conjecture″.
In the mentioned talk, Feigin proposed the existence of a certain family of homomorphisms on the quantized enveloping algebra $U_q(g)$ to the ring of skew-polynomials which will led us to a deffinition of lattice $W-$algebras.
These ”homomorphisms” has been turned to a very useful tool for to study the fraction field of quantized enveloping algebras. [6]
There have been many attempt for to construct lattice $W$-algebras in Feigin's sence, which ensures the simplicity of the construction process of lattice $W$-algebra; for example the best known articles in the subject has been written by Kazuhiro Hikami and Rei Inoue who tried to obtain the algebra structure by using lax operators and generalized R matrices. [7] [8]
Or Alexander Belov and Alexander Antonov and Karen Chaltikian, who first tried to follow Feigin's construction but finaly they also solved part of the conjecture by getting help of lax operators, and it made very difficult to follow their publication.[9] [10]
But here in this paper we will proceed and will introduce the most simplest way of constructing such kind of algebras by just employing Feigin's homomorphisms and screening operators by defining a Poisson bracket on our variables just based on our Cartan matrix. [1] [2]
In [2], Yaroslav Pugai has constructed lattice $W_3$ algebras already, but here we will introduce its weaker version based on a Poisson bracket as mentioned before, constructed on just Cartan matrix $A_n$, which will make our job more easier and more elegant.
For to do this, let us set $C$ an arbitrary symmetrizable Cartan matrix of rank $r$ and let $n = n_+$ be the standard maximal nilpotent sub-algebra of the Kac-Moody algebra associated with $C$.
So $n$ is generated by elements $E_1, \cdots, E_r$ which are satisfying in Serre relations. [11] Where $r$ stands for rank$(C)$.
In [1], we proved that screening operators $S_{X_i^{ji}} = \sum\limits_{\substack{j\in\mathbb{Z} \\ \text{for } i \text{ fixed}}}^{n} X_i^{ji}$; for $X_i^{ji}$ generators of the $q-$commutative ring $\mathbb{C}_q[X_i^{ji}] := \frac{\mathbb{C}[X_i^{ji}]}{\left\langle X_i^{ji}X_k^{jk} - q^{<\alpha_i,\alpha_j>}X_k^{jk}X_i^{ji}\right\rangle}$ and for $<\alpha_i, \alpha_j>= a_{ij}$ the $ij$'s components of our Cartan matrix $C$; are satisfying in quantum Serre relations $\mathrm{ad}_q(X_i)^{1-a_{ij}}(X_j)$ for adjoint action $\mathrm{ad}_q(X_i)(X_j) = X_iX_j - q^{a_{ij}}X_jX_i$ and $X_i \in (U_q)_\alpha$, $X_j \in (U_q)_\beta$ [5], for $(U_q)_\alpha = \{u \in U_q(g) | q^{\mathfrak{h}}uq^{-\mathfrak{h}} = q^{\alpha(\mathfrak{h})}u \quad \text{for all } \mathfrak{h} \in \overset{\vee}{P}\}$ and $U_q(g) = \bigoplus\limits_{\alpha \text{ in } Q}(U_q)_\alpha$, for $Q = \bigoplus\limits_{i\in I}\mathbb{Z}_{\alpha_i}$ the root lattice and for $\overset{\vee}{P}$ a free abelian group of rank $2|I| - \mathrm{rank}C$ with $\mathbb{Z}-$basis $\{h_i | i \in I\} \cup \{d_s | s = 1, \cdots, |I| - \mathrm{rank}C\}$ and $\mathfrak{h} = \mathbb{F} \otimes_{\mathbb{Z}} \overset{\vee}{P}$ be the $\mathbb{F}-$linear space spanned by $\overset{\vee}{P}$. [5]
$\overset{\vee}{P}$ will be called dual weight lattice and $\mathfrak{h}$ the Cartan subalgebra. And $\mathbb{F}$ will stand for our ground field.[5]
Here for our Cartan matrix $C$, the quantum Serre relation will be

$$\mathrm{ad}_q(X_i)^{1-(-1)}(X_j) = \mathrm{ad}_q^2(X_i)(X_j)$$
$$= X_i^2 X_j - [2]_q X_i X_j X_i + X_j X_i^2$$
$$= X_i^2 X_j - (q+q^{-1}) X_i X_j X_i + X_j X_i^2$$

Where $[2]_q$ stands for quantum number $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ in general.
And again as what we had in [1], we can define

$$U_q(n) := \langle S_{X_i^{ji}}, S_{X_k^{jk}} \mid (\mathrm{ad}_q(S_{X_i^{ji}}))^2 (S_{X_k^{jk}}) = 0 \rangle$$

and for $\mathbb{C}_q[X]$ the quantum polynomial ring in one variable and twisted tensor product $\bar{\otimes}$, we can define

$$U_q(n) \bar{\otimes} \mathbb{C}_q[X_l^{jl}] := \langle S_{X_i^{ji}}, S_{X_k^{jk}}, X_l^{jl} \mid (\mathrm{ad}_q(S_{X_i^{ji}}))^2 (S_{X_k^{jk}}) = 0$$
$$, S_{X_i^{ji}} X_l^{jl} = q^2 X_l^{jl} S_{X_i^{ji}}, S_{X_k^{jk}} X_l^{jl} = q^{-1} X_l^{jl} S_{X_k^{jk}} \rangle$$

such that we have the following embeding

$$U_q(n) \hookrightarrow U_q(n) \bar{\otimes} \mathbb{C}_q[X_l^{jl}] \hookrightarrow U_q(n) \bar{\otimes} \mathbb{C}_q[X_l^{jl}] \bar{\otimes} \mathbb{C}_q[X_m^{jm}]$$

where $\mathbb{C}_q[X_l^{jl}] \bar{\otimes} \mathbb{C}_q[X_m^{jm}] = \mathbb{C}\langle X_l^{jl}, X_m^{jm} \mid X_l^{jl} X_m^{jm} = q^{a_{lm}} X_m^{jm} X_l^{jl}$.[1]
Which will ensure the well definedness of our definition of lattice $W-$algebras.

## 2. Weak Faddeev-Takhtajan-Volkov algebras

As it has been mentioned already in [1], the main tools that we use, will be difference equations, screening operators, Feigin's homomorphisms, adjoint actions, partial differential equations and Cartan matrices, etc
We know that from an abstract view $g = sl_{m+1}$ is an algebra related to the Cartan matrix $(a_{ij})$, for $a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$ and so for $sl_2$ it will consist of just one row and one column, i.e. we have $A_1 = (2)$ and let us denote by $C\langle X \rangle$ the skew polynomial ring on generators $X = (X_i)_i$ labeled by $i \in \{-\infty, \cdots -1, 0, 1, \cdots, +\infty\}$ and defining $q-$commutation relations $X_i X_j = q^2 X_j X_i$ for if $i \leq j$ with all having the same color.

Definition . *Let's define our Poisson bracket as follows in the case of $sl_2$:*

$$\begin{cases} \{X_i, X_j\} := 2 X_i X_j & \text{if } i < j \\ \{X_i, X_i\} := 0 \end{cases} \tag{1}$$

The main problem is to find solutions of the system of difference equations from infinite number of non-commutative variables in quantum case and commutative variables in classical case. It is significant that commutation relations (1) depend on the sign of the difference $(i-j)$ only and is based on our Cartan matrix. We should try to find all solutions of the system:

$$\begin{cases} \mathfrak{D}_x^{(n)} \lhd \tau_1 = 0 \\ H_x^{(n)} \lhd \tau_1 = 0 \end{cases} \tag{2}$$

Let us define our system of variables as follows

$$\begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & X_1^{(11)} & X_1^{(21)} & X_1^{(31)} & X_1^{(41)} & \cdots \\ \cdots & X_2^{(12)} & X_2^{(22)} & X_2^{(32)} & X_2^{(42)} & \cdots \end{array}$$

$$\begin{array}{cccccc} \cdots & X_3^{(13)} & X_3^{(23)} & X_3^{(33)} & X_3^{(43)} & \cdots \\ \cdots & X_4^{(14)} & X_4^{(24)} & X_4^{(34)} & X_4^{(44)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

And let us equip this system of variables with lexicographic ordering, i.e. $j_{k_m} i < j_{k_n} i$ if $j_{k_m} < j_{k_n}$ and $j i_{k_m} < j i_{k_n}$ if $i_{k_m} < i_{k_n}$. And we need this kind of ordering because we have different kind of set of variables with a proper coloring such that each set has its own color different of its neighbors. We have $\tau_1 := \tau_1[\cdots, X_1^{(11)}, X_1^{(21)}, X_1^{(31)}, \cdots, X_2^{(12)}, X_2^{(22)}, X_2^{(32)}, \cdots]$, a multi-variable function depend on $\{X_i^{(ji)}\}$'s for $i, j \in \{-\infty, \cdots, 1, \cdots, n, \cdots, +\infty\}$ and $\mathfrak{D}_x^{(n)}$ comes from

$$\{S_{X_i^{ji}}, \tau_1\}_p = S_{X_i^{ji}} \tau_1 - p^{\deg \tau_1 < \alpha_i, \alpha_j >} \tau_1 S_{X_i^{ji}} \tag{3}$$

where $< \alpha_i, \alpha_j >= a_{ij}$ is related to our Cartan matrix and $S_{X_i^{ji}}$ is a screening operator on one of our variable sets, i.e. $S_{X_i^{ji}} = \sum_{j \in \mathbb{Z}} X_i^{ji}$. Then we will obtain the whole set of solutions by using the following shift operator:

$$\begin{aligned} \tau_2 &= \tau_1[X_1^{(11)} \to X_1^{(21)}, X_1^{(21)} \to X_1^{(31)}, \cdots], \\ \tau_3 &= \tau_2[X_1^{(21)} \to X_1^{(31)}, X_1^{(31)} \to X_1^{(41)}, \cdots] \\ &\vdots \end{aligned} \tag{4}$$

Definition . *Let us define our lattice W-algebra based on its generators according to [2] [1]. Generators of lattice W-algebra associated with simple Lie algebra g constitute of the functional basis of the space of invariants*

$$\tau_i := Inv_{U_q(n_+)}(\mathbb{C}_q[X_i^{ji} | i \in \mathbb{Z}]) \tag{5}$$

*with additional requirements*

$$H_{X_i^{ji}}(\tau_i) = 0 \qquad \textit{and} \qquad D_{X_i^{ji}}(\tau_i) = 0 \tag{6}$$

*where $H_{X_i^{ji}}$ and $D_{X_i^{ji}}$ will be specified later.*

Equation (4) means that the generators have to satisfy in quantum Serre relations and the first equation (5) means that they should have zero degree.

Here in this paper we just will work on $g = \mathrm{sl}_n$ and will use $\tau_i^{(n)}$ instead of $\tau_i$.

Where $(n)$ sits for $n$ in $\mathrm{sl}_n$.

## 2.1. Lattice $W_2$ algebra

Let us first consider the $sl_2$ case for to open out the concepts of (2) and (4). And also for to simplifying out notations, let us consider our set of variables as $X_i := X_i^{ji}$.

And as it has shown in [1], it is enough just to work with $S_{X_i^{ji}} =: S_{X_i} = \sum_{i=1}^{3} X_i$, because the other parts for $i > 3$ and $i < 1$ will tend to zero.

By setting $q = e^{-\mathfrak{h}}$, for the Planck constant $\mathfrak{h}$, we will try to find generators of our lattice $W_2$-algebra, in the case of $\mathrm{sl}_2$.

**First step:**

First let us try to find $D_X^{(2)}$.

For to do this and for simplicity, we will set $\tau_1 := \tau_1[\cdots, X_1, X_2, X_3, , \cdots]$. And as it has been defined already, we have

$$\begin{aligned} D_X^{(2)} &:= \{S_{X_i}, \tau_1\} \\ &= \{X_1 + X_2 + X_3, \tau_1\} \\ &= \{X_1, \tau_1\} + \{X_2, \tau_1\} + \{X_3, \tau_1\} \\ &= (D_{X_1} + D_{X_2} + D_{X_3})\tau_1 \end{aligned} \tag{7}$$

Now for to understand what is (6), we note that partial $D_{X_i} = \{X_i, \tau_1\}$ and also note that our function $\tau_1[\cdots, X_1, X_2, X_3, \cdots]$ is a polynomial function consist of powers of $X_i$. What I mean is that, it is enough to find $D_{X_i}$ on just powers of $X_j$ for different values of $j \in \mathbb{Z}$.
So

$$(6) = \sum_j (\{X_1, X_j^n\} + \{X_2, X_j^n\} + \{X_3, X_j^n\}) \tag{8}$$

Where according to rules which has been showed out in [1], we have

$$\begin{aligned} \{X_1, X_j^n\} &= X_1 X_j^n - q^{2n} X_j^n X_1 \\ &= \begin{cases} 0, & \text{if } j > 1 \\ (1 - q^{4n}) X_1 X_j^n, & \text{if } j < 1 \\ (1 - q^{2n}) X_1 X_j^n, & \text{if } j = 1 \end{cases} \end{aligned}$$

Where by setting $q = e^{-\mathfrak{h}}$ and letting $\mathfrak{h} = 1$ at the end, we will have:

First case: $j > 1$;

$$\{X_1, X_j^n\} = 0;$$

Second case: $j < 1$;

$$\begin{aligned} \{X_1, X_j^n\} &= (1 - e^{-4n\mathfrak{h}}) X_1, X_j^n \\ &\sim (1 - (1 - 4n\mathfrak{h})) X_1, X_j^n \\ &= 4n\mathfrak{h} X_1, X_j^n \sim 4n X_1, X_j^n \\ &= 4 X_1 X_j \tfrac{\partial X_j^n}{\partial X_j}. \end{aligned}$$

Third case: $j = 1$;

$$\begin{aligned} \{X_1, X_1^n\} &= (1 - q^{2n}) X_1 X_1^n \\ &= (1 - e^{-2n\mathfrak{h}}) X_1 X_1^n \\ &\sim (1 - (1 - 2n\mathfrak{h})) X_1 X_1^n \\ &= 2n\mathfrak{h} X_1 X_1^n \sim 2n X_1 X_1^n \\ &= 2 X_1^2 \tfrac{\partial X_1^n}{\partial X_1}. \end{aligned}$$

And so we have

$$\begin{aligned} (7) = &\{X_1, X_1^n\} + \textstyle\sum_{j<1} \{X_1, X_j^n\} + \sum_{j>1} \{X_1, X_j^n\} \\ &+ \{X_2, X_2^n\} + \textstyle\sum_{j<2} \{X_2, X_j^n\} + \sum_{j>2} \{X_2, X_j^n\} \\ &+ \{X_3, X_3^n\} + \textstyle\sum_{j<3} \{X_3, X_j^n\} + \sum_{j>3} \{X_3, X_j^n\} \\ &= 2X_1^2 \tfrac{\partial}{\partial X_1} + 0 + 0 \\ &+ 2X_2^2 \tfrac{\partial}{\partial X_2} + 4X_2 X_1 \tfrac{\partial}{\partial X_1} + 0 \\ &+ 2X_3^2 \tfrac{\partial}{\partial X_3} + 4X_3 X_2 \tfrac{\partial}{\partial X_2} + 4X_3 X_1 \tfrac{\partial}{\partial X_1} \\ &= 2X_1(X_1 + 2X_2 + 2X_3) \tfrac{\partial}{\partial X_1} + 2X_2(X_2 + 2X_3) \tfrac{\partial}{\partial X_2} \\ &+ 2X_3^2 \tfrac{\partial}{\partial X_3}. \end{aligned}$$

So we found $D_X^{(2)}$ which is as follows and we can omit 2, because finally we will make the action equal to zero and we can cancel 2 out from both sides. So we have

$$D_X^{(2)} = X_1(X_1 + 2X_2 + 2X_3)\frac{\partial}{\partial X_1} + X_2(X_2 + 2X_3)\frac{\partial}{\partial X_2} + X_3^2\frac{\partial}{\partial X_3} \tag{9}$$

**Second step:**

Now we will try to find $H_X^{(2)}$.

For to find $H_X^{(2)}$, we note that it resembles degree of our polynomial function. So if for example $H_X^{(2)}$ acts on $X_1^n X_2^m X_3^l$, then we should get $(n + m + l)$.

So let us define

$$H_X^{(2)} := \sum_i X_i \frac{\partial}{\partial X_i} \tag{10}$$

and then we have;

$$\begin{aligned} H_X^{(2)}(X_1^n X_2^m X_3^l) &= (\textstyle\sum_i X_i \frac{\partial}{\partial X_i})(X_1^n X_2^m X_3^l) \\ &= \textstyle\sum_i X_i \frac{\partial X_1^n X_2^m X_3^l}{\partial X_i} \\ &= X_1 \frac{\partial X_1^n X_2^m X_3^l}{\partial X_1} + X_2 \frac{\partial X_1^n X_2^m X_3^l}{\partial X_2} + X_3 \frac{\partial X_1^n X_2^m X_3^l}{\partial X_3} \\ &= nX_1^n X_2^m X_3^l + mX_1^n X_2^m X_3^l + lX_1^n X_2^m X_3^l \\ &= (n+m+l)X_1^n X_2^m X_3^l. \end{aligned}$$

Which gives us

$$H_X^{(2)}(X_1^n X_2^m X_3^l) = (n+m+l)X_1^n X_2^m X_3^l$$

and in the other side we have

$$\begin{aligned} (n+m+l)X_1^n X_2^m X_3^l &= nX_1 X_1^{n-1} X_2^m X_3^l + mX_1^n X_2 X_2^{m-1} X_3^l + lX_1^n X_2^m x_3 X_3^{l-1} \\ &= X_1 \frac{X_2^m X_3^l \partial X_1^n}{\partial X_1} + X_2 \frac{X_1^n X_3^l \partial X_2^m}{\partial X_2} + X_3 \frac{X_1^n X_2^m \partial X_3^l}{\partial X_3} \\ &= X_1 \frac{\partial}{\partial X_1} + X_2 \frac{\partial}{\partial X_2} + X_3 \frac{\partial}{\partial X_3} \end{aligned}$$

Which gives us

$$(n+m+l)X_1^n X_2^m X_3^l = \sum_i X_i \frac{\partial}{\partial X_i}$$

And it shows that (2.9) is well defined.

Now the only thing which remains is just to find the solutions of the following system of 2-linear homogeneous equations in one unknown $\tau_1$:

$$\begin{cases} (X_1(X_1 + 2X_2 + 2X_3)\frac{\partial}{\partial X_1} + X_2(X_2 + 2X_3)\frac{\partial}{\partial X_2} + X_3^2\frac{\partial}{\partial X_3})\tau_1[\cdots, X_1, X_2, X_3, \\ \quad \cdots] = 0, \\ (X_1\frac{\partial}{\partial X_1} + X_2\frac{\partial}{\partial X_2} + X_3\frac{\partial}{\partial X_3})\tau_1[\cdots, X_1, X_2, X_3, \cdots] = 0; \end{cases} \tag{11}$$

Now the goal is to find such $\tau_1[\cdots, X_1, X_2, X_3, \cdots]$ which satisfies in our system of equations (10). The second equation ensures that the solution has degree 0 and also the partial differentials will fix us a multi-variable function dependent on just $X_1, X_2, X_3$.

The system of PDEs (10) can be solved using the procedure described in Chapter V, Sec IV of [3]. And after all it became clear that the system (10) has only one functional dependent nontrivial solution:

$$\tau_1^{(2)}[X_1, X_2, X_3] = \frac{(X1 + X2)(X2 + X3)}{X2(X1 + X2 + X3)} = \frac{(\sum_{1 \le i_1 \le 2} X_{i_1}^{(1)})(\sum_{1 \le i_1 \le 2} X_{i_1+1}^{(1)})}{X_2^{(1)}(\sum_{1 \le i_1 \le 3} X_{i_1}^{(1)})} \tag{12}$$

And again as before, (2) goes back to 2 in $Sl_2$ and 1 is a default index which will be used later for to employ shifting operator.

According to the number of variables, we will have two shifts and then everything will be in a loop.

So here in $sl_2$ case we have three solutions for our system of linear equations (2.10) which belong to the fraction ring of polynomial functions.

$$\begin{cases} \tau_1^{(2)}[X_1, X_2, X_3] = \frac{(\sum_{1\le i_1\le 2} X_{i_1}^{(1)})(\sum_{1\le i_1\le 2} X_{i_1+1}^{(1)})}{X_2^{(1)}(\sum_{1\le i_1\le 3} X_{i_1}^{(1)})}; \\ \tau_2^{(2)}[X_2, X_3, X_4] = \frac{(\sum_{2\le i_1\le 3} X_{i_1}^{(1)})(\sum_{2\le i_1\le 3} X_{i_1+1}^{(1)})}{X_2^{(1)}(\sum_{2\le i_1\le 4} X_{i_1}^{(1)})}; \\ \tau_3^{(2)}[X_3, X_4, X_5] = \frac{(\sum_{3\le i_1\le 4} X_{i_1}^{(1)})(\sum_{3\le i_1\le 4} X_{i_1+1}^{(1)})}{X_2^{(1)}(\sum_{3\le i_1\le 5} X_{i_1}^{(1)})}; \end{cases} \tag{13}$$

And as it already has mentioned we go to define our non-commutative Poisson algebra according to definition of Poisson brackets given by Poisson himself [4] with the difference that here we work on $q-$commutative ring $\frac{\mathbb{C}[X_i^{ji}]}{X_i^{ji}X_k^{jk} - q^{<\alpha_i,\alpha_k>}X_k^{jk}X_i^{ji}}$, based on the generators which are the solutions of PDEs system (2.10).
For to do this we will use the following bracket based on

$$\tau_i^{(n)}[\cdots, X_1, X_2, X_3, \cdots] \qquad \text{and} \qquad \tau_j^{(n)}[\cdots, X_1, X_2, X_3, \cdots]$$

So we have to define our Poisson brackets as follows:

$$F_j^{(n)} := \{\tau_i^{(n)}, \tau_j^{(n)}\} = \sum_i \frac{\partial \tau_i^{(n)}}{\partial X_i} \sum_j \frac{\partial \tau_j^{(n)}}{\partial X_j} \{X_i, X_j\} \tag{14}$$

Where $\{X_i, X_j\}$ is our previously defined Poisson bracket on our set of variables.
For instance in the case of $sl_2$ we have

$$\begin{aligned} \{\tau_1^{(2)}, \tau_2^{(2)}\} = & \Big(\frac{\partial \tau_1^{(2)}}{\partial X_1}\Big)\Big(\frac{\partial \tau_2^{(2)}}{\partial X_2}\{X_1, X_2\} + \frac{\partial \tau_2^{(2)}}{\partial X_3}\{X_1, X_3\} + \frac{\partial \tau_2^{(2)}}{\partial X_2}\{X_1, X_4\}\Big) \\ & + \Big(\frac{\partial \tau_1^{(2)}}{\partial X_2}\Big)\Big(\frac{\partial \tau_2^{(2)}}{\partial X_2}\{X_2, X_2\} + \frac{\partial \tau_2^{(2)}}{\partial X_3}\{X_2, X_3\} + \frac{\partial \tau_2^{(2)}}{\partial X_2}\{X_2, X_4\}\Big) \\ & + \Big(\frac{\partial \tau_1^{(2)}}{\partial X_3}\Big)\Big(\frac{\partial \tau_2^{(2)}}{\partial X_2}\{X_3, X_2\} + \frac{\partial \tau_2^{(2)}}{\partial X_3}\{X_3, X_3\} + \frac{\partial \tau_2^{(2)}}{\partial X_2}\{X_3, X_4\}\Big) \\ & = \Big(\frac{\partial \tau_1^{(2)}}{\partial X_1}\Big)\Big(\frac{\partial \tau_2^{(2)}}{\partial X_2}(2X_1X_2) + \frac{\partial \tau_2^{(2)}}{\partial X_3}(2X_1X_3) + \frac{\partial \tau_2^{(2)}}{\partial X_2}(2X_1X_4)\Big) \\ & + \Big(\frac{\partial \tau_1^{(2)}}{\partial X_2}\Big)\Big(\frac{\partial \tau_2^{(2)}}{\partial X_2}(0) + \frac{\partial \tau_2^{(2)}}{\partial X_3}(2X_2X_3) + \frac{\partial \tau_2^{(2)}}{\partial X_2}(2X_2X_4)\Big) \\ & + \Big(\frac{\partial \tau_1^{(2)}}{\partial X_3}\Big)\Big(\frac{\partial \tau_2^{(2)}}{\partial X_2}(-2X_3X_2) + \frac{\partial \tau_2^{(2)}}{\partial X_3}(0) + \frac{\partial \tau_2^{(2)}}{\partial X_2}(2X_3X_4)\Big) \\ & = 2\frac{X_1X_2^2X_3^2X_4(X_1+X_2+X_3+X_4)}{(X_1+X_2)^2(X_2+X_3)^3(X_3+X_4)^2} \end{aligned}$$

So we have

$$F_2^{(2)} = \{\tau_1^{(2)}, \tau_2^{(2)}\} = \frac{2X_1X_2^2X_3^2X_4(X_1+X_2+X_3+X_4)}{(X_1+X_2)^2(X_2+X_3)^3(X_3+X_4)^2} \tag{15}$$

And it is enough to find our brackets on just first generator, because then we are able to find other brackets based on the other generators, so for $\tau_3^{(2)}$ we have in a same process as follows

$$F_3^{(2)} = \{\tau_1^{(2)}, \tau_3^{(2)}\}$$

$$= \Big(\frac{\partial\tau_1^{(2)}}{\partial X_1}\Big)\Big(\frac{\partial\tau_3^{(2)}}{\partial X_3}\{X_1,X_3\} + \frac{\partial\tau_3^{(2)}}{\partial X_4}\{X_1,X_4\} + \frac{\partial\tau_3^{(2)}}{\partial X_5}\{X_1,X_5\}\Big)$$

$$+\Big(\frac{\partial\tau_1^{(2)}}{\partial X_2}\Big)\Big(\frac{\partial\tau_3^{(2)}}{\partial X_3}\{X_2,X_3\} + \frac{\partial\tau_3^{(2)}}{\partial X_4}\{X_2,X_4\} + \frac{\partial\tau_3^{(2)}}{\partial X_5}\{X_2,X_5\}\Big)$$

$$+\Big(\frac{\partial\tau_1^{(2)}}{\partial X_3}\Big)\Big(\frac{\partial\tau_3^{(2)}}{\partial X_3}\{X_3,X_3\} + \frac{\partial\tau_3^{(2)}}{\partial X_4}\{X_3,X_4\} + \frac{\partial\tau_3^{(2)}}{\partial X_5}\{X_3,X_5\}\Big)$$

$$= \Big(\frac{\partial\tau_1^{(2)}}{\partial X_1}\Big)\Big(\frac{\partial\tau_3^{(2)}}{\partial X_3}(2X_1X_3) + \frac{\partial\tau_3^{(2)}}{\partial X_4}(2X_1X_4) + \frac{\partial\tau_3^{(2)}}{\partial X_5}(2X_1X_5)\Big)$$

$$+\Big(\frac{\partial\tau_1^{(2)}}{\partial X_2}\Big)\Big(\frac{\partial\tau_3^{(2)}}{\partial X_3}(2X_2X_3) + \frac{\partial\tau_3^{(2)}}{\partial X_4}(2X_2X_4) + \frac{\partial\tau_3^{(2)}}{\partial X_5}(2X_2X_5)\Big)$$

$$+\Big(\frac{\partial\tau_1^{(2)}}{\partial X_3}\Big)\Big(\frac{\partial\tau_3^{(2)}}{\partial X_3}(0) + \frac{\partial\tau_3^{(2)}}{\partial X_4}(2X_3X_4) + \frac{\partial\tau_3^{(2)}}{\partial X_5}(2X_3X_5)\Big)$$

$$= \frac{-2X_1X_2X_3^2X_4X_5}{(X_1+X_2)(X_2+X_3)^2(X_3+X_4)^2(X_4+X_5)} \tag{16}$$

We have to note that we almost are done with our Poisson algebra in $\mathrm{sl}_2$ case, but for our further plan i.e. to find our Volterra system, the differential-difference chain of non-linear equations

$$\begin{cases} H = \sum_i [\ln(\tau_i)]; \\ \dot{\tau_j} = \{\tau_j, H\} = \tau_j \times \sum_i \Gamma_i; \end{cases} \tag{17}$$

Where $\Gamma_i$ stands for $\frac{\tau_1,\tau_i}{\tau_1\tau_i}$ [2]. Which means that we have to write down the brackets $\{\tau_1,\tau_i\}$ in terms of their decompositions to $\tau_j$'s for $1 \le j \le i$.
So we need to write it as decomposition of our generators and it will be done by using the Mathematica coding which we have produced in Appendix C.
And the result will be as follows

$$\begin{cases} F_2^{(2)} = \{\tau_1^{(2)},\tau_2^{(2)}\} = 2(1-\tau_1^{(2)})(1-\tau_2^{(2)})(-1+\tau_1^{(2)}+\tau_2^{(2)}); \\ F_3^{(2)} = \{\tau_1^{(2)},\tau_3^{(2)}\} = -2(1-\tau_1^{(2)})(1-\tau_2^{(2)})(1-\tau_3^{(2)}); \\ F_i^{(2)} = \{\tau_1^{(2)},\tau_i^{(2)}\} = 0 \qquad \text{for } |i-1| \ge 3; \end{cases} \tag{18}$$

This result are weaker than Faddeev-Takhtajan-Volkov algebra which has been mentioned in [2] and if we continue this for $sl_3$, then we will have again a weaker version of what which has been mentioned in [2].

## 2.2. Lattice $W_3$ algebra

In this case we will use the following defined Poisson bracket based on Cartan matrix $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, but for to do this according to our previous ordering and list of variables, let us for simplicity set our variables as follows
Set $X_i^{(1i)} := X_i$ and $X_i^{(2i)} := Y_i$.

DEFINITION . *Let's define our Poisson bracket as follows in the case of $sl_3$:*

$$\begin{cases} \{X_i, X_j\} := 2X_iX_j & \text{if } i < j; \\ \{Y_i, Y_j\} := 2Y_iY_j & \text{if } i < j; \\ \{X_i, X_i\} := 0; \\ \{Y_i, Y_i\} := 0; \\ \{X_i, Y_j\} := X_iY_j & \text{if } i > j; \\ \{X_i, Y_j\} := -X_iY_j & \text{if } i \le j; \end{cases} \tag{19}$$

And instead of (2.1) we will have the following $q-$commutation relations

$$\begin{cases} X_iX_j = q^2X_jX_i & \text{if } i \le j; \\ Y_iY_j = q^2Y_jY_i & \text{if } i \le j; \\ X_iY_j = q^{-1}Y_jX_i & \text{if } i \le j; \end{cases} \tag{20}$$

And we will get the following equations in a same manner as in $\mathrm{sl}_2$:

First case: $i < j$;

$$\begin{aligned} \{X_i, Y_j^n\} &= X_iY_j^n - q^{-n}Y_j^nX_i \\ &= X_iY_j^n - q^0X_iX_j^n \\ &= 0 \end{aligned}$$

Second case: $i \ge j$;

$$\begin{aligned} \{X_i, Y_j^n\} &= X_iY_j^n - q^{-n}Y_j^nX_i \\ &= (1 - q^{-2n})X_iY_j^n \\ &= (1 - e^{2n\mathfrak{h}})X_iY_j^n \\ &\sim (1 - (1 + 2n\mathfrak{h}))X_iY_j^n \\ &= -2n\mathfrak{h}X_iY_j^n \\ &\sim -2nX_iY_j^n \\ &= -2X_iY_j\frac{\partial Y_j^n}{\partial Y_j} \end{aligned}$$

$$\begin{cases} \{X_i, X_j^n\} = 0 & \text{if } i \le j; \\ \{X_i, X_j^n\} = 4X_iX_j\frac{\partial X_j^n}{\partial X_j} & \text{if } i > j; \\ \{X_i, Y_j^n\} = 0 & \text{if } i < j; \\ \{X_i, Y_j^n\} = -2X_iY_j\frac{\partial Y_j^n}{\partial Y_j} & \text{if } i \ge j; \\ \{Y_j, X_i^n\} = -2Y_jX_i\frac{\partial X_i^n}{\partial X_i} & \text{if } i \le j; \end{cases} \tag{21}$$

According to (2.20) we will try to find $H_X^{(3)}$ as follows

$$\begin{aligned} &\{X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}Y_1^{\beta_1}Y_2^{\beta_2}Y_3^{\beta_3}, X_0\} \\ &= X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}Y_1^{\beta_1}Y_2^{\beta_2}Y_3^{\beta_3}X_0 - X_0X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}Y_1^{\beta_1}Y_2^{\beta_2}Y_3^{\beta_3} \\ &= (1 - q^{2\alpha_1+2\alpha_2+2\alpha_3-\beta_1-\beta_2-\beta_3})X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}Y_1^{\beta_1}Y_2^{\beta_2}Y_3^{\beta_3}X_0 \\ &\sim (1 - (1 - n\mathfrak{h}(2\alpha_1 + 2\alpha_2 + 2\alpha_3 - \beta_1 - \beta_2 - \beta_3)))X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}Y_1^{\beta_1}Y_2^{\beta_2}Y_3^{\beta_3}X_0 \\ &= (2\alpha_1 + 2\alpha_2 + 2\alpha_3 - \beta_1 - \beta_2 - \beta_3)n\mathfrak{h}X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}Y_1^{\beta_1}Y_2^{\beta_2}Y_3^{\beta_3}X_0 \\ &\sim (2\alpha_1 + 2\alpha_2 + 2\alpha_3 - \beta_1 - \beta_2 - \beta_3)nX_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}Y_1^{\beta_1}Y_2^{\beta_2}Y_3^{\beta_3}X_0 \\ &= (2X_1\frac{\partial}{\partial X_1} + 2X_2\frac{\partial}{\partial X_2} + 2X_3\frac{\partial}{\partial X_3} - Y_1\frac{\partial}{\partial Y_1} - Y_2\frac{\partial}{\partial Y_2} - Y_3\frac{\partial}{\partial Y_3})\tau_1^{(3)}. \end{aligned}$$

Now let us as usual suppose $i > j$ and then we will define the following quantities Here we have for $X_i$s:

$$\begin{aligned} {}_{X_j}D_{X_i} &:= \{X_i, X_j^n\} \\ &= X_iX_j^n - q^{2n}X_j^nX_i \end{aligned}$$

$$= (1-q^{4n})X_iX_j^n$$
$$= (1-e^{-4n\mathfrak{h}})X_iX_j^n$$
$$\sim (1-(1-4n\mathfrak{h}))X_iX_j^n$$
$$= 4n\mathfrak{h}X_iX_j^n$$
$$\sim 4nX_iX_j^n$$
$$= 4nX_iX_j\tfrac{\partial X_j^n}{\partial X_j}.$$

And the same will be for $Y_i$s.
And for the different quantities $X_i$ and $Y_j$s we have:

First case: for $i > j$;

$$
\begin{aligned}
{}_{Y_j}D_{X_i} &:= \{X_i, Y_j^n\}\\
&= X_iY_j^n - q^{-n}Y_j^nX_i\\
&= (1-q^{-2n})X_iY_j^n\\
&= (1-e^{-2n\mathfrak{h}})X_iY_j^n\\
&\sim (1-(1-2n\mathfrak{h}))X_iY_j^n\\
&= 2n\mathfrak{h}X_iY_j^n\\
&\sim 2nX_iY_j^n\\
&= 2X_iY_j\tfrac{\partial Y_j^n}{\partial Y_j}.
\end{aligned}
$$

Second case: for $i \le j$;

According to what has just mentioned we have

$$
\begin{aligned}
{}_{Y_j}D_1^Y &:= {}_{Y_j}D_{Y_1}\\
&= 4Y_1Y_j\tfrac{\partial Y_j^n}{\partial Y_j}.
\end{aligned}
$$

And

$$
\begin{aligned}
{}_{Y_1}D_1^Y &:= {}_{Y_1}D_{Y_1}\\
&= 2Y_1^2\tfrac{\partial Y_1^n}{\partial Y_1}.
\end{aligned}
$$

And in a same way we can find the desired results for ${}_{Y_j}D_2^Y$ and ${}_{Y_j}D_3^Y$,
So let us define

$$
\begin{cases}
{}_YD_1^Y := {}_{Y_1}D_1^Y + {}_{Y_j}^{j<1}D_1^Y + {}_{Y_j}^{j>1}D_1^Y;\\
{}_YD_2^Y := {}_{Y_2}D_2^Y + {}_{Y_j}^{j<2}D_2^Y + {}_{Y_j}^{j>2}D_2^Y;\\
{}_YD_3^Y := {}_{Y_3}D_3^Y + {}_{Y_j}^{j<3}D_3^Y + {}_{Y_j}^{j>3}D_3^Y;
\end{cases}
\tag{22}
$$

And then we will have

$${}_YD_1^Y = Y_1^2\frac{\partial}{\partial Y_1} + \sum_{j<1} 2Y_1Y_j\frac{\partial}{\partial Y_j} + 0$$

And

$${}_YD_2^Y = Y_2^2\frac{\partial}{\partial Y_2} + \sum_{j<2} 2Y_2Y_j\frac{\partial}{\partial Y_j} + 0$$

And

$${}_YD_3^Y = Y_3^2\frac{\partial}{\partial Y_3} + \sum_{j<2} 2Y_3Y_j\frac{\partial}{\partial Y_j} + 0$$

And finally we get

$$
\begin{aligned}
{}_YD_Y^{(3)} &:= {}_YD_1 + {}_YD_2 + {}_YD_3\\
&= Y_1(Y_1+2Y_2+2Y_3)\tfrac{\partial}{\partial Y_1} + Y_2(Y_2+2Y_3)\tfrac{\partial}{\partial Y_2} + Y_3^2\tfrac{\partial}{\partial Y_3}.
\end{aligned}
$$

For $j \ge 1$ we have

$${}_{X_1}D_{Y_j} := \{Y_j, X_1^n\}$$

$$\begin{aligned}
&= Y_jX_1^n - q^{-n}X_1^nY_j\\
&= (1-q^{-2n})Y_jX_1^n\\
&= (1-e^{2n\mathfrak{h}})Y_jX_1^n\\
&\sim (1-(1+2n\mathfrak{h}))Y_jX_1^n\\
&= -2n\mathfrak{h}Y_jX_1^n\\
&\sim -2nY_jX_1^n\\
&= -2Y_jX_1\tfrac{\partial}{\partial X_1}\\
&= -2X_1(Y_1+Y_2+Y_3)\tfrac{\partial}{\partial X_1}.
\end{aligned}$$

For $j \geq 2$ we have

$$\begin{aligned}
{}_{X_2}D_{Y_j} &:= \{Y_j, X_2^n\}\\
&= Y_jX_2^n - q^{-n}X_2^nY_j\\
&= (1-q^{-2n})Y_jX_2^n\\
&= (1-e^{2n\mathfrak{h}})Y_jX_2^n\\
&\sim (1-(1+2n\mathfrak{h}))Y_jX_2^n\\
&= -2n\mathfrak{h}Y_jX_2^n\\
&\sim -2nY_jX_2^n\\
&= -2Y_jX_2\tfrac{\partial}{\partial X_2}\\
&= -2X_2(Y_2+Y_3)\tfrac{\partial}{\partial X_2}.
\end{aligned}$$

For $j \geq 3$ we have

$$\begin{aligned}
{}_{X_3}D_{Y_j} &:= \{Y_j, X_3^n\}\\
&= Y_jX_3^n - q^{-n}X_3^nY_j\\
&= (1-q^{-2n})Y_jX_3^n\\
&= (1-e^{2n\mathfrak{h}})Y_jX_3^n\\
&\sim (1-(1+2n\mathfrak{h}))Y_jX_3^n\\
&= -2n\mathfrak{h}Y_jX_3^n\\
&\sim -2nY_jX_3^n\\
&= -2Y_jX_3\tfrac{\partial}{\partial X_3}\\
&= -2X_3Y_3\tfrac{\partial}{\partial X_3}.
\end{aligned}$$

And after all these, let us define

$$\begin{aligned}
{}_XD_Y^{(3)} &:= {}_{X_1}D_{Y_j} + {}_{X_2}D_{Y_j} + {}_{X_3}D_{Y_j}\\
&= -2X_1(Y_1+Y_2+Y_3)\tfrac{\partial}{\partial X_1} - 2X_2(Y_2+Y_3)\tfrac{\partial}{\partial X_2}\\
&\quad -2X_3Y_3\tfrac{\partial}{\partial X_3}.
\end{aligned}$$

And finally let us define

$$\begin{aligned}
D_Y^{(3)} &:= {}_YD_Y^{(3)} + {}_XD_Y^{(3)}\\
&= Y_1(Y_1+2Y_2+2Y_3)\tfrac{\partial}{\partial Y_1} + Y_2(Y_2+2Y_3)\tfrac{\partial}{\partial Y_2}\\
&\quad +Y_3^2\tfrac{\partial}{\partial Y_3} - 2X_1(Y_1+Y_2+Y_3)\tfrac{\partial}{\partial X_1} - 2X_2(Y_2\\
&\quad +Y_3)\tfrac{\partial}{\partial X_2} - 2X_3Y_3\tfrac{\partial}{\partial X_3}.
\end{aligned}$$

**Next step:**

Now let us try to find $D_X^{(3)}$:

For $i > 1$, let us define ${}_{Y_1}D_{X_i}$ as follows:

$$\begin{aligned}
{}_{Y_1}D_{X_i} &:= \{X_i, Y_1^n\}\\
&= X_iY_1^n - q^{-n}Y_1^nX_i\\
&= (1-q^{-2n})X_iY_1^n\\
&= (1-e^{2n\mathfrak{h}})X_iY_1^n\\
&\sim (1-(1+2n\mathfrak{h}))X_iY_1^n\\
&= -2n\mathfrak{h}X_iY_1^n\\
&\sim -2nX_iY_1^n = -2X_iY_1\tfrac{\partial}{\partial Y_1}\\
&= -2Y_1(X_2+X_3)\tfrac{\partial}{\partial Y_1}.
\end{aligned}$$

For $i > 2$ we have

$$\begin{aligned}
{}_{Y_2}D_{X_i} &:= \{X_i, Y_2^n\} \\
&= X_i Y_2^n - q^{-n} Y_2^n X_i \\
&= (1 - q^{-2n}) X_i Y_2^n \\
&= (1 - e^{2n\mathfrak{h}}) X_i Y_2^n \\
&\sim (1 - (1 + 2n\mathfrak{h})) X_i Y_2^n \\
&= -2n\mathfrak{h} X_i Y_2^n \\
&\sim -2n X_i Y_2^n \\
&= -2 X_i Y_2 \tfrac{\partial}{\partial Y_2} \\
&= -2 Y_2 X_3 \tfrac{\partial}{\partial Y_2}.
\end{aligned}$$

For $i > 3$ we have 0.

Let us again have the following definitions

$$_{Y_1}D_2^X :=_{Y_1} D_{X_2} = -2Y_1 X_2 \frac{\partial Y_1^n}{\partial Y_1};$$

$$_{Y_1}D_3^X :=_{Y_1} D_{X_3} = -2Y_1 X_3 \frac{\partial Y_1^n}{\partial Y_1};$$

$$_{Y_2}D_3^X :=_{Y_2} D_{X_3} = -2Y_2 X_3 \frac{\partial Y_1^n}{\partial Y_2};$$

Now let us define

$$_X D_Y^{(3)} :=_{Y_1} D_2^X +_{Y_1} D_3^X +_{Y_2} D_3^X = -Y_1(X_2 + X_3)\frac{\partial}{\partial Y_1} - Y_2 X_3 \frac{\partial}{\partial Y_2};$$

And now as before we have

$$\begin{aligned}
{}_{X_j}D_1^X &:=_{X_j} D_{X_1} \\
&= 4X_1 X_j \tfrac{\partial X_j^n}{\partial X_j}. \\
{}_{X_1}D_1^X &:=_{X_1} D_{X_1} \\
&= 2X_1^2 \tfrac{\partial X_1^n}{\partial X_1}.
\end{aligned}$$

And in a same way we are able to define for $_{X_j}D_2^X$ and $_{X_j}D_3^X$. So let us define

$$\begin{cases}
_X D_1^X :=_{X_1} D_1^X +_{X_j}^{j<1} D_1^X +_{X_j}^{j>1} D_1^X; \\
_X D_2^X :=_{X_2} D_2^X +_{X_j}^{j<2} D_2^X +_{X_j}^{j>2} D_2^X; \\
_X D_3^X :=_{X_3} D_3^X +_{X_j}^{j<3} D_3^X +_{X_j}^{j>3} D_3^X;
\end{cases} \tag{23}$$

Then we will have

$$_X D_1^X = X_1^2 \frac{\partial}{\partial X_1} + \sum_{j<1} 2X_1 X_j \frac{\partial}{\partial X_j} + 0$$

And

$$_X D_2^X = X_2^2 \frac{\partial}{\partial X_2} + \sum_{j<2} 2X_2 X_j \frac{\partial}{\partial X_j} + 0$$

And

$$_X D_3^X = X_3^2 \frac{\partial}{\partial X_3} + \sum_{j<2} 2X_3 Y_j \frac{\partial}{\partial X_j} + 0$$

So we will have

$$_X D_X^{(3)} :=_X D_1 +_X D_2 +_X D_3$$

$$= X_1(X_1 + 2X_2 + 2X_3)\frac{\partial}{\partial X_1} + X_2(X_2 + 2X_3)\frac{\partial}{\partial X_2} + X_3^2\frac{\partial}{\partial X_3}.$$

And therefore as in (10) we will have the following system of $PDE$s

$$\begin{cases} (X_1(X_1 + 2X_2 + 2X_3)\frac{\partial \tau_1^{(3)}}{\partial X_1} + X_2(X_2 + 2X_3)\frac{\partial \tau_1^{(3)}}{\partial X_2} + X_3^2\frac{\partial \tau_1^{(3)}}{\partial X_2} \\ -Y_1(X_1 + X_2 + X_3)\frac{\partial \tau_1^{(3)}}{\partial Y_1} - Y_2(X_2 + X_3)\frac{\partial f}{\partial Y_2} - Y_3X_3\frac{\partial \tau_1^{(3)}}{\partial Y_3}) = 0; \\ (2X_1\frac{\partial \tau_1^{(3)}}{\partial X_1} + 2X_2\frac{\partial \tau_1^{(3)}}{\partial X_2} + 2X_3\frac{\partial \tau_1^{(3)}}{\partial X_3} - Y_1\frac{\partial \tau_1^{(3)}}{\partial Y_1} - Y_2\frac{\partial \tau_1^{(3)}}{\partial Y_2} - Y_3\frac{\partial \tau_1^{(3)}}{\partial Y_3}) = 0; \\ D_Y^{(3)} = (Y_1(Y_1 + 2Y_2 + 2Y_3)\frac{\partial \tau_1^{(3)}}{\partial Y_1} + Y_2(Y_2 + 2Y_3)\frac{\partial \tau_1^{(3)}}{\partial Y_2} + Y_3^2\frac{\partial \tau_1^{(3)}}{\partial Y_2} \\ -Y_1(X_1 + X_2 + X_3)\frac{\partial \tau_1^{(3)}}{\partial Y_1} - Y_2(X_2 + X_3)\frac{\partial \tau_1^{(3)}}{\partial Y_2} - Y_3X_3\frac{\partial \tau_1^{(3)}}{\partial Y_3}) = 0; \\ (2X_1\frac{\partial \tau_1^{(3)}}{\partial X_1} + 2X_2\frac{\partial \tau_1^{(3)}}{\partial X_2} + 2X_3\frac{\partial \tau_1^{(3)}}{\partial X_3} - Y_1\frac{\partial \tau_1^{(3)}}{\partial Y_1} - Y_2\frac{\partial \tau_1^{(3)}}{\partial Y_2} - Y_3\frac{\partial \tau_1^{(3)}}{\partial Y_3}) = 0; \end{cases} \tag{24}$$

And according to appendix A we have the following functional dependent nontrivial solution for the whole system of $PDE$s (23)

$$\tau_1^{(3)} = \frac{(\Sigma_{1\le i\le j\le 2} \quad X_iY_j)(\Sigma_{1\le i\le j\le 2} \quad X_{i+1}Y_{j+1})}{X_2Y_2(\Sigma_{1\le i\le j\le 3} \quad X_iY_j)}; \tag{25}$$

And again as before, (3) goes back to 3 in the $Sl_3$ and 1 is a default index which later we will use it for to employ our shifting operators.
According to the number of variables, we will have 6 shifts and then after that it will be in a loop. So here in $sl_3$ case we have six solutions which belong to the fraction ring of polynomial functions.

$$\begin{cases} \tau_1^{(3)}[X_1, Y_1, X_2, Y_2, X_3, Y_3] = \frac{X_2Y_2(X_3Y_3+X_2(Y_2+Y_3)+X_1(Y_1+Y_2+Y_3))}{(X_2Y_2+X_1(Y_1+Y_2))(X_3Y_3+X_2(Y_2+Y_3))}; \\ \tau_2^{(3)}[Y_1, X_2, Y_2, X_3, Y_3, X_4] = \frac{X_3Y_2(X_2Y_1+(X_3+X_4)(Y_1+Y_2)+X_4Y_3)}{(X_2Y_1+X_3(Y_1+Y_2))(X_3Y_2+X_4(Y_2+Y_3))}; \\ \tau_3^{(3)}[X_2, Y_2, X_3, Y_3, X_4, Y_4] = \frac{X_3y_3(X_4Y_4+X_3(Y_3+Y_4)+X_2(Y_2+Y_3+Y_4))}{(X_3Y_3+X_2(Y_2+Y_3))(X_4Y_4+X_3(Y_3+Y_4))}; \\ \tau_4^{(3)}[Y_2, X_3, Y_3, X_4, Y_4, X_5] = \frac{X_4Y_3(X_3Y_2+(X_4+X_5)(Y_2+Y_3)+X_5Y_4)}{(X_3Y_2+X_4(Y_2+Y_3))(X_4Y_3+X_5(Y_3+Y_4))}; \\ \tau_5^{(3)}[X_3, Y_3, X_4, Y_4, X_5, Y_5] = \frac{X_4Y_4(X_5Y_5+X_4(Y_4+Y_5)+X_3(Y_3+Y_4+Y_5))}{(X_4Y_4+X_3(Y_3+Y_4))(X_5Y_5+X_4(Y_4+Y_5))}; \\ \tau_6^{(3)}[Y_3, X_4, Y_4, X_5, Y_5, X_6] = \frac{X_5Y_4(X_4Y_3+(X_5+X_6)(Y_3+Y_4)+X_6Y_5)}{(X_4Y_3+X_5(Y_3+Y_4))(X_5Y_4+X_6(Y_4+Y_5))}; \end{cases} \tag{26}$$

Where $\tau_1^{(3)} := \tau_1^{(3)}[\cdots, X_1, Y_1, X_2, Y_2, X_3, Y_3\cdots]$.
Again by setting $X_i^{(1i)} := X_i$ and $X_i^{(2i)} := Y_i$ and $X_i^{(3i)} := Z_i$ and according to (2.4) we have to write down the following brackets as a composition of $\tau_i^{(3)}$s, because of algebra structure and it will be done by using Mathematica coding in appendix A.

$$\begin{cases} F_2^{(3)} = \{\tau_1^{(3)}, \tau_2^{(3)}\} = -(1-\tau_1^{(3)})(1-\tau_2^{(3)})(\tau_1^{(3)}\tau_2^{(3)}); \\ F_3^{(3)} = \{\tau_1^{(3)}, \tau_3^{(3)}\} = (1-\tau_1^{(3)})(1-\tau_3^{(3)})(\tau_1^{(3)}\tau_2^{(3)} + \tau_2^{(3)}\tau_3^{(3)} - \tau_2^{(3)}); \\ F_4^{(3)} = \{\tau_1^{(3)}, \tau_4^{(3)}\} = -(1-\tau_1^{(3)})(1-\tau_4^{(3)}) \\ \qquad (\tau_1^{(3)}\tau_2^{(3)} + \tau_2^{(3)}\tau_3^{(3)} + \tau_3^{(3)}\tau_4^{(3)} - \tau_1^{(3)} - \tau_2^{(3)} - \tau_3^{(3)} - \tau_4^{(3)} + 1); \\ F_5^{(3)} = \{\tau_1^{(3)}, \tau_5^{(3)}\} = (1-\tau_1^{(3)})(1-\tau_5^{(3)})(\tau_2^{(3)}\tau_3^{(3)} + \tau_3^{(3)}\tau_4^{(3)} - \tau_2^{(3)} \\ \qquad -\tau_3^{(3)} - \tau_4^{(3)} + 1); \\ F_6^{(3)} = \{\tau_1^{(3)}, \tau_6^{(3)}\} = -(1-\tau_1^{(3)})(1-\tau_6^{(3)})(\tau_3^{(3)}\tau_4^{(3)} - \tau_4^{(3)} - \tau_3^{(3)} + 1); \\ F_i^{(3)} = \{\tau_1^{(3)}, \tau_i^{(3)}\} = 0 \qquad\qquad \text{for } |i-1| \ge 6; \end{cases} \tag{27}$$

### 2.3. Lattice $W_4$ algebra; main generator

In this case we will use the following defined Poisson bracket based on Cartan matrix

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

But for to do this according to our previous ordering and list of variables, and the same as what we din in $sl_3$ case, let us for simplicity set our set of variables as follows:
Set $X_i^{(1i)} := X_i$ and $X_i^{(2i)} := Y_i$ and $X_i^{(3i)} := Z_i$ and so on.

Definition . *Let's define our Poisson bracket as follows in the case of $sl_3$:*

$$\begin{cases} \{X_i, X_j\} := 2X_iX_j & \textit{if } i < j; \\ \{Y_i, Y_j\} := 2Y_iY_j & \textit{if } i < j; \\ \{Z_i, Z_j\} := 2Z_iZ_j & \textit{if } i < j; \\ \{X_i, X_i\} := 0; \\ \{Y_i, Y_i\} := 0; \\ \{Z_i, Z_i\} := 0; \\ \{X_i, Y_j\} := X_iY_j & \textit{if } i > j; \\ \{X_i, Y_j\} := -X_iY_j & \textit{if } i \le j; \\ \{X_i, Z_j\} := 0; \\ \{Y_i, Z_j\} := Y_iZ_j & \textit{if } i > j; \\ \{Y_i, Z_j\} := -Y_iZ_j & \textit{if } i \le j; \end{cases} \tag{28}$$

And instead of (1) we will have the following $q-$commutation relations for $j \in \{1, 2, 3\}$ and as always $i \in \{1, 2, 3\}$:

$$\begin{cases} X_iX_j = q^2 X_jX_i & \text{if } i \le j \\ Y_iY_j = q^2 Y_jY_i & \text{if } i \le j \\ Z_iZ_j = q^2 Z_jZ_i & \text{if } i \le j \\ X_iY_j = q^{-1} Y_jX_i & \text{if } i \le j \\ Y_iZ_j = q^{-1} Z_jY_i & \text{if } i \le j \\ X_iZ_j = Z_jX_i \end{cases} \tag{29}$$

And by using the same approach as what we did for $sl_2$ and $sl_3$, it became clear that the equations $D_X^{(4)}$, $D_Y^{(4)}$ and $D_Z^{(4)}$ and also $H_X^{(4)}$, $H_Y^{(4)}$ and $H_Z^{(4)}$ will have the following forms:

$$\mathfrak{D}_X^{(4)} = X_1(X_1 + 2X_2 + 2X_3)\frac{\partial \tau_1^{(4)}}{\partial X_1} + X_2(X_2 + 2X_3)\frac{\partial \tau_1^{(4)}}{\partial X_2} + X_3^2 \frac{\partial \tau_1^{(4)}}{\partial X_3} - Y_1(X_2 + X_3) \tag{30}$$

$$\frac{\partial \tau_1^{(4)}}{\partial Y_1} - Y_2X_3\frac{\partial \tau_1^{(4)}}{\partial Y_2};$$

$$\mathfrak{D}_Y^{(4)} = Y_1(Y_1 + 2Y_2 + 2Y_3)\frac{\partial \tau_1^{(4)}}{\partial Y_1} + Y_2(Y_2 + 2Y_3)\frac{\partial \tau_1^{(4)}}{\partial Y_2} + Y_3^2 \frac{\partial \tau_1^{(4)}}{\partial Y_3} - X_1(Y_1 + Y_2 + Y_3) \tag{31}$$

$$\frac{\partial \tau_1^{(4)}}{\partial X_1} - X_2(Y_2 + Y_3)\frac{\partial \tau_1^{(4)}}{\partial X_2} - X_3Y_3\frac{\partial \tau_1^{(4)}}{\partial X_3} - Z_1(Y_2 + Y_3)\frac{\partial \tau_1^{(4)}}{\partial z_1} - Z_2y_3\frac{\partial \tau_1^{(4)}}{\partial Z_2};$$

$$\mathfrak{D}_Z^{(4)} = Z_1(Z_1 + 2Z_2 + 2Z_3)\frac{\partial \tau_1^{(4)}}{\partial Z_1} + Z_2(Z_2 + 2Z_3)\frac{\partial \tau_1^{(4)}}{\partial Z_2} + Z_3^2 \frac{\partial \tau_1^{(4)}}{\partial Z_3} - Y_1(Z_1 + Z_2 + Z_3) \tag{32}$$

$$\frac{\partial\tau_1^{(4)}}{\partial Y_1} - Y_2(Z_2+Z_3)\frac{\partial\tau_1^{(4)}}{\partial Y_2} - Y_3Z_3\frac{\partial\tau_1^{(4)}}{\partial Y_3};$$

$$H_X^{(4)} = 2X_1\frac{\partial\tau_1^{(4)}}{\partial X_1} + 2X_2\frac{\partial\tau_1^{(4)}}{\partial X_2} + 2X_3\frac{\partial\tau_1^{(4)}}{\partial X_3} - Y_1\frac{\partial\tau_1^{(4)}}{\partial Y_1} - Y_2\frac{\partial\tau_1^{(4)}}{\partial Y_2} - Y_3\frac{\partial\tau_1^{(4)}}{\partial Y_3}; \quad (33)$$

$$H_Y^{(4)} = 2Y_1\frac{\partial\tau_1^{(4)}}{\partial Y_1} + 2Y_2\frac{\partial\tau_1^{(4)}}{\partial Y_2} + 2Y_3\frac{\partial\tau_1^{(4)}}{\partial Y_3} - X_1\frac{\partial\tau_1^{(4)}}{\partial X_1} - X_2\frac{\partial\tau_1^{(4)}}{\partial X_2} - X_3\frac{\partial\tau_1^{(4)}}{\partial X_3} - z_1\frac{\partial\tau_1^{(4)}}{\partial Z_1} \quad (34)$$

$$-Z_2\frac{\partial\tau_1^{(4)}}{\partial Z_2} - Z_3\frac{\partial\tau_1^{(4)}}{\partial Z_3};$$

$$H_Z^{(4)} = 2Z_1\frac{\partial\tau_1^{(4)}}{\partial Z_1} + 2Z_2\frac{\partial\tau_1^{(4)}}{\partial Z_2} + 2Z_3\frac{\partial\tau_1^{(4)}}{\partial Z_3} - Y_1\frac{\partial\tau_1^{(4)}}{\partial Y_1} - Y_2\frac{\partial\tau_1^{(4)}}{\partial Y_2} - Y_3\frac{\partial\tau_1^{(4)}}{\partial Y_3}; \quad (35)$$

And the functional dependent nontrivial solutions for the whole system of first order partial differential equation is as follows:

$$\tau_1^{(4)} = \frac{(\Sigma_{1\le i\le j\le m\le 2} \quad x_iy_jz_m)(\Sigma_{1\le i\le j\le m\le 2} \quad x_{i+1}y_{j+1}z_{m+1})}{x_2y_2z_2(\Sigma_{1\le i\le j\le m\le 3} \quad x_iy_jz_m)}; \quad (36)$$

And again as before, (4) goes back to 4 in the $Sl_4$ and 1 is a default index which later we will use it for to employ our shifting operators.
According to the number of variables, we will have 9 shifts and then after that it will be in a loop. So here in $sl_4$ case we have nine solutions:
$\tau_1^{(4)} := \tau_1^{(4)}[X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3, Z_3];$
$\tau_2^{(4)} := \tau_1^{(4)}[X_1 \to Y_1, Y_1 \to Z_1, Z_1 \to X_2, X_2 \to Y_2, Y_2 \to Z_2, Z_2 \to X_3, X_3 \to Y_3, Y_3 \to Z_3];$
$\tau_3^{(4)} := \tau_2^{(4)}[Y_1 \to Z_1, Z_1 \to X_2, X_2 \to Y_2, Y_2 \to Z_2, Z_2 \to X_3, X_3 \to Y_3, Y_3 \to Z_3, Z_3 \to X_4];$
$\tau_4^{(4)} := \tau_3^{(4)}[Z_1 \to X_2, X_2 \to Y_2, Y_2 \to Z_2, Z_2 \to X_3, X_3 \to Y_3, Y_3 \to Z_3, Z_3 \to X_4, X_4 \to Y_4];$
$\tau_5^{(4)} := \tau_4^{(4)}[X_2 \to Y_2, Y_2 \to Z_2, Z_2 \to X_3, X_3 \to Y_3, Y_3 \to Z_3, Z_3 \to X_4, X_4 \to Y_4, Y_4 \to Z_4];$
$\tau_6^{(4)} := \tau_5^{(4)}[Y_2 \to Z_2, Z_2 \to X_3, X_3 \to Y_3, Y_3 \to Z_3, Z_3 \to X_4, X_4 \to Y_4, Y_4 \to Z_4, Z_4 \to X_5];$
$\tau_7^{(4)} := \tau_6^{(4)}[Z_2 \to X_3, X_3 \to Y_3, Y_3 \to Z_3, Z_3 \to X_4, X_4 \to Y_4, Y_4 \to Z_4, Z_4 \to X_5, X_5 \to Y_5];$
$\tau_8^{(4)} := \tau_7^{(4)}[X_3 \to Y_3, Y_3 \to Z_3, Z_3 \to X_4, X_4 \to Y_4, Y_4 \to Z_4, Z_4 \to X_5, X_5 \to Y_5, Y_5 \to Z_5];$
$\tau_9^{(4)} := \tau_8^{(4)}[Y_3 \to Z_3, Z_3 \to X_4, X_4 \to Y_4, Y_4 \to Z_4, Z_4 \to X_5, X_5 \to Y_5, Y_5 \to Z_5, Z_5 \to X_6];$
which belong to the fraction ring of polynomial functions.

## 2.4. Lattice $W_5$ algebra; main generator

In this case we will use the following defined Poisson bracket based on Cartan matrix

$$A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

But for to do this according to our previous ordering and list of variables, and the same as what we din in $sl_4$ case, let us for simplicity set our set of variables as follows:
Set $X_i^{(1i)} := X_i$ and $X_i^{(2i)} := Y_i$ and $X_i^{(3i)} := Z_i$ and $X_i^{(4i)} := K_i$ and so on.

Definition . *Let's define our Poisson bracket as follows in the case of $sl_4$:*

$$\begin{cases} \{X_i,X_j\} := 2X_iX_j & \textit{if } i<j; \\ \{Y_i,Y_j\} := 2Y_iY_j & \textit{if } i<j; \\ \{Z_i,Z_j\} := 2Z_iZ_j & \textit{if } i<j; \\ \{K_i,K_j\} := 2K_iK_j & \textit{if } i<j; \\ \{X_i,X_i\} := 0; \\ \{Y_i,Y_i\} := 0; \\ \{Z_i,Z_i\} := 0; \\ \{K_i,K_i\} := 0; \\ \{X_i,Y_j\} := X_iY_j & \textit{if } i>j; \\ \{X_i,Y_j\} := -X_iY_j & \textit{if } i\le j; \\ \{X_i,Z_j\} := 0; \\ \{X_i,K_j\} := 0; \\ \{Y_i,Z_j\} := Y_iZ_j & \textit{if } i>j; \\ \{Y_i,Z_j\} := -Y_iZ_j & \textit{if } i\le j; \\ \{Y_i,K_j\} := Y_iK_j & \textit{if } i>j; \\ \{Y_i,K_j\} := -Y_iK_j & \textit{if } i\le j; \end{cases} \tag{37}$$

And instead of (2.1) we will have the following $q-$commutation relations for $j\in\{1,2,3\}$ and as always $i\in\{1,2,3\}$:

$$\begin{cases} x_ix_j = q^2x_jx_i & \text{if } i\le j \\ y_iy_j = q^2y_jy_i & \text{if } i\le j \\ z_iz_j = q^2z_jz_i & \text{if } i\le j \\ k_ik_j = q^2k_jk_i & \text{if } i\le j \\ x_iy_j = q^{-1}y_jx_i & \text{if } i\le j \\ y_iz_j = q^{-1}z_jy_i & \text{if } i\le j \\ z_ik_j = q^{-1}k_jz_i & \text{if } i\le j \\ x_iz_j = z_jx_i \\ y_ik_j = k_jy_i \\ x_ik_j = k_jx_i \end{cases}$$

And by using the same approach as what we did for $sl_2$ and $sl_3$ and $sl_4$ , it became clear that the equations $D_X^{(5)}$, $D_Y^{(5)}$, $D_Z^{(5)}$ and $D_K^{(5)}$ and also $H_X^{(5)}$, $H_Y^{(5)}$, $H_Z^{(5)}$ and $H_K^{(5)}$ will have the following forms:

$$\mathfrak{D}_X^{(5)} = X_1(X_1+2X_2+2X_3)\frac{\partial\tau_1^{(5)}}{\partial X_1} + X_2(X_2+2X_3)\frac{\partial\tau_1^{(5)}}{\partial X_2} + X_3^2\frac{\partial\tau_1^{(5)}}{\partial X_3} - Y_1(X_2+X_3) \tag{38}$$

$$\frac{\partial\tau_1^{(5)}}{\partial Y_1} - Y_2X_3\frac{\partial\tau_1^{(5)}}{\partial Y_2};$$

$$\mathfrak{D}_Y^{(5)} = Y_1(Y_1+2Y_2+2Y_3)\frac{\partial\tau_1^{(5)}}{\partial Y_1} + Y_2(Y_2+2Y_3)\frac{\partial\tau_1^{(5)}}{\partial Y_2} + Y_3^2\frac{\partial\tau_1^{(5)}}{\partial Y_3} - X_1(Y_1+Y_2+Y_3) \tag{39}$$

$$\frac{\partial\tau_1^{(5)}}{\partial X_1} - X_2(Y_2+Y_3)\frac{\partial\tau_1^{(5)}}{\partial X_2} - X_3Y_3\frac{\partial\tau_1^{(5)}}{\partial X_3} - Z_1(Y_2+Y_3)\frac{\partial\tau_1^{(5)}}{\partial z_1} - Z_2y_3\frac{\partial\tau_1^{(5)}}{\partial Z_2};$$

$$\mathfrak{D}_Z^{(5)} = Z_1(Z_1+2Z_2+2Z_3)\frac{\partial\tau_1^{(5)}}{\partial Z_1} + Z_2(Z_2+2Z_3)\frac{\partial\tau_1^{(5)}}{\partial Z_2} + Z_3^2\frac{\partial\tau_1^{(5)}}{\partial Z_3} - Y_1(Z_1+Z_2+Z_3) \tag{40}$$

$$\frac{\partial\tau_1^{(5)}}{\partial Y_1} - Y_2(Z_2+Z_3)\frac{\partial\tau_1^{(5)}}{\partial Y_2} - Y_3Z_3\frac{\partial\tau_1^{(5)}}{\partial Y_3} - K_1(Z_2+Z_3)\frac{\partial\tau_1^{(5)}}{\partial k_1} - K_2Z_3\frac{\partial\tau_1^{(5)}}{\partial K_2};$$

$$\mathfrak{D}_K^{(5)} = K_1(K_1+2K_2+2K_3)\frac{\partial\tau_1^{(5)}}{\partial K_1} + K_2(K_2+2K_3)\frac{\partial\tau_1^{(5)}}{\partial K_2} + K_3^2\frac{\partial\tau_1^{(5)}}{\partial Z_3} - Z_1(K_1+K_2 \tag{41}$$

$$+K_3)\frac{\partial\tau_1^{(5)}}{\partial Z_1} - Z_2(K_2+K_3)\frac{\partial\tau_1^{(5)}}{\partial z_2} - Z_3X_3\frac{\partial\tau_1^{(5)}}{\partial Z_3};$$

$$H_X^{(5)} = 2X_1\frac{\partial\tau_1^{(5)}}{\partial X_1} + 2X_2\frac{\partial\tau_1^{(5)}}{\partial X_2} + 2X_3\frac{\partial\tau_1^{(5)}}{\partial X_3} - Y_1\frac{\partial\tau_1^{(5)}}{\partial Y_1} - Y_2\frac{\partial\tau_1^{(5)}}{\partial Y_2} - Y_3\frac{\partial\tau_1^{(5)}}{\partial Y_3}; \tag{42}$$

$$H_Y^{(5)} = 2Y_1\frac{\partial\tau_1^{(5)}}{\partial Y_1} + 2Y_2\frac{\partial\tau_1^{(5)}}{\partial Y_2} + 2Y_3\frac{\partial\tau_1^{(5)}}{\partial Y_3} - X_1\frac{\partial\tau_1^{(5)}}{\partial X_1} - X_2\frac{\partial\tau_1^{(5)}}{\partial X_2} - X_3\frac{\partial\tau_1^{(5)}}{\partial X_3} - Z_1\frac{\partial\tau_1^{(5)}}{\partial Z_1} \tag{43}$$

$$-Z_2\frac{\partial\tau_1^{(5)}}{\partial Z_2} - Z_3\frac{\partial\tau_1^{(5)}}{\partial Z_3};$$

$$H_Z^{(5)} = 2Z_1\frac{\partial\tau_1^{(5)}}{\partial Z_1} + 2Z_2\frac{\partial\tau_1^{(5)}}{\partial Z_2} + 2Z_3\frac{\partial\tau_1^{(5)}}{\partial z_3} - Y_1\frac{\partial\tau_1^{(5)}}{\partial Y_1} - y_2\frac{\partial\tau_1^{(5)}}{\partial Y_2} - Y_3\frac{\partial\tau_1^{(5)}}{\partial Y_3} - K_1\frac{\partial\tau_1^{(5)}}{\partial K_1} \tag{44}$$

$$-K_2\frac{\partial\tau_1^{(5)}}{\partial K_2} - K_3\frac{\partial\tau_1^{(5)}}{\partial K_3};$$

$$H_K^{(5)} = 2K_1\frac{\partial\tau_1^{(5)}}{\partial K_1} + 2K_2\frac{\partial\tau_1^{(5)}}{\partial K_2} + 2K_3\frac{\partial\tau_1^{(5)}}{\partial K_3} - Z_1\frac{\partial\tau_1^{(5)}}{\partial Z_1} - Z_2\frac{\partial\tau_1^{(5)}}{\partial Z_2} - Z_3\frac{\partial\tau_1^{(5)}}{\partial Z_3}; \tag{45}$$

And the functional dependent nontrivial solutions for the whole system of first order partial differential equation is as follows:

$$\tau_1^{(5)} = \frac{(\Sigma_{1\le i\le j\le m\le l\le 2} \quad x_iy_jz_mk_l)(\Sigma_{1\le i\le j\le m\le l\le 2} \quad x_{i+1}y_{j+1}z_{m+1}k_{l+1})}{x_2y_2z_2k_2(\Sigma_{1\le i\le j\le m\le l\le 3} \quad x_iy_jz_mk_l)}; \tag{46}$$

And again as before, (5) goes back to 5 in the $Sl_5$ and 1 is a default index which later we will use it for to employ our shifting operators.

According to the number of variables, we will have 12 shifts and then after that it will be in a loop. So here in $sl_5$ case we have twelve solutions just as what we did in $sl_4$, and here skip to write them down.

## 2.5. Lattice $W_n$ algebra; main generator

Here for $sl_n$, we skip to write down all steps which we have done in previous sections and just will write down our main generator of the lattice $W_n$ algebra.

The functional dependent nontrivial solution for the whole system of first order partial differential equations will be as what comes in follow:

$$\tau_1^{(n)} = \tag{47}$$

$$\frac{(\Sigma_{1\le i_1\le i_2\cdots\le i_{n-1}\le 2} \quad x_{i_1}^{(1)}x_{i_2}^{(2)}\cdots x_{i_{n-1}}^{(n-1)})(\Sigma_{1\le i_1\le i_2\cdots\le i_{n-1}\le 2} \quad x_{i_1+1}^{(1)}x_{i_2+1}^{(2)}\cdots x_{i_{n-1}+1}^{(n-1)})}{x_2^{(1)}\cdots x_2^{(n-1)}(\Sigma_{1\le i_1\le i_2\cdots\le i_{n-1}\le 3} \quad x_{i_1}^{(1)}x_{i_2}^{(2)}\cdots x_{i_{n-1}}^{(n-1)})};$$

We should notice that $x_{i_j}^{(j)}$s are different of each other for any $j\in\{1,\cdots n-1\}$

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