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ON THE PERIODIC ZETA-FUNCTION

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Abstract

We present an universality theorem for the periodic zeta-function which is defined by a Dirichlet series with periodic coefficients satisfying a certain dependence condition. This simplifies the problem and allows to elucidate the universality of the periodic zeta-function.

Keywords: analytic function, Dirichlet series, periodic zeta-function, universality.

О ПЕРИОДИЧЕСКОЙ ДЗЕТА-ФУНКЦИИ

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Аннотация

В статье доказана теорема универсальности для периодической дзета функции, которая опредеяется рядом Дирихле с периодическими коэффициентами, удовлетворяющими некоторому условию зависимости. Это упрощает задачу и разрешает осветить универсапьность периодической дзета функции.

Ключевые слова: аналитическая функция, периодическая дзета функция, ряд Дирихле, универсальность.

1. Introduction

Let $s = \sigma + it$ be a complex variable, and $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. The periodic zeta-function $\zeta(s; \mathfrak{a})$ is defined, for $\sigma > 1$, by the series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

In view of the periodicity of the sequence $\mathfrak{a}$, we have that, for $\sigma > 1$,

$$\zeta(s; \mathfrak{a}) = \frac{1}{q^s} \sum_{l=1}^{q} a_l \zeta\left(s, \tfrac{l}{q}\right), \tag{1}$$

where $\zeta(s; \alpha)$ is the classical Hurwitz zeta-function with parameter α, $0 < \alpha \leqslant 1$, which is defined, for $\sigma > 1$, by the series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and is continued analytically to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. Therefore, equality (1) gives a meromorphic continuation for $\zeta(s; \mathfrak{a})$ to the whole complex plane. If

$$a \stackrel{def}{=} \frac{1}{q} \sum_{l=1}^{q} a_l = 0,$$

then the function $\zeta(s; \mathfrak{a})$ is entire one. Otherwise, $\zeta(s; \mathfrak{a})$ has a simple pole at the point $s = 1$ with residue a.

Let χ be a Dirichlet character modulo q, and $L(s, \chi)$ denote the corresponding Dirichlet L-function. It is well known, that for $(b, q) = 1$,

$$\zeta\left(s, \tfrac{b}{q}\right) = \frac{q^s}{\varphi(q)} \sum_{\chi = \chi(\mathrm{mod} q)} \overline{\chi}(b) L(s, \chi),$$

where summing runs over all $\varphi(q)$ Dirichlet characters modulo q, and $\varphi(q)$ is the Euler function. Therefore, denoting by (l, q) the greatest common divisor of the numbers l and q, we find from (1) that

$$\begin{aligned} \zeta(s; \mathfrak{a}) &= \frac{1}{q^s} \sum_{l=1}^{q} a_l \zeta\left(s, \frac{\frac{l}{(l,q)}}{\frac{q}{(l,q)}}\right) \\ &= \frac{1}{q^s} \sum_{l=1}^{q} \frac{a_l \left(\frac{q}{(l,q)}\right)^s}{\varphi\left(\frac{q}{(l,q)}\right)} \sum_{\chi = \chi\left(\mathrm{mod} \frac{q}{(l,q)}\right)} \overline{\chi}\left(\tfrac{l}{(l,q)}\right) L(s, \chi) \end{aligned}$$

$$= \sum_{l=1}^{q} \frac{a_l}{\varphi\left(\frac{q}{(l,q)}\right)(l,q)^s} \sum_{\chi=\chi\left(\bmod \frac{q}{(l,q)}\right)} \overline{\chi}\left(\frac{l}{(l,q)}\right) L(s,\chi). \tag{2}$$

In [9] J. Steuding, assuming that $q > 2$, a_m is not a multiple of Dirichlet character modulo q, and that $a_m = 0$ for $(m,q) > 1$, obtained the universality of the function $\zeta(s;\mathfrak{a})$. Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$, $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, and $H(K)$ and $H_0(K)$, $K \in \mathcal{K}$, be the classes of continuous functions on K and of continuous non-vanishing functions on K, respectively, which are analytic in the interior of K. Then J. Steuding proved the following statement, Theorem 11.8 of [9].

Теорема 1. *Suppose that q and $\mathfrak{a}$ are as above. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

Here and in the sequel, meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

If $a_m \equiv c$, $m \in \mathbb{N}$, with $c \in \mathbb{C} \setminus \{0\}$, then the sequence $\mathfrak{a}$ is periodic with $q = 1$. In this case, we have that

$$\zeta(s;\mathfrak{a}) = c\zeta(s). \tag{3}$$

If a_m is a multiple of a Dirichlet character χ modulo q, i.e., $a_m = c\chi(m)$, $m \in \mathbb{N}$, with a certain constant $c \in \mathbb{C} \setminus \{0\}$, then, clearly,

$$\zeta(s;\mathfrak{a}) = cL(s,\chi). \tag{4}$$

Since the functions $\zeta(s)$ and $L(s,\chi)$ are universal in the Voronin sense [1, 5, 9, 11], we have that, in the cases (3) and (4), the function $\zeta(s;\mathfrak{a})$ is also universal.

Теорема 2. *Suppose that $a_m = c \neq 0$ or a_m is a multiple of Dirichlet character modulo q. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

We observe that the approximated function $f(s)$ in Theorem 2 is different from that in Theorem 1: in Theorem 1, $f(s)$ is not necessarily non-vanishing.

The aim of this note is to consider an example of the function $\zeta(s;\mathfrak{a})$ with prime q, where

$$a_q = \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l. \tag{5}$$

This example elucidates the situation.

We say that $\zeta(s;\mathfrak{a})$ is universal if the inequality of universality

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a})-f(s)|<\varepsilon\right\}>0$$

with every $\varepsilon>0$ is satisfied for all $K\in\mathcal{K}$ and $f(s)\in H_0(K)$. If this inequality is satisfied for all $K\in\mathcal{K}$ and $f(s)\in H(K)$, we say that $\zeta(s;\mathfrak{a})$ is strongly universal. Let, for brevity,

$$b(q,\chi)=\sum_{l=1}^{q-1}a_l\chi(l),\qquad \chi=\chi(\mathrm{mod}q).$$

We suppose that $a_m\not\equiv 0$, $m\in\mathbb{N}$. Then the following statement is true.

Teopema 3. *Suppose that q is a prime number, and that the periodic sequence $\mathfrak{a}=\{a_m: m\in\mathbb{N}\}$ with minimal period q satisfies equality* (5)*.*
1° *If the sequence $\mathfrak{a}$ satisfies at least one of hypotheses*
i) $a_m\equiv c,\quad m\in\mathbb{N}$;
ii) a_m is a multiple of a Dirichlet character modulo q;
iii) $q=2$;
iv) only one number $b(q,\chi)\neq 0$,
then the function $\zeta(s;\mathfrak{a})$ is universal.
2° *If at least two numbers $b(q,\chi)\neq 0$, then the function $\zeta(s;\mathfrak{a})$ is strongly universal.*

The proof of assertion 2° of Theorem 3 is based on the Voronin theorem on joint universality of Dirichlet L-functions.

2. Voronin theorem

We remind that two Dirichlet characters are equivalent if they are generated by the same primitive characters.

Teopema 4. *Suppose that $\chi_1,\dots,\chi_r$ are pairwise non-equivalent Dirichlet characters and $L(s,\chi_1),\dots,L(s,\chi_r)$ are the corresponding Dirichlet L-functions. For $j=1,\dots,r$, let $K_j\in\mathcal{K}$, and $f_j(s)\in H(K_j)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|L(s+i\tau;\chi_j)-f_j(s)|<\varepsilon\right\}>0.$$

Theorem 4, for circles in place of the sets K_j, was obtained by S.M. Voronin and applied for the functional independence of Dirichlet L-functions in [10]. A full proof of this case is given in [4]. The Voronin theorem in the form of Theorem 4 can be found in [9] and [6]. Modifications of Theorem 3 also were obtained by S.M. Gonek [3] and B. Bagchi [1, 2].

A very good survey on universality of zeta and L-functions is given in [7].

3. Mergelyan theorem

Approximation theory of analytic functions is one of the most important fields of mathematics, and has a long and rich history. In universality of zeta-functions, a very useful is the Mergelyan theorem on the approximation of analytic functions by polynomials. This theorem is a generalization of results of many authors, and is a final point in the field.

We state the Mergelyan theorem in a convenient for us form.

ТЕОРЕМА 5. *Suppose that $K \subset \mathbb{C}$ is a compact subset with connected complement, and that $F(s)$ is a function continuous on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

Proof of Theorem 5 is given in [8], see also [12].

4. Proof of Theorem 3

The cases i) and ii) of assertion 1° are contained in Theorem 2.

Since q is prime, we have that $(l, q) = 1$ for $l = 1, \dots, q-1$, and $(l, q) = q$ for $l = q$. Therefore, we deduce from (2) that

$$\begin{aligned} \zeta(s; \mathfrak{a}) &= \frac{a_q}{q^s} \sum_{\chi=\chi(\mathrm{mod} 1)} \overline{\chi}(1) L(s, \chi) + \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l \sum_{\chi=\chi(\mathrm{mod} q)} \overline{\chi}(l) L(s, \chi) \\ &= \frac{a_q}{q^s}\zeta(s) + \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l \sum_{\chi=\chi(\mathrm{mod} q)} \overline{\chi}(l) L(s, \chi). \end{aligned} \tag{6}$$

It is well known that if χ_0 is the principal character modulo q, and p denotes a prime number, then

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \tfrac{1}{p^s}\right) = \zeta(s) \left(1 - \tfrac{1}{p^s}\right)$$

in our case. This and (6) show that

$$\begin{aligned} \zeta(s; \mathfrak{a}) &= \frac{1}{q^s} \left(a_l - \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l \right) \zeta(s) + \frac{\zeta(s)}{\varphi(q)} \sum_{l=1}^{q-1} a_l \\ &+ \frac{1}{\varphi(q)} \sum_{l=1}^{q-1} a_l \sum_{\chi=\chi(\mathrm{mod} q)} \overline{\chi}(l) L(s, \chi) \end{aligned}$$

$$= \frac{\zeta(s)}{\varphi(q)}\sum_{l=1}^{q-1} a_l + \frac{1}{\varphi(q)}\sum_{l=1}^{q-1} a_l \sum_{\substack{\chi=\chi(\mathrm{mod} q)\\ \chi\neq\chi_0}} \overline{\chi}(l)L(s,\chi) \tag{7}$$

in view of (5). In the set $\{\chi : \chi = \chi(\mathrm{mod} q)\}$, replace the principal character χ_0 modulo q by the character $\hat{\chi}(\mathrm{mod} 1)$, and preserve the notation $\chi = \chi(\mathrm{mod} q)$. Then (7) we can rewrite in the form

$$\zeta(s;\mathfrak{a}) = \frac{1}{\varphi(q)}\sum_{l=1}^{q-1} a_l \sum_{\chi=\chi(\mathrm{mod} q)} \overline{\chi}(l)L(s,\chi) = \frac{1}{\varphi(q)}\sum_{\chi=\chi(\mathrm{mod} q)} L(s,\chi)b(q,\chi). \tag{8}$$

Now we consider the case iii) of 1°, i.e., $q = 2$. By (8), we find that, in this case,

$$\zeta(s;\mathfrak{a}) = \zeta(s)b(2,\hat{\chi}) = a_1\zeta(s).$$

Thus we obtain the case i).

We note that at least one number $b(q,\chi)$ in (8) is non-zero. Suppose that only one of the numbers $b(q,\chi)$ is non-zero. Let $b(q,\hat{\chi}) \neq 0$. Then, by (8),

$$\zeta(s;\mathfrak{a}) = \frac{1}{\varphi(q)}\zeta(s)b(q,\hat{\chi}) = \frac{1}{\varphi(q)}\zeta(s)\sum_{l=1}^{q-1} a_l.$$

Thus,

$$a_1 = \cdots = a_{q-1} = \frac{1}{\varphi(q)}\sum_{l=1}^{q-1} a_l = a_q,$$

and we have again the case i).

Now let $b(q,\chi) \neq 0$ for some $\chi = \chi(\mathrm{mod} q)$. Then (8) gives

$$\zeta(s;\mathfrak{a}) = \frac{1}{\varphi(q)}L(s,\chi)\sum_{l=1}^{q-1} a_l\overline{\chi}(l).$$

Hence,

$$a_l = \frac{1}{\varphi(q)}\chi(l)\sum_{l=1}^{q-1} a_l\overline{\chi}(l)$$

for all $l \in \mathbb{N}$, and

$$a_1\overline{\chi}(1) = \cdots = a_{q-1}\overline{\chi}(q-1) = \frac{1}{\varphi(q)}\sum_{l=1}^{q-1} a_l\overline{\chi}(l).$$

Thus, $a_l = a_1\chi(l)$, $l \in \mathbb{N}$, and we have the case ii).

It remains to prove 2°. Denote by $H(D)$ the space of analytic functions on the strip D equipped with the topology of uniform convergence on compacta. Preserving the above notation, i.e., in place of $\chi_0(\text{mod} q)$ using $\hat{\chi}(\text{mod} 1)$, we define the operator $F : H^{\varphi(q)}(D) \to H(D)$ by the formula

$$F\left(g_\chi(s) : \chi(\text{mod} q)\right) = \frac{1}{\varphi(q)} \sum_{\chi=\chi(\text{mod} q)} g_\chi(s) b(q,\chi),$$
$$\left(g_\chi(s) : \chi(\text{mod} q)\right) \in H^{\varphi(q)}(D).$$

First we will prove that, for every $K \in \mathcal{K}$ and polynomial $p = p(s)$, there exists $(g_\chi(s) : \chi(\text{mod} q)) \in F^{-1}\{p\}$ such that $g_\chi(s) \neq 0$ on K for all $\chi(\text{mod} q)$.

Suppose that

$$b(q,\chi_j) \neq 0, \qquad j = 1, 2.$$

Since the set K is bounded, there exists a constant $C \in \mathbb{C}$ such that

$$p(s) + C \neq 0 \qquad \text{on } K,$$

and

$$-C - \frac{1}{\varphi(q)} \sum_{\substack{\chi=\chi(\text{mod} q)\\ \chi\neq\chi_1,\chi_2}} b(q,\chi) \neq 0.$$

We take

$$g_{\chi_1}(s) = \varphi(q) b^{-1}(q,\chi_1)(p(s) + C),$$

$$g_{\chi_2}(s) = \varphi(q) b^{-1}(q,\chi_2) \left(-C - \frac{1}{\varphi(q)} \sum_{\substack{\chi=\chi(\text{mod} q)\\ \chi\neq\chi_1,\chi_2}} b(q,\chi) \right)$$

and $g_\chi(s) = 1$ for $\chi \neq \chi_1, \chi_2$. Then we have that $g_\chi(s) \neq 0$ on K for all $\chi = \chi(\text{mod} q)$, and

$$F\left(g_\chi(s) : \chi = \chi(\text{mod} q)\right) = p(s).$$

Let, for brevity,

$$M = \sum_{l=1}^{q-1} |a_l|,$$

and let $\tau \in \mathbb{R}$ satisfy the inequality

$$\sup_{\chi=\chi(\text{mod} q)} \sup_{s\in K} |L(s + i\tau,\chi) - g_\chi(s)| < \frac{\varepsilon}{2M}, \tag{9}$$

where $g_\chi(s)$ has the above properties. Then, for such τ, by (8)

$$\sup_{s\in K} |\zeta(s + i\tau; \mathfrak{a}) - p(s)|$$

$$= \sup_{s\in K}\left|F\left(L(s+i\tau,\chi):\chi=\chi(\mathrm{mod}q)\right)-F\left(g_\chi(s):\chi=\chi(\mathrm{mod}q)\right)\right|$$
$$\leqslant \sup_{s\in K}\frac{M}{\varphi(q)}\sum_{\chi=\chi(\mathrm{mod}q)}|L(s+i\tau,\chi)-g_\chi(s)|$$
$$\leqslant \sup_{\chi=\chi(\mathrm{mod}q)}\sup_{s\in K}|L(s+i\tau,\chi)-g_\chi(s)|<\frac{\varepsilon}{2}. \tag{10}$$

Since q is prime, the characters $\chi=\chi(\mathrm{mod}q)$, where χ_0 is replaced by $\hat{\chi}$, are pairwise non-equivalent. Therefore, by Theorem 1, the set of $\tau\in\mathbb{R}$ satisfying (9) has a positive lower density. However, (9) implies (10). Thus

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a})-p(s)|<\frac{\varepsilon}{2}\right\}>0 \tag{11}$$

for every polynomial $p(s)$.

It remains to replace the polynomial $p(s)$ by $f(s)$. By the Mergelyan theorem (Theorem 5), we can find a polynomial $p(s)$ such that

$$\sup_{s\in K}|f(s)-p(s)|<\frac{\varepsilon}{2}. \tag{12}$$

If $\tau\in\mathbb{R}$ satisfies
$$\sup_{s\in K}|\zeta(s+i\tau,\mathfrak{a})-p(s)|<\frac{\varepsilon}{2},$$
then, in view of (12),
$$\sup_{s\in K}|\zeta(s+i\tau,\mathfrak{a})-f(s)|<\varepsilon.$$
This shows that

$$\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau,\mathfrak{a})-p(s)|<\frac{\varepsilon}{2}\right\}$$
$$\subset\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau,\mathfrak{a})-f(s)|<\frac{\varepsilon}{2}\right\}.$$

Then, by (11),

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau;\mathfrak{a})-f(s)|<\varepsilon\right\}>0.$$

The theorem is proved.

REFERENCES

1. Bagchi B. The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series. Ph. D. Thesis. Calcutta: Indian Statistical Institute, 1981.

2. Bagchi B. A joint universality theorem for Dirichlet L-functions// Math. Z. 1982. V. 181. P. 319–334.
3. Gonek S.M. Analytic properties of zeta and L-functions. Ph. D. Thesis. University of Michigan, 1979.
4. Karatsuba A.A., Voronin S.M. The Riemann-Zeta Function. New York: de Gruyter, 1992.
5. Laurinčikas A. Limit Theorems for the Riemann Zeta-Function. Dordrecht, Boston, London: Kluwer Academic Publishers, 1996.
6. Laurinčikas A. On joint universality of Dirichlet L-functions// Chebyshevskii Sb. 2011. V. 12, No. 1. P. 129–139.
7. Matsumoto K. A survey on the theory of universality for zeta and L-functions. // Proceedings of the 7th China-Japan Number Theory Conference (submitted).
8. S. N. Mergelyan, Uniform approximations to functions of a complex variable// Usp. Matem. Nauk. 1952. V. 7. P. 31–122 (in Russian).
9. Steuding J. Value-Distribution of L-functions. Lecture Notes in Math. V. 1877. Berlin, Heidelberg: Springer Verlag, 2007.
10. Voronin S.M. The functional independence of Dirichlet L-functions// Acta Arith. 1975. V. 27. P. 493–503.
11. Voronin S.M. Theorem on the "universality"of the Riemann zeta-function// Izv. Akad. Nauk SSSR. 1975. V. 39. P. 475–486 (in Russian) ≡ Math. USSR Izv. 1975. V. 9. P. 443–453.
12. Walsh J.L. Interpolation and Approximation by Rational Functions on the Complex Domain. Amer. Math. Soc. Colloq. Publ. V. 20. Providence: American Mathematical Society, 1960.

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