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Ядра весовых структур — это в точности слабо идемпотентно полные категории

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Аннотация

Мы доказываем, что аддитивные категории, являющиеся ядрами весовых структур — это в точности все слабо идемпотентно полные категории, т. е, категории, в которых расщепимые мономорфизмы соответствуют прямым слагаемым. Мы также приводим ряд условий, эквивалентных слабой идемпотентной полноте (часть их полностью нова), и обсуждаем слабо идемпотентные пополнения аддитивных категорий.

Ключевые слова: слабо идемпотентно полная категория, идемпотентное пополнение, слабое ретракт-замыкание, триангулированная категория, весовая структура, ядро.

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The hearts of weight structures are the weakly idempotent complete categories[1]

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Abstract

This paper proves that additive categories that occur as hearts of weight structures are precisely the *weakly idempotent complete* categories, that is, the categories where all split monomorphisms give direct sum decompositions. The work also gives several other conditions equivalent to weak idempotent completeness (some of them are completely new) and discusses weak idempotent completions of additive categories.

Keywords: weakly idempotent complete category, idempotent completion, weak retraction-closure, triangulated category, weight structure, heart.

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1. Introduction

The goal of this note is to study additive categories that can occur as hearts of weight structures on triangulated categories.[2] Actually, an answer to this question can be extracted from Theorem 4.3.2(I,II) of [3]; yet the corresponding calculation of hearts does not contain all the detail. For this reason, in the current paper we study the corresponding *weakly idempotent complete* (additive) categories and *weak idempotent completions* in detail. Another notion important for this paper is the *weak retraction-closure* (of a subcategory; see Definition 1).

Let us briefly describe the contents of the paper. An additive category $\underline{B}$ is said to be weakly idempotent complete if any $\underline{B}$-split monomorphism gives a direct sum decomposition (see Definition 2(1)); obviously, any idempotent complete category is weakly idempotent complete. $\underline{B}$ is said to be weakly retraction-closed in $\underline{B}' \supset \underline{B}$ if for a $\underline{B}'$-isomorphism $Y \cong X \bigoplus Z$ the object Z belongs to $\operatorname{Obj} \underline{B}$ whenever X and Y do. In §3 we prove that $\underline{B}$ is weakly idempotent complete if and only if it is weakly retraction-closed in a (weakly) idempotent complete category $\underline{B}' \supset \underline{B}$. These conditions are equivalent to the existence of $\underline{B}'' \subset \underline{B}$ and an idempotent complete $\underline{B}' \supset \underline{B}$ such that $\underline{B}$ equals the corresponding weak retraction-closure of $\underline{B}''$ in $\underline{B}'$. Moreover, the weak retraction-closure of $\underline{B}$ in the idempotent completion $\operatorname{Kar}(\underline{B})$ gives a canonical weak idempotent completion $\operatorname{wKar}(\underline{B})$ of $\underline{B}$, and we prove that the universality of the Kar-construction also yields that of the wKar-one. Furthermore, $\underline{B}$ is weakly idempotent complete if and only if any contractible bounded $\underline{B}$-complex splits (into a direct sum of isomorphisms; see Proposition 1(7)).

In §4 we recall some basics on weight structures. Recall that these are given by classes $\underline{C}_{w\leqslant 0}$ and $\underline{C}_{w\geqslant 0}$ of objects of a triangulated category $\underline{C}$; the heart $\underline{Hw}$ of w is the additive subcategory $\underline{C}_{w\leqslant 0} \cap \underline{C}_{w\geqslant 0}$. The aforementioned Theorem 4.3.2(I,II) of [3] (along with the somewhat stronger Corollary 2.1.2 of [5]) gives an almost complete characterization of *bounded* weight structures. Loc. cit. implies that any (additive) *connective* subcategory $\underline{B}$ of $\underline{C}$ gives a canonical bounded weight structure w on the smallest strictly full triangulated subcategory $\underline{D}$ of $\underline{C}$ that contains $\underline{B}$, and $\underline{Hw}$ consists of $\underline{D}$-retracts of objects of $\underline{B}$. Now, Theorem 1 implies that $\underline{Hw}$ is equivalent to $\operatorname{wKar}(\underline{B})$; thus $\underline{Hw}$ is equivalent to $\underline{B}$ whenever $\underline{B}$ is weakly idempotent complete. Moreover, the results of §3 easily imply that weakly idempotent complete categories are precisely the ones that occur as weakly retraction-closed subcategories of triangulated categories; they are also the categories equivalent to hearts of (bounded) weight structures. Furthermore, we prove that a full embedding $\underline{B} \to \underline{B}'$ induces an equivalence of $K^b(\underline{B})$ with $K^b(\underline{B}')$ if and only if $\underline{B}'$ is essentially a subcategory of $\operatorname{wKar}(\underline{B})$, and $K_0^{\mathrm{add}}(\underline{B}) \cong K_0^{\mathrm{add}}(\underline{B}')$ if this is the case.

[2] Recall that weight structures are certain "cousins" of t-structures (see Remark 4 below) that were introduced in [3] and [8]; in the latter paper they were called co-t-structures. Weight structures have several interesting applications to representation theory, motives, and algebraic topology; see [5] for some references.

2. On additive categories and (weak) retraction-closures

All categories and functors (including embedding ones) in this paper will be additive.

- Given a category C and $X, Y \in \operatorname{Obj} C$ we will write $C(X,Y)$ for the set of morphisms from X to Y in C.
- For categories C' and C we write $C' \subset C$ if C' is a full subcategory of C.
- Given a category C and $X, Y \in \operatorname{Obj} C$, we say that X is a *retract* of Y if id_X can be factored through Y.[3]
- A class of objects D in (an additive category) $\underline{B}$ is said to be *retraction-closed* in $\underline{B}$ if it contains all $\underline{B}$-retracts of its elements.
- For any $(\underline{B}, D)$ as above we will write $\mathrm{Kar}_{\underline{B}}(D)$ for the class of all $\underline{B}$-retracts of elements of D.
- We will say that $\underline{B}$ is *idempotent complete* if any idempotent endomorphism gives a direct sum decomposition in it; cf. Definition 1.2 of [1].
- The *idempotent completion* $\mathrm{Kar}(\underline{B})$ (no lower index) of $\underline{B}$ is the category of "formal images" of idempotents in $\underline{B}$. Respectively, its objects are the pairs (B,p) for $B \in \operatorname{Obj} \underline{B}$, $p \in \underline{B}(B,B)$, $p^2 = p$, and the morphisms are given by the formula

$$\mathrm{Kar}(\underline{B})((X,p),(X',p')) = \{f \in \underline{B}(X,X') : \ p' \circ f = f \circ p = f\}.$$

  The correspondence $B \mapsto (B, \mathrm{id}_B)$ (for $B \in \operatorname{Obj} \underline{B}$) fully embeds $\underline{B}$ into $\mathrm{Kar}(\underline{B})$, and it is well known that $\mathrm{Kar}(\underline{B})$ is essentially the smallest idempotent complete category containing $\underline{B}$; see Proposition 1.3 of ibid.

Now we will give definitions that appear to be (more or less) new.

Definition 1. *Let $\underline{B}'$ be an (additive) subcategory of $\underline{B}$.*

1. We will write $\mathrm{wKar}_{\underline{B}}(\underline{B}')$ *for the full subcategory of $\underline{B}$ whose objects are those $Z \in \operatorname{Obj} \underline{B}$ such that there exist $X, Y \in \operatorname{Obj} \underline{B}'$ with $X \bigoplus Z \cong Y$. We will call* $\mathrm{wKar}_{\underline{B}}(\underline{B}')$ *the weak retraction-closure of $\underline{B}'$ in $\underline{B}$.*

2. We will say that $\underline{B}'$ is weakly retraction-closed in $\underline{B}$ if $\mathrm{wKar}_{\underline{B}}(\underline{B}') = \underline{B}$.

Below we will need the following simple statements.

Lemma 1. *Let $\underline{B}'$ be a subcategory of $\underline{B}$.*

1. If $\underline{B}'$ is retraction-closed in $\underline{B}$ then it is also weakly retraction-closed in $\underline{B}$.

2. $\mathrm{wKar}_{\underline{B}}(\underline{B}')$ *is weakly retraction-closed in $\underline{B}$.*

Proof. 1. Obvious.

2. For an object Z of $\underline{B}$ and $X, Y \in \operatorname{Obj} \mathrm{wKar}_{\underline{B}}(\underline{B}')$ such that $X \bigoplus Z \cong Y$ we should prove that Z is an object of $\mathrm{wKar}_{\underline{B}}(\underline{B}')$ as well. Now we recall Definition 1(1) and choose $X_1, X_2, Y_1, Y_2 \in \operatorname{Obj} \underline{B}'$ such that $X \bigoplus X_1 \cong X_2$ and $Y \bigoplus Y_1 \cong Y_2$. Then $Z \bigoplus (X_2 \oplus Y_1) \cong Y \bigoplus Y_1 \bigoplus X_1 \cong Y_2 \bigoplus X_1$. Since both $X_2 \bigoplus Y_1$ and $Y_2 \bigoplus X_1$ are objects of $\underline{B}'$, we obtain the result. □

[3] Clearly, if C is triangulated then X is a retract of Y if and only if X is its direct summand.

3. On weakly idempotent complete categories

Let us give some more definitions. Throughout this paper $\underline{B}$ will be an (additive) category.

Definition 2. *1. We will say that $\underline{B}$ is weakly idempotent complete if any split $\underline{B}$-monomorphism $i : X \to Y$ (that is, id_X equals $p \circ i$ for some $p \in \underline{B}(Y, X)$) is isomorphic to the monomorphism $\mathrm{id}_X \bigoplus 0 : X \to X \bigoplus Z$ for some object Z of $\underline{B}$.*

2. Assume that $\underline{B}$ is essentially small. Then the split Grothendieck group $K_0^{\mathrm{add}}(\underline{B})$ is the abelian group whose generators are the isomorphism classes of objects of $\underline{B}$, and the relations are of the form $[B] = [A] + [C]$ for all $A, B, C \in \mathrm{Obj}\,\underline{B}$ such that $B \cong A \bigoplus C$.

Now we prove that this definition is equivalent to several other ones.

Proposition 1. *The following assumptions on $\underline{B}$ are equivalent.*

1. *$\underline{B}$ is weakly idempotent complete.*
2. *$\underline{B}$ is weakly retraction-closed in any (additive) category $\underline{B}'$ containing $\underline{B}$ as a strictly full subcategory.*
3. *$\underline{B}'$ is a weakly retraction-closed subcategory of some weakly idempotent complete category $\underline{B}'$.*
4. *The obvious embedding of $\underline{B}$ into the category $\mathrm{wKar}(\underline{B}) = \mathrm{wKar}_{\mathrm{Kar}(\underline{B})}(\underline{B})$ (see Definition 1 (1)) is an equivalence.*
5. *$\underline{B}$ is equivalent to the category $\mathrm{wKar}(\underline{B}'')$ for some (additive) category $\underline{B}''$.*
6. *There exist additive categories $\underline{B}'' \subset \underline{B} \subset \underline{B}'$ such that $\underline{B}'$ is idempotent complete and $\underline{B} = \mathrm{wKar}_{\underline{B}'}(\underline{B}'')$.*
7. *If a bounded $\underline{B}$-complex is contractible (i.e., it is zero in $K^b(\underline{B})$) then it splits, that is, it has the form $\bigoplus \mathrm{id}_{N^i}[-i]$ for some $N^i \in \mathrm{Obj}\,\underline{B}$.*

Proof. Obviously, condition 1 implies condition 2, and condition 4 implies condition 5. Next, replacing $\underline{B}$ by its isomorphism-closure in $\mathrm{Kar}(\underline{B})$ we obtain that condition 2 implies condition 4. Moreover, if $\underline{B}$ is equivalent to the category $\mathrm{wKar}(\underline{B}'')$ then we can replace $\underline{B}''$ and $\mathrm{Kar}(\underline{B}'')$ by equivalent categories so that $\underline{B}'' \subset \underline{B} \subset \underline{B}'$, $\underline{B}'$ is equivalent to $\mathrm{Kar}(\underline{B}'')$, and $\underline{B}$ is a strict subcategory of $\underline{B}'$. Hence condition 5 implies condition 6.

Next, applying Lemma 1(2) we obtain that condition 5 implies condition 3; note that $\underline{B}'$ is weakly idempotent complete since it is idempotent complete.

Now assume that $\underline{B}'$ is a weakly retraction-closed subcategory of a weakly idempotent complete category $\underline{B}'$. We should prove that any split $\underline{B}$-monomorphism $i : X \to Y$ is isomorphic to the monomorphism $\mathrm{id}_X \bigoplus 0 : X \to X \bigoplus Z$ for some object Z of $\underline{B}$. Since $\underline{B}'$ is weakly idempotent complete, we obtain that Z as desired exists in the category $\underline{B}' \supset \underline{B}$. Since $\underline{B}$ is a weakly retraction-closed subcategory of $\underline{B}'$ and $X \bigoplus Z \cong Y$, we obtain $Y \in \mathrm{Obj}\,\underline{B}$; hence $\underline{B}$ is weakly idempotent complete indeed.

Lastly we prove the equivalence of conditions 1 and 7. If $p \circ i = \mathrm{id}_X$ for some $\underline{B}$-morphisms $X \stackrel{i}{\to} Y \stackrel{p}{\to} X$ then the complex

$$\dots \to 0 \to X \stackrel{i}{\to} Y \stackrel{\mathrm{id}_Y - i \circ p}{\longrightarrow} Y \stackrel{p}{\to} X \to 0 \to \dots$$

is easily seen to be split in $K(\mathrm{Kar}\,\underline{B})$; hence it is zero in $K(\underline{B})$ as well. If it is also split in $K(\underline{B})$ then i and p come from a $\underline{B}$-isomorphism $Y \cong X \bigoplus Z$; thus condition 7 follows from condition 1.

Let us establish the converse implication by the induction on the essential length of a complex $M=(M^i)$; that is, we look for the minimal $l\geqslant 0$ such that the terms M^i are zero for $i<m$ and $i>n$, where $n-m=l$. Contractible complexes (over an arbitrary additive category) obviously splits if its essentiall length is at most 1. Now, assume that M is contractible of length $l\geqslant 2$, and all contractible complexes of length less than l split. Clearly, the contracting homotopy provides a factorization of id_{M^m} through the boundary $d^m: M^m\to M^{m+1}$. Hence the complex M is isomorphic to $\mathrm{Cone}(\mathrm{id}_{M^m})[-1-m]\bigoplus M'$, where M' is of length $l-1$. Obviously, M' is contractible as well and we conclude by applying the inductive assumption. □

Remark 1. *1. The notions of a weakly retraction-closed subcategory and of the weak retraction-closure are obviously self-dual.*

Hence conditions 2, 4, 5, 6 of our proposition are self-dual as well; this is also true for condition 7 (that is, these assumptions are fulfilled for $\underline{B}$ if and only if they are valid for $\underline{B}^{op}$). Thus the notion of weak idempotent completeness is self-dual. Hence weak idempotent completions can also be characterized by the duals of conditions 1 and 3 in Proposition 1. In particular, we obtain Lemma 7.1 of [6].

2. We will call the category $\mathrm{wKar}(\underline{B})=\mathrm{wKar}_{\mathrm{Kar}(\underline{B})}(\underline{B})$ the weak idempotent completion of $\underline{B}$ following Remark 7.8 of [6]. We will justify this terminology and also prove and extend the claim made in loc. cit. in Corollary 1 below.

3. Let R be an (associative unital) ring. Let us describe certain categories that fulfil the assumptions of Proposition 1(6).

Take $\underline{B}''$ to be the category of free left finitely generated R-modules and $\underline{B}'$ to be the category of all left R-modules. Then the corresponding category $\underline{B}\cong\mathrm{wKar}(\underline{B}'')$ is just the category of finitely generated stably free left R-modules.[4]

This example demonstrates that weakly idempotent complete categories do not have to be idempotent complete and gives a nice example of weak idempotent completions (along with weak retraction-closures).

4. The argument used in the proof of the implication (1) $\Longrightarrow$ (7) easily implies that any bounded above or below contractible $\underline{B}$-complex splits as well.

On the other hand, Proposition 10.9 of [6] says that arbitrary (unbounded) contractible $\underline{B}$-complexes split if and only if B is idempotent complete.

These statements (along with our arguments above) are closely related to Remark 1.12 of [9].

Corollary 1. *Let $F:\underline{B}_1\to\underline{B}_2$ be an additive functor.*

1. Then there exists a natural "idempotent complete version" $\mathrm{Kar}(F):\mathrm{Kar}(\underline{B}_1)\to\mathrm{Kar}(\underline{B}_2)$ that restricts to a functor $\mathrm{wKar}(F):\mathrm{wKar}(\underline{B}_1)\to\mathrm{wKar}(\underline{B}_2)$.

2. Consequently, if $\underline{B}_2$ is (weakly) idempotent complete then F extends to an additive functor from $\mathrm{Kar}(\underline{B}_1)$ (resp. from $\mathrm{wKar}(\underline{B}_1)$) into $\underline{B}_2$.

3. Assume that $\underline{B}$ is essentially small. Then $\mathrm{Kar}(\underline{B})$ also is, and $\mathrm{wKar}(\underline{B})$ consists of those $M\in\mathrm{Obj}\,\mathrm{Kar}(\underline{B})$ such that the class of M in $K_0(\mathrm{Kar}(\underline{B}))$ (see Definition 2(2)) belongs to the image of the obvious homomorphism $K_0^{\mathrm{add}}(\underline{B})\to K_0^{\mathrm{add}}(\mathrm{Kar}(\underline{B}))$.

Proof. 1. It is easily seen that F yields a canonical additive functor $\mathrm{Kar}(F)$ that sends (B,p) for $B\in\mathrm{Obj}\,\underline{B}_1$, $p\in\underline{B}_1(B,B)$, $p^2=p$ into $(F(B),f(p))$ indeed.

Next, if $X\bigoplus Z\cong Y$ in $\mathrm{Kar}(\underline{B}_1)$ then $\mathrm{Kar}(F)(X)\bigoplus\mathrm{Kar}(F)(Z)\cong\mathrm{Kar}(F)(Y)$. Thus if an object Z of $\mathrm{Kar}(\underline{B}_1)$ belongs to $\mathrm{wKar}(\underline{B}_1)$ then $\mathrm{Kar}(F)(Z)$ belongs to $\mathrm{wKar}(\underline{B}_2)$ indeed.

[4] The authors are deeply grateful to Vladimir Sosnilo for this nice observation.

2. If $\underline{B}_2$ is idempotent complete then it is equivalent to the category $\mathrm{Kar}(\underline{B}_2)$; hence one can modify $\mathrm{Kar}(F)$ to obtain the extension in question.

Similarly, if $\underline{B}_2$ is weakly idempotent complete then it is equivalent to the category $\mathrm{wKar}(\underline{B}_2)$ (see condition 4 in Proposition 1); thus one can modify $\mathrm{wKar}(F)$ to obtain the result.

3. The essential smallness of $\mathrm{Kar}(\underline{B})$ obviously follows from that of $\underline{B}$.

Next we note that the definition of $K_0(\mathrm{Kar}(\underline{B}))$ immediately implies the following: we have $[M] = [N_1] - [N_2]$ for some objects N_i of $\underline{B}$ (being more precise, here we consider the objects (N_i, id_{N_i}) of $\mathrm{Kar}(\underline{B})$) whenever there exists $B \in \mathrm{Obj}\,\mathrm{Kar}(\underline{B})$ such that $M \bigoplus B \bigoplus N_2 \cong N_1 \bigoplus B$. Since B is a retract of an object of $\underline{B}$, this is equivalent to the existence of $B' \in \mathrm{Obj}\,\underline{B}$ such that $M \bigoplus B' \bigoplus N_2 \cong N_1 \bigoplus N \bigoplus B'$. Our assertion follows immediately. □

Remark 2. *Let us now relate the terminology in the current paper to that in earlier ones.*

It appears that the term "weakly idempotent complete"for a category $\underline{B}$ was introduced in [6, Definition 7.2]. In [7] (probably, this is where this notion was originally introduced) it was said that retracts have complements (in $\underline{B}$),[5] *whereas in Definition 1.11 of [9] it was said that $\underline{B}$ is semi-saturated. Most of the conditions in Proposition 1 and Theorem* 1 were not mentioned in these papers.

Recall also that weak idempotent completions were called *small envelopes* in Definition 4.3.1(3) of [3] and *semi-saturations* in §1.12.1 of of [9].

4. Weight structures: short reminder

Let us start from the definition of a weight structure (note however that the only axiom of weight structures that we will mention explicitly in this text is the axiom (i)). The symbol $\underline{C}$ in this paper will always denote some triangulated category.

It will be convenient for us to use the following notation below: for $D, E \subset \mathrm{Obj}\,\underline{C}$ we will write $D \perp E$ if $\underline{C}(X, Y) = \{0\}$ for all $X \in D$ and $Y \in E$.

Definition 3. *I. A couple $(\underline{C}_{w\leqslant 0}, \underline{C}_{w\geqslant 0})$ of classes of objects of $\underline{C}$ will be said to give a weight structure w on $\underline{C}$ if the following conditions are fulfilled.*

(i) $\underline{C}_{w\leqslant 0}$ *and* $\underline{C}_{w\geqslant 0}$ *are retraction-closed in* $\underline{C}$ *(i.e., contain all* $\underline{C}$*-retracts of their objects).*

(ii) **Semi-invariance with respect to translations.**

$\underline{C}_{w\leqslant 0} \subset \underline{C}_{w\leqslant 0}[1]$ *and* $\underline{C}_{w\geqslant 0}[1] \subset \underline{C}_{w\geqslant 0}$.

(iii) **Orthogonality.**

$\underline{C}_{w\leqslant 0} \perp \underline{C}_{w\geqslant 0}[1]$.

(iv) **Weight decompositions**.

For any $M \in \mathrm{Obj}\,\underline{C}$ there exists a distinguished triangle

$$L_w M \to M \to R_w M \to L_w M[1]$$

such that $L_w M \in \underline{C}_{w\leqslant 0}$ and $R_w M \in \underline{C}_{w\geqslant 0}[1]$.

We will also need the following definitions.

Definition 4. *Assume that a triangulated category $\underline{C}$ is endowed with a weight structure w, $i \in \mathbb{Z}$.*

1. *The full category $\underline{Hw} \subset \underline{C}$ whose objects are $\underline{C}_{w=0} = \underline{C}_{w\geqslant 0} \cap \underline{C}_{w\leqslant 0}$ is called the heart of w.*

2. *$\underline{C}_{w\geqslant i}$ (resp. $\underline{C}_{w\leqslant i}$, resp. $\underline{C}_{w=i}$) will denote the class $\underline{C}_{w\geqslant 0}[i]$ (resp. $\underline{C}_{w\leqslant 0}[i]$, resp. $\underline{C}_{w=0}[i]$).*

[5] Recall that the main Proposition of ibid. says that weakly idempotent complete categories closed with respect to countable coproducts are idempotent complete.

3. *We will say that* $(\underline{C}, w)$ *is bounded and* $\underline{C}$ *is a bounded weighted category if* $\operatorname{Obj}\underline{C} = \cup_{i\in\mathbb{Z}}\underline{C}_{w\geqslant i} = \cup_{i\in\mathbb{Z}}\underline{C}_{w\leqslant i}$.

4. *Let* $\underline{D}$ *be a full triangulated subcategory of* $\underline{C}$.

   We will say that w *restricts to* $\underline{D}$ *whenever the couple* $w_{\underline{D}} = (\underline{C}_{w\leqslant 0}\cap\operatorname{Obj}\underline{D},\ \underline{C}_{w\geqslant 0}\cap\operatorname{Obj}\underline{D})$ *is a weight structure on* $\underline{D}$.

5. *We will say that the subcategory* $\underline{H}\subset\underline{C}$ *is connective (in* $\underline{C}$*) if* $\operatorname{Obj}\underline{H}\perp(\cup_{i>0}\operatorname{Obj}(\underline{H}[i]))$.[6]

6. *The smallest strictly full triangulated subcategory of* $\underline{C}$ *containing* $\underline{H}$ *will be called the subcategory strongly generated by* $\underline{H}$ *in* $\underline{C}$.

7. *We will say that a class* $\mathcal{P}\subset\operatorname{Obj}\underline{C}$ *is extension-closed if* $\mathcal{P}$ *contains* 0 *and for any* $\underline{C}$*-distinguished triangle* $A\to C\to B\to A[1]$ *the object* B *belongs to* $\mathcal{P}$ *whenever (both)* A *and* C *do.*

Remark 3. *1. A simple (and still quite useful) example of a weight structure comes from the stupid filtration on the homotopy category of cohomological complexes* $K(\underline{B})$ *for an arbitrary additive* $\underline{B}$*; it can also be restricted to the subcategory* $K^b(\underline{B})$ *of bounded complexes (see Definition 4(4)). In this case* $K(\underline{B})_{w_{st}\leqslant 0}$ *(resp.* $K(\underline{B})_{w_{st}\geqslant 0}$*) is the class of complexes that are homotopy equivalent to complexes concentrated in degrees* $\geqslant 0$ *(resp.* $\leqslant 0$*); see Remark 1.2.3(1) of [5] for more detail.*

The heart of the weight structure w_{st} *is the retraction-closure of* $\underline{B}$ *in* $K(\underline{B})$*; hence it is equivalent to* $\operatorname{Kar}(\underline{B})$ *(since both* $K^-(\underline{B})$ *and* $K^+(\underline{B})$ *are idempotent complete).*

The restriction of w_{st} *to* $K^b(\underline{B})$ *will be denoted by* w^b_{st}*; in Theorem* 1 below we will demonstrate that its heart $\underline{Hw}^b_{st}$ is equivalent to $\operatorname{wKar}(\underline{B})$.

2. In this note we use the "homological convention" for weight structures. This is the convention used by several papers of the first author (including [5] and [4]). However, in [3] the so-called cohomological convention was used; in this convention the functor [1] "shifts weights"by -1. This is one of the reasons for us not to cite ibid. below; another one is that the exposition of the theory of weight complexes (that we will apply in the proof of Theorem 1) in §3 of [3] is rather inaccurate.

Let us now recall the relation of connective subcategories to weight structures.

Proposition 2. *Let* $\underline{C}$ *be a triangulated category.*

I. Assume that w *is a weight structure on* $\underline{C}$.

1. Then the classes $\underline{C}_{w\leqslant 0}$, $\underline{C}_{w\geqslant 0}$, *and* $\underline{C}_{w=0}$ *are extension-closed; consequently, they are additive.*

2. Let v *be another weight structure for* $\underline{C}$*; suppose that* $\underline{C}_{w\leqslant 0}\subset\underline{C}_{v\leqslant 0}$ *and* $\underline{C}_{w\geqslant 0}\subset\underline{C}_{g\leqslant 0}$. *Then* $w = v$ *(i.e., the inclusions are equalities).*

II. Under the assumptions of Definition 4(5) there exists a unique weight structure $w_{\underline{B}}$ *on the category* $\underline{D} = \langle\underline{B}\rangle$ *whose heart contains* $\underline{B}$*. Moreover, this weight structure is bounded and* $\underline{D}_{w_{\underline{B}}=0} = \operatorname{Kar}_{\underline{C}}(\operatorname{Obj}\underline{B})$.

Proof. I.1. See Proposition 1.2.4(3) and Remark 1.2.3(4) of [5].

2. This is Proposition 1.2.4(7) of loc. cit.

II. Immediate from Corollary 2.1.2 of ibid. □

[6] In earlier texts of the first author connective subcategories were called *negative* ones. Moreover, in several papers (mostly, on representation theory and related matters) a connective subcategory satisfying certain additional assumptions was said to be *silting*; this notion generalizes the one of *tilting*.

5. On hearts of weight structures

THEOREM 1. *The following assumptions on (an additive category) $\underline{B}$ are equivalent as well.*

1. *$\underline{B}$ is weakly idempotent complete.*
2. *There exists a triangulated category $\underline{C}$ such that $\underline{B}$ is its weakly retraction-closed subcategory.*
3. *$\underline{B}$ is equivalent to the heart of a weight structure.*
4. *$\underline{B}$ is equivalent to the heart of a bounded weight structure.*
5. *$\underline{B}$ is equivalent to the heart $\underline{Hw}^b_{st}$ of the weight structure w^b_{st} on the category $K^b(\underline{B})$ (see Remark 3(1)).*
6. *For any category $\underline{B}''$ such that the embedding $\underline{B}'' \to \mathrm{wKar}(\underline{B}'')$ factors through a fully faithful functor $\underline{B} \to \mathrm{wKar}(\underline{B}'')$ and $\underline{B}''$ is connective in a triangulated category $\underline{C}$ (see Definition 4(5)), there exists a unique weight structure w on the triangulated subcategory $\underline{D}$ of $\underline{C}$ strongly generated by $\underline{B}''$ such that the heart $\underline{Hw}$ is naturally equivalent to $\underline{B}$ (that is, $\underline{Hw}$ contains $\underline{B}''$ and the embedding $\underline{B}'' \to \underline{Hw}$ factors through an equivalence of $\underline{B}$ with $\underline{Hw}$).*

PROOF. Clearly, condition 5 implies condition 4, and 4 implies 3. Next, we can take $\underline{B}'' = \underline{B}$ in condition 6. Since $\underline{B}$ is connective in the category $\underline{C} = \underline{D} = K^b(\underline{B})$ and strongly generates it, we obtain that condition 6 implies condition 5.

Now, axiom (i) of Definition 3 implies that $\underline{Hw}$ is retraction-closed in $\underline{C}$ (note that it is an additive subcategory by Proposition 2(I.1)). Thus condition 3 implies condition 2.

Furthermore, any triangulated category is easily seen to be weakly idempotent complete since for X and Y as in Definiton 2(1) we have $Y \cong X \bigoplus \mathrm{Cone}(X \to Y)$. Thus condition 2 implies that $\underline{B}$ is weakly idempotent complete (i.e., that condition 1 is fulfilled); see Proposition 1(3).

Thus it remains to verify that any weakly idempotent complete category $\underline{B}$ fulfils condition 6. The existence and the uniqueness of a weight structure w on $\underline{D}$ such that $\underline{B}'' \subset \underline{Hw}$ follows immediately from Proposition 2(II); we also obtain the existence of a fully faithful functor $\underline{Hw} \to \mathrm{Kar}(\underline{B}'')$, whereas the latter category is clearly equivalent to $\mathrm{Kar}(\underline{B})$. Moreover, $\underline{Hw}$ is weakly idempotent complete (recall that we have just proved that our condition 3 implies condition 1); hence Corollary 1 implies that the embedding $\underline{B} \to \underline{B}''$ factors through a full embedding of $\underline{B}$ into $\underline{Hw}$.

Since $\underline{B}$ is weakly idempotent complete, it remains to verify that for any $M \in \underline{D}_{w=0}$ there exist objects X and Y of $\underline{B}$ such that $M \bigoplus X \cong Y$. We will deduce this statement from the existence of splittings of contractible complexes in $K^b(\underline{Hw})$; for this purpose we invoke the theory of *(weak) weight complex functors* as provided by Proposition 1.3.4 of [4].

Part 6 of loc. cit. associates to M its weight complex $t(M) \in \mathrm{Obj}\, K(\underline{Hw})$. Parts 4 and 10 of loc. cit. imply that $t(M) \cong M$ (in the homotopy category $K(\underline{Hw})$). On the other hand, parts 4 and 9 easily yield that $t(M)$ is homotopy equivalent to a complex $N \in \mathrm{Obj}\, K^b(\underline{B}'') \subset K(\underline{Hw})$. Hence there exists a $K(\underline{Hw})$-morphism $f : M \to N$ such that $\mathrm{Cone}(f)$ is contractible. Since $\mathrm{Cone}(f) \in \mathrm{Obj}\, K^b(\underline{Hw})$ and we have already proved that $\underline{Hw}$ is weakly idempotent complete, we obtain that $\mathrm{Cone}(f)$ splits in $K^b(\underline{Hw})$. Now, if $N^i \in \mathrm{Obj}\, \underline{B}''$ are the terms of N, then this splitting yields $M \bigoplus_{j\in\mathbb{Z}} N^{2i-1} \cong \bigoplus_{j\in\mathbb{Z}} N^{2i}$. This concludes the proof. □

REMARK 4. *Weight structures are well known to be closely related to t-structures (as introduced in §1.3 of [2]). However, the properties of weight structures are significantly distinct from that of t-structures. Recall in particular that the hearts of t-structures are precisely the abelian categories. Hence there are plenty of additive categories that are hearts of some weight structures and cannot occur as hearts of t-structures; cf. Remark* 1(3).

The following statement gives one more characterization of weak idempotent completions as well as certain Grothendieck group isomorphisms.

Corollary 2. *1. If* $\underline{B} \subset \underline{B}'$ *then the corresponding embedding* $K^b(\underline{B}) \to K^b(\underline{B}')$ *is an equivalence if and only if the embedding* $\underline{B} \to \mathrm{wKar}(\underline{B})$ *factors through a fully faithful functor from* $\underline{B}'$ *into* $\mathrm{wKar}(\underline{B})$.

2. Consequently, if $\underline{B}'$ *is essentially small and* $\underline{B} \subset \underline{B}' \subset \mathrm{wKar}(\underline{B})$ *then the obvious homomorphism* $K_0^{\mathrm{add}}(\underline{B}) \to K_0^{\mathrm{add}}(\underline{B}')$ *(see Definition* 2(2)) is bijective.

Proof. 1. Assume that the embedding $K^b(\underline{B}) \to K^b(\underline{B}')$ is an equivalence. Then we can assume that the stupid weight structure on $K^b(\underline{B}')$ (see Remark 3(1)) gives a weight structure v on $\underline{C} = K^b(\underline{B})$. Since the classes $\underline{C}_{v\leqslant 0}$ and $\underline{C}_{v\geqslant 0}$ are closed with respect to isomorphisms (see the axiom (i) in Definition 3), we obtain $\underline{C}_{w_{st}^b\leqslant 0} \subset \underline{C}_{v\leqslant 0}$ and $\underline{C}_{w_{st}^b\geqslant 0} \subset \underline{C}_{v\geqslant 0}$. Thus $w_{st}^b = v$ according to Proposition 2(I.2). Since $\underline{Hw}_{st}^b$ is equivalent to $\mathrm{wKar}(\underline{B})$ (see condition 5 in Theorem 1), this clearly gives a fully faithful functor from $\underline{B}'$ into $\mathrm{wKar}(\underline{B})$.

Now let us prove the converse implication. We should prove that $\underline{D} = K^b(\underline{B}')$, where $\underline{D}$ is the closure of $K^b(\underline{B})$ in $K^b(\underline{B}')$ with respect to isomorphisms, if $\underline{B}'$ is equivalent to a subcategory of $\mathrm{wKar}(\underline{B})$. Now, $\underline{D}$ is a strictly full subcategory of $K^b(\underline{B}')$ that essentially contains $\mathrm{wKar}(\underline{B})$ (see condition 6 in Theorem 1). Since $K^b(\underline{B}')$ is clearly strongly generated (see Definition 4(6)) by $\underline{B}'$, we easily obtain the equality in question.

2. According to assertion 1, the embedding $K^b(\underline{B}) \to K^b(\underline{B}')$ is an equivalence in this case. Thus it suffices to recall that for any essentially small (additive) category $\underline{A}$ the group $K_0^{\mathrm{add}}(\underline{A})$ can be computed as a certain triangulated Grothendieck group of the category $K^b(\underline{A})$; see Definition 2 and Theorem 1 of [10]. □

Remark 5. *1. One can also prove that a category* $\underline{B}' \supset \underline{B}$ *is essentially a subcategory of* $\mathrm{Kar}(\underline{B})$ *if and only if* $K(\underline{B}) \cong K(\underline{B}')$*; this is also equivalent to* $K^+(\underline{B}) \cong K^+(\underline{B}')$ *and* $K^-(\underline{B}) \cong K^-(\underline{B}')$*. To prove the "if"implications here one can apply stupid weight structure arguments similar to the one in the proof of Corollary 2, and one can invoke either Remark 3.3.2(2) of [4] or Remark 1.12.4 of [9] to obtain the converse implications.*

2. This observation along with Proposition 1(7), Proposition 10.9 of [6] (see Remark 1(4)), and Remark 3 justifies the following vague claim: $\mathrm{Kar}(\underline{B})$ *is the "extension"of* $\underline{B}$ *corresponding to unbounded* $\underline{B}$*-complexes, wheras* $\mathrm{wKar}(\underline{B})$ *"corresponds to"*$K^b(\underline{B})$.

3. One can also prove Corollary 2*(2) more explicitly; cf. Corollary* 1*(3).*

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