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Суммы Клоостермана по простым числам и разрешимость одного сравнения с обратными вычетами — II

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Аннотация

В настоящей статье продолжены исследования, связанные с распределением обратных вычетов по заданному модулю. Ранее автором был получен ряд нетривиальных оценок коротких сумм Клоостермана с простыми числами, отвечающих произвольному модулю q. Следствием таких оценок стали результаты о распределении вычетов $\overline{p}$, обратных простым числам "короткого" промежутка: $p\overline{p} \equiv 1 \pmod{q}$, $1 < p \leqslant N$, $N \leqslant q^{1-\delta}$, $\delta > 0$, и, более общо, о распределении по модулю q величин $g(p) = a\overline{p}+bp$, где a, b – целые числа, $(ab, q) = 1$.

Еще одно приложение найденных оценок связано с задачей о представимости произвольного заданного вычета $m \pmod{q}$ суммою $g(p_1) + \ldots + g(p_k)$ при фиксированных a, b и $k \geqslant 3$, и простых $1 < p_1, \ldots, p_k \leqslant N$. Для количества таких представлений автором была найдена формула, поведение предполагаемого главного члена которой определяется аналогом "сингулярного ряда" классического кругового метода, т.е. некоторой величиной κ, зависящей от q и набора k, a, b, m. При фиксированных k, a, b, m она является мультипликативной функцией q. В случае, когда модуль q не делится на 2 или 3, эта величина строго положительна, так что формула для искомого числа представлений является асимптотической.

В настоящей работе исследуется поведение κ в случае, когда $q = 3^n$. Оказывается, что при любых $n \geqslant 1$, $k \geqslant 3$ существуют "исключительные" тройки a, b, m, для которых $\kappa = 0$. Цель работы состоит в описании всех таких троек и нижней оценки величины κ для "неисключительных" троек.

Ключевые слова: сравнения, разрешимость, обратные вычеты, суммы Клоостермана, простые числа, сингулярный ряд.

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Kloosterman sums with primes and the solvability of one congruence with inverse residues — II[1]

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Abstract

In the paper, we continue to study the distribution of inverse residues to given modulus. Earlier, the author obtained a series of non-trivial estimates for incomplete Kloosterman sums over prime numbers with an arbitrary modulus q. One of the applications of such estimates are some assertions concerning the distribution of inverse residues $\overline{p}$ to prime numbers lying in a "short" segment: $p\overline{p} \equiv 1 \pmod{q}$, $1 < p \leqslant N$, $N \leqslant q^{1-\delta}$, $\delta > 0$, and, more general, concerning the distribution of the quantities $g(p) = a\overline{p} + bp$ with respect to modulus q, where a, b are some integers, $(ab, q) = 1$.

Another application is connected with the problem of the representation of a given residue $m \pmod{q}$ by the sum $g(p_1) + \ldots + g(p_k)$ for fixed a, b and $k \geqslant 3$, in primes $1 < p_1, \ldots, p_k \leqslant N$. For the number of such representations, the author have found the formula, where the behavior of the expected main term is controlled by some analogous of the "singular series" that appears in classical circle method, that is, by some function κ depending on q and the tuple k, a, b, m. For fixed k, a, b, m, this function is multiplicative with respect to q. In the case when q is not divisible by 2 or 3, this function is strictly positive, and therefore the formula for the number of the representations becomes asymptotic.

In this paper, we study the behavior of κ for $q = 3^n$. It appears that, for any $n \geqslant 1$, $k \geqslant 3$, there exist the "exceptional" triples a, b, m such that $\kappa = 0$. The main purpose is to describe all such triples and to obtain the lower estimate for κ for all non-exceptional triples.

Keywords: congruences, solvability, inverse residues, Kloosterman sums, prime numbers, singular series.

Bibliography: 17 titles.

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To academician V. P. Platonov in occasion with his 80th anniversary.

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1. Introduction

In the present paper, we continue the study of the solvability of some congruences with inverse residues to modulo q started in [1] and [2]. As in [2], the main subject now is the congruence

$$g(p_1) + \ldots + g(p_k) \equiv m \pmod{q}, \tag{1}$$

where $q \geqslant 2$ is an arbitrary integer, $g(x) \equiv a\overline{x} + bx \pmod{q}$, k, a, b, m are any fixed integers satisfying the conditions $k \geqslant 3$, $1 \leqslant a, b, m \leqslant q$, $(ab, q) = 1$. The variables $p_1, \ldots, p_k$ run through prime numbers from the interval $(1, N]$. This interval is assumed to be "short", that is, we are interested in the case when $N \leqslant q^{1-\delta}$ for some positive δ.

The key role in such problems is played by the estimates of Kloosterman sums with primes, that is, of the exponential sums of the type

$$W_q(a, b; X) = \sum_{p \leqslant X,\, p \nmid q} \exp\biggl(\frac{2\pi i}{q}(a\overline{p} + bp)\biggr),$$

(for such estimates and their applications, see: [3]-[16]). The estimates given in [1], [2] lead to the following assertion (see [2]):

Theorem A. *Let $0 < \varepsilon < 0.01$ be an arbitrary fixed constant and let $k \geqslant 3$ be any fixed integer. Suppose that $q \geqslant q_0(\varepsilon, k)$. Further, let $(ab, q) = 1$ and $g(x) \equiv a\overline{x} + bx \pmod{q}$. Finally, let*

$$\gamma_k = \frac{2(k+33)}{3k+64} \quad \textit{if} \quad 3 \leqslant k \leqslant 16 \quad \textit{and} \quad \gamma_k = \frac{3k+50}{4(k+12)} \quad \textit{if} \quad k \geqslant 17,$$

and suppose that $q^{\gamma_k+\varepsilon} \leqslant N \leqslant q$. Then the number $I_k(N) = I_k(N, q, a, b, m)$ of solutions of (1) in primes $p_j \leqslant N$, $(p_j, q) = 1$, satisfies the relation

$$I_k(N) = \frac{\pi^k(N)}{q}\bigl(\varkappa_k(q) + O(\Delta_k)\bigr). \tag{2}$$

Here $\varkappa_k(q) = \varkappa_k(a, b, m; q)$ is some non-negative multiplicative function of q for any fixed tuple k, a, b and m. Moreover,

a) *for any $k \geqslant 7$ we have $\Delta_k = (\ln\ln N)^B(\ln N)^{-A}$,*

$$A = \frac{1}{2} + \frac{29}{2}(k-7), \quad B = 2^k - 1;$$

b) *for any $k \geqslant 3$ we have $\Delta_k = q^{-\varepsilon}$, if Generalized Riemann hypothesis is true.*

Remark. One can check that $\gamma_k \leqslant 1 - \frac{1}{73}$ for any $k \geqslant 3$.

The ascertaining of the conditions of when (2) becomes the asymptotic formula is connected with the detailed study of the multiplicative function $\varkappa_k(q) = \varkappa_k(a, b, m; q)$. In this direction, in [2], we prove the following assertion:

Theorem B. *Suppose that q is coprime to 6 and let $k \geqslant 3$ be any fixed integer. Then, for any triple (a, b, m) with the conditions $1 \leqslant a, b, m \leqslant q$, $(ab, q) = 1$ the following inequalities hold:*

$$\varkappa_k(a, b, m; q) \geqslant \begin{cases} c_1 \exp\Bigl(-\dfrac{c_2\sqrt{\ln q}}{\ln\ln q}\Bigr), & \textit{if} \;\; k = 3, \\ c_3(\ln\ln q)^{-6}, & \textit{if} \;\; k = 4, \\ 10^{-5}, & \textit{for any} \;\; k \geqslant 5 \end{cases}$$

where the constants c_j, $j = 1, 2, 3$, are absolute.

Remark. The exponent (-6) (for $k = 4$) and constant 10^{-5} (for $k \geqslant 5$) are not optimal.

Now the purpose is to study the behavior of $\varkappa_k(q)$ in the case when $q = 3^n$. Unlike the case $(q, 6) = 1$, for any pair k, n there exists the set $\Omega_k(3^n)$ of "exceptional triples" (a, b, m) such that $\varkappa_k(a, b, m; 3^n) = 0$. In what follows, we shall refer these sets as "exceptional sets".

The main results of this paper is the description of all exceptional sets $\Omega_k(3^n)$, $k \geqslant 3$, $n \geqslant 1$. Namely, we prove here the following assertion:

Theorem. *In the cases $k \geqslant 8$, $n \geqslant 1$ and $3 \leqslant k \leqslant 7$, $n = 1$, the set $\Omega_k(3^n)$ consists of the triples satisfying the conditions*

$$1 \leqslant a, b, m \leqslant 3^n, \quad (ab, 3) = 1, \quad a + b \equiv 0 \pmod 3, \quad m \not\equiv 0 \pmod 3. \tag{3}$$

In particular, $|\Omega_k(3^n)| = 4 \cdot 3^{3(n-1)}$.

In the case $3 \leqslant k \leqslant 7$, the set $\Omega_k(3^2)$ consists of the triples listed in (13)-(16); *in particular, $|\Omega_k(3^2)| = 18(14 - k)$;*

Finally, in the case $3 \leqslant k \leqslant 7$, $n \geqslant 3$, the set $\Omega_k(3^n)$ consists of the triples coinciding modulo 3^2 with the triples from the set $\Omega_k(3^2)$; in particular, $|\Omega_k(3^n)| = 2 \cdot 3^{3n-4}(14 - k)$.

At the same time, for any $k \geqslant 3, n \geqslant 1$ and for any "non-exceptional" triple $(a, b, m) \notin \Omega_k(3^n)$ one has

$$\varkappa_k(a, b, m; 3^n) \ > \ \frac{1}{50}.$$

2. Complete Kloosterman sums to prime power moduli

In [2], we establish the following formula for $\varkappa_k(p^n) = \varkappa_k(a, b, m; p^n)$:

$$\varkappa_k(p^n) \ = \ 1 + A_k(p) + A_k(p^2) + \ldots + A_k(p^n).$$

Here

$$A_k(p^n) \ = \ A_k(a, b, m; p^n) \ = \ \frac{1}{\varphi^k(p^n)} \sum_{f=1}^{p^n}{}' e^{-2\pi i \frac{fm}{p^n}} S^k(fa, fb; p^n),$$

and $S(a, b; q)$ is a complete Kloosterman sum, that is,

$$S(a, b; q) \ = \ \sum_{\substack{x=1 \\ (x,q)=1}}^{q} \exp\left(\frac{2\pi i}{q}(a\overline{x} + bx)\right).$$

To study the properties of $\varkappa_k(q)$, we need some explicit expressions and the estimates for the quantities $S(a, b; p^n)$, $A_k(a, b, m; 3^n)$, $n \geqslant 2$.

Lemma 2.1. *Let $p \geqslant 3$ and $(ab, p) = 1$. If ab is a quadratic non-residue modulo p then $S(a, b; p^n) = 0$ for any $n \geqslant 2$. Otherwise, setting q for p^n and ν for any solution of the congruence $ab \equiv \nu^2 \pmod q$, we have*

$$S(a, b; q) \ = \ S(\nu, \nu; q) \ = \begin{cases} 2\sqrt{q} \cos \dfrac{4\pi\nu}{q} & \text{for even } n, \\ 2\sqrt{q}\left(\frac{\nu}{q}\right) \cos\left(\frac{4\pi\nu}{q} + \frac{\pi s}{2}\right) & \text{for odd } n, \end{cases}$$

where $s = 0$ for $p \equiv 1 \pmod 4$ and $s = 1$ for $p \equiv 3 \pmod 4$.

For the proof, see [17].

Corollary. *Under the conditions of Lemma 2.1, for any $n \geqslant 2$ and $q = p^n$ we have*

$$|S(a,b;q)| < 2\sqrt{q}.$$

Lemma 2.2. *For any a, b coprime to 3 the following equalities hold:*

$$S(a,b;3) = \begin{cases} 2, & \text{if } a+b \equiv 0 \pmod 3, \\ -1, & \text{otherwise}; \end{cases}$$

$$S(a,b;3^2) = \begin{cases} 3(-1)^b \cos\frac{\pi}{9}(4a+b), & \text{if } b-a \equiv 0 \pmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Setting $\omega = e^{2\pi i/3}$ we get

$$S(a,b;3) = \sum_{x=1}^{2} \omega^{a\overline{x}+bx} = \omega^{a+b} + \omega^{2(a+b)} = \begin{cases} 2, & \text{if } a+b \equiv 0 \pmod 3, \\ -1, & \text{otherwise.} \end{cases}$$

Further, setting $x = y + 3z$ in the sum $S(a,b;3^2)$ we obtain

$$S(a,b;3^2) = \sum_{y=1}^{3}{}' e^{\frac{2\pi i}{3^2}(a\overline{y}+by)} \sum_{z=1}^{3} \omega^{z(b-a)}.$$

So, $S(a,b;3^2) = 0$ for $b - a \not\equiv 0 \pmod 3$. Finally, if $a = b + 3n$ for some integer n then

$$S(a,b;3^2) = 3\left(e^{\frac{2\pi i}{3^2}(a+b)} + e^{\frac{2\pi i}{3^2}(5a+2b)}\right) = 3e^{\frac{2\pi i}{3^2}\left(3a+\frac{3b}{2}\right)} \cos\frac{\pi}{9}(4a+9).$$

Since

$$\frac{2}{9}\left(3a + \frac{3b}{2}\right) = \frac{2}{9}\left(3b + \frac{3b}{2} + 9n\right) = b + 2n \equiv b \pmod 2,$$

we arrive at the assertion of the lemma. □

In this section, we use the explicit formulas for Kloosterman sums $S(a,b;p^n)$, $p \geqslant 3$, $n \geqslant 1$ to provide the explicit formulas for the values

$$A_k(p^n) = A_k(a,b,m;p^n) = \frac{1}{\varphi^k(p^n)} \sum_{f=1}^{p^n}{}' e^{-2\pi i \frac{fm}{p^n}} S^k(fa,fb;p^n).$$

Lemma 2.3. *Suppose that $(ab,3) = 1$. If $a + b \equiv 0 \pmod 3$ then*

$$A_k(3) = \begin{cases} 2, & \text{when } m \equiv 0 \pmod 3, \\ -1, & \text{when } m \not\equiv 0 \pmod 3; \end{cases} \qquad A_k(3^n) = 0 \quad \text{for} \quad n \geqslant 2.$$

If $a + b \not\equiv 0 \pmod 3$ then

$$A_k(3) = \frac{(-1)^{k+m}}{2^{k-1}} \cos\frac{\pi m}{3} = \frac{(-1)^k}{2^{k-1}} \cos\frac{2\pi m}{3},$$

$$A_k(3^2) = \sum_{f=1}^{3^2}{}' (-1)^{kbf} \cos\frac{2\pi m f}{9} \cos^k \frac{\pi f}{9}(4a+b).$$

Proof. In the case $a+b\equiv 0 \pmod 3$, Lemma 2.2 implies that

$$A_k(3) \;=\; \frac{1}{2^k}\cdot 2^k\sum_{f=1}^{2}\omega^{-mf} \;=\; \begin{cases} 2, & \text{when } m\equiv 0 \pmod 3,\\ -1, & \text{when } m\not\equiv 0 \pmod 3.\end{cases}$$

It is easy to check that the conditions $a+b\equiv 0 \pmod 3$ and $\left(\dfrac{ab}{3}\right) = -1$ are equivalent, so we have $A_k(3^n)=0$ for any $n\geqslant 2$ by Lemma 2.1. In the case $a+b\not\equiv 0 \pmod 3$, Lemma 2.2 implies:

$$A_k(3) \;=\; \frac{1}{2^k}\sum_{f=1}^{2}\omega^{-mf}(-1)^k \;=\; \frac{(-1)^k}{2^k}\,(\omega^{-m}+\omega^{-2m}) \;=\; \frac{(-1)^k}{2^k}\,e^{-\pi m}\bigl(e^{\frac{\pi i m}{3}} + e^{-\frac{\pi i m}{3}}\bigr) \;=\; \frac{(-1)^{k+m}}{2^k}\,\cos\frac{\pi m}{3}.$$

Finally we have

$$A_k(3^2) \;=\; \frac{1}{6^k}\cdot 6^k\sum_{f=1}^{3^2}{}' e^{-2\pi i\frac{mf}{9}}(-1)^{kbf}\cos^k\frac{\pi f}{9}(4a+b) \;=\; \sum_{f=1}^{3^2}{}'(-1)^{kbf}\cos\frac{2\pi mf}{9}\cos^k\frac{\pi f}{9}(4a+b).$$

Lemma is proved. □

Lemma 2.4. *Suppose that $s\geqslant n\geqslant 2$, $k\geqslant 5$. Then the following inequality holds:*

$$\Bigl|\sum_{r=n}^{s}A_k(3^r)\Bigr| \;<\; \frac{2\cdot 3^{k+n-kn/2-1}}{1-3^{1-k/2}}.$$

Lemma 2.5. *Suppose that $n\geqslant 5$. Then, for any $s\geqslant n$ and for any a,b satisfying the condition $\left(\dfrac{ab}{3}\right)=1$ the following inequality holds:*

$$\Bigl|\sum_{r=n}^{s}A_3(3^r)\Bigr| \;\leqslant\; \begin{cases} \dfrac{3}{16}\cdot 3^{(8-n)/2}, & \textit{when } n\equiv 0 \pmod 2,\\[2mm] \dfrac{5}{16}\cdot 3^{(7-n)/2}, & \textit{when } n\equiv 1 \pmod 2;\end{cases}$$

Lemma 2.6. *Suppose that $n\geqslant 4$. Then*

$$\Bigl|\sum_{r=n}^{s}A_4(3^r)\Bigr| \;\leqslant\; \frac{1}{8}\cdot 3^{5-n}.$$

Lemma 2.7. *Suppose $n\geqslant 3$. Then, for any $s\geqslant n$, the following inequalities hold:*

$$\Bigl|\sum_{r=n}^{s}A_5(3^r)\Bigr| \;\leqslant\; \begin{cases} \dfrac{35}{416}\cdot 3^{3(4-n)/2}, & \textit{when } n\equiv 0 \pmod 2,\\[2mm] \dfrac{55}{416}\cdot 3^{(11-3n)/2}, & \textit{when } n\equiv 1 \pmod 2;\end{cases}$$

Lemmas 2.4, 2.5, 2.6 and 2.7 are particular cases of Lemmas 6.1, 6.3–6.5 from [2], respectively.

3. The singular series $\varkappa_k(a,b,m;q)$ for $q=3^n$

In the cases $q = 2^n, 3^n$, $n = 1,2,3,\ldots$, the behavior of the series $\varkappa_k(a,b,m;q)$ is more sophisticated than in the case $(q,6)=1$. In particular, for "small" k and any n there exist some "exceptional" set $\Omega_k(q)$ of triples (a,b,m) such that $\varkappa_k(a,b,m)=0$. These exceptional sets can be completely described in the case $q=3^n$ and partially in the case $q=2^n$. In what follows, we consider only the case $p=3$ which is more easy.

LEMMA 3.1. *Suppose that* $(ab,3)=1$, $a+b\equiv 0 \pmod 3$. *Then, for any* $n\geqslant 1$ *and* $q=3^n$ *we have*

$$\varkappa_k(a,b,m;q) \;=\; \begin{cases} 0, & \textit{when}\;\; m\not\equiv 0 \pmod 3,\\ 3, & \textit{when}\;\; m\equiv 0 \pmod 3.\end{cases}$$

PROOF. Obviously, the conditions $(ab,3)=1$, $a+b\equiv 0 \pmod 3$ are equivalent to the condition $\left(\dfrac{ab}{3}\right) = -1$. In view of Lemma 2.1, in this case we have $\varkappa_k(q) = 1 + A_k(3)$. By Lemma 2.3, $A_k(3)=-1$ for $m\not\equiv 0 \pmod 3$ and $A_k(3)=2$ for $m\equiv 0 \pmod 3$. Thus lemma follows. □

COROLLARY. *For any* $n\geqslant 1$, *the set* $\Omega_k(3^n)$ *contains all the triples* (a,b,m) *satisfying the following conditions:* $1\leqslant a,b,m\leqslant 3^n$, $(ab,3)=1$, $a+b\equiv 0 \pmod 3$, $m\not\equiv 0 \pmod 3$. *The number of such triples is equal* $4\cdot 3^{3(n-1)}$.

LEMMA 3.2. *Suppose that* $k\geqslant 8$, $(ab,3)=1$ *and* $a+b\not\equiv 0 \pmod 3$. *Then, for any* $n\geqslant 1$, $q=3^n$ *and for any* m, $1\leqslant m\leqslant q$, *one has* $\varkappa_k(a,b,m;q) > \dfrac{1}{50}$.

PROOF. Suppose first that $n=1$. Since $\overline{x}\equiv x \pmod 3$ for any $x\in\mathbb{Z}_3^*$ then $g(x)$ becomes a linear function: $g(x)\equiv a\overline{x}+bx\equiv (a+b)x$. Hence, the congruence (1) is equivalent to

$$(a+b)(x_1+\ldots+x_k)\;\equiv\; m \pmod 3$$

or to

$$x_1+\ldots+x_k\;\equiv\;\mu \pmod 3,\qquad \mu\;\equiv\; m(a+b)^*\;\equiv\; m(a+b) \pmod 3. \tag{4}$$

At the same time, the number of solutions $(x_1,\ldots,x_k)$ of (4) such that $(x_j,3)=1$, $j=1,\ldots,k$, is equal to

$$V_k(3) \;=\; \frac{1}{3}\sum_{c=1}^{3}\sum_{x_1,\ldots,x_k=1}^{2} e^{2\pi i\,\frac{c}{3}(x_1+\ldots+x_k-\mu)} \;=\; \frac{1}{3}\sum_{c=1}^{3}\Bigl(\sum_{x=1}^{2} e^{2\pi i\,\frac{cx}{3}}\Bigr)^k e^{-2\pi i\,\frac{c\mu}{3}} \;=$$

$$=\;\frac{1}{3}\sum_{c=1}^{3}\omega^{-c\mu}\bigl(\omega^c+\omega^{2c}\bigr)^k \;=\; \frac{1}{3}\Bigl(2^k \;+\; (-1)^k\sum_{c=1}^{2}\omega^{-c\mu}\Bigr) \;=\; \frac{1}{3}\left(2^k+(-1)^k\delta\right),$$

where $\omega=e^{2\pi i/3}$,

$$\delta \;=\; \begin{cases} 2, & \text{when}\;\; m\equiv 0 \pmod 3,\\ -1, & \text{when}\;\; m\not\equiv 0 \pmod 3.\end{cases}$$

Thus we obtain

$$V_k(3) \;=\; \frac{\varphi^k(3)}{3}\Bigl(1+\frac{(-1)^k\delta}{2^k}\Bigr),\quad \varkappa_k(3) \;=\; 1+\frac{(-1)^k\delta}{2^k} \;\geqslant\; 1-\frac{1}{2^{k-1}} \;\geqslant\; 1-\frac{1}{16} \;=\; \frac{15}{16}.$$

Further, let $n=2$. Since $\left(\dfrac{ab}{3}\right)=1$, then Lemma 2.2 implies

$$\varkappa_k(3^2) \;=\; 1+\frac{(-1)^{k+m}}{2^k}\cos\frac{\pi m}{3} \;+\; {\sum_{f=1}^{3^2}}'(-1)^{kbf}\cos\Bigl(\frac{2\pi mf}{3}\Bigr)\cos^k\frac{\pi f}{3^2}(4a+b). \tag{5}$$

The direct tabulation over all triples (a,b,m), $1\leqslant a,b,m\leqslant 9$, $(ab,3)=1$ shows that

$$\varkappa_k(3^2)\;>\;0.0351562\ldots\quad\text{for}\;\;k=8,9,\tag{6}$$
$$\varkappa_k(3^2)\;>\;0.0966797\ldots\quad\text{for}\;\;k=10,11,\tag{7}$$
$$\varkappa_k(3^2)\;>\;0.171387\ldots\quad\text{for}\;\;k=12.\tag{8}$$

Moreover, (4) implies the inequality

$$\bigl|\varkappa_k(3^2)-1\bigr|\;\leqslant\;\frac{1}{2^k}\;+\;\sum_{f=1}^{3^2}{}'\Bigl|\cos\frac{\pi f}{3^2}(4a+b)\Bigr|^k.$$

The condition $a+b\not\equiv 0 \pmod 3$ implies that $4a+b\not\equiv 0 \pmod 3$, that is, $(4a+b,3)=1$. Hence, both the quantities f and $(4a+b)f$ run through the reduced residual system modulo 3^2. Thus,

$$\bigl|\varkappa_k(3^2)-1\bigr|\;\leqslant\;\frac{1}{2^k}\;+\;\sum_{f=1}^{3^2}{}'\Bigl|\cos\frac{\pi f}{3^2}\Bigr|^k\;=\;\frac{1}{2^k}\;+\;2\Bigl(\cos^k\frac{\pi}{9}\;+\;\cos^k\frac{2\pi}{9}\;+\;\cos^k\frac{4\pi}{9}\Bigr).\tag{9}$$

Denote the right-hand-side of (9) by $g(k)$. Since $g(k)$ is decreasing function of k then for any $k\geqslant 13$ we have

$$\bigl|\varkappa_k(3^2)-1\bigr|\;=\;g(k)\;\leqslant\;g(13)\;=\;0.953621\ldots,\quad \varkappa_k(3^2)\;>\;0.0463788\ldots\;>\;\frac{1}{22}.$$

The last inequality together with (6)-(8) yields:

$$\varkappa_k(3^2)\;>\;0.0463788\ldots\;>\;\frac{1}{22}\quad\text{for any}\;\;k\geqslant 8.$$

Finally, let $n\geqslant 3$. Then, for $k\geqslant 10$, Lemma 2.4 implies the inequality

$$\Bigl|\sum_{r=3}^{n}A_k(3^r)\Bigr|\;<\;\frac{18\cdot 3^{-k/2}}{1-3^{1-k/2}}.$$

Setting $h(k)$ for right-hand-side and using (9) we find

$$\bigl|\varkappa_k(q)-1\bigr|\;\leqslant\;g(k)+h(k)$$

for any $n\geqslant 3$ and $q=3^n$. Since $h(k)\leqslant h(13)<0.0142895\ldots$ for any $k\geqslant 13$, we get

$$\bigl|\varkappa_k(q)-1\bigr|\;\leqslant\;g(13)+h(13)\;<\;0.967911\ldots,\quad \varkappa_k(q)\;>\;0.0320892\ldots\;>\;\frac{1}{32}.$$

For $10\leqslant k\leqslant 12$, the inequalities (7), (8) together with the bound

$$\varkappa_k(3^n)>\varkappa_k(3^2)-h(k)$$

imply

$$\varkappa_{10}(3^n)\;>\;0.0966797\ldots-0.075\;>\;0.021\;>\;\frac{1}{50},$$
$$\varkappa_{11}(3^n)\;>\;0.0966797\ldots-0.0430737\;>\;0.053606\;>\;\frac{1}{19},$$
$$\varkappa_{12}(3^n)\;>\;0.171387\ldots-0.0247934\;>\;0.1465936\;>\;\frac{1}{7}.$$

To conclude the proof, it remains to consider the cases $k = 8, 9$. By Lemma 2.4,

$$\varkappa_k(3^n) \,=\, \varkappa_k(3^2) \,+\, \sum_{r=4}^{n} A_k(3^r) \,\geqslant\, \varkappa_k(3^2) \,-\, \frac{54\cdot 3^{-k}}{1-3^{1-k/2}}.$$

The direct calculation with all triples (a, b, m), $1 \leqslant a, b, m \leqslant 3^3$, $(ab, 3) = 1$, $a + b \not\equiv 0 \pmod 3$ shows that

$$\varkappa_k(3^2) \,>\, 0.0347222\ldots, \quad k = 8, 9.$$

Therefore,

$$\varkappa_8(3^n) \,>\, 0.0347222\ldots - 0.00854701 \,>\, 0.026175 \,>\, \frac{1}{39},$$
$$\varkappa_9(3^n) \,>\, 0.0347222\ldots - 0.00280343 \,>\, 0.031918 \,>\, \frac{1}{32}.$$

This proves the lemma. □

Corollary. *For any $k \geqslant 8$, the exceptional set $\Omega_k(3^n)$ consists precisely of the triples pointed in Corollary of Lemma* 3.1.

Now we proceed to study the values of $\varkappa_k(a, b, m; 3^n)$ for $3 \leqslant k \leqslant 7$ and $n \geqslant 2$. In view of Corollary to Lemma 3.1, we will assume that $a + b \not\equiv 0 \pmod 3$ or, that is the same, $a \equiv b \pmod 3$.

Lemma 3.3. *Suppose that $3 \leqslant k \leqslant 7$. Then the set $\Omega_k(3^2)$ contains $18(8 - k)$ triples (a, b, m), $a \equiv b \pmod 3$, $(ab, 3) = 1$ listed in* (10)-(13).

Proof. Indeed, let $\mathcal{G} = \mathcal{G}_{a,b} = \mathcal{G}_{b,a}$ be the set of values of the function $g(x) = a\overline{x} + bx$ on $\mathbb{Z}_9^*$. The direct computation shows that

$$\begin{aligned}
&\mathcal{G} = \{1, 8\} \quad \text{for the pairs } (a; b) \in \mathcal{E}_1, \quad \mathcal{E}_1 \,=\, \bigl\{(1; 7), (7; 1), (2; 8), (8; 2), (4; 4), (5; 5)\bigr\},\\
&\mathcal{G} = \{2, 7\} \quad \text{for the pairs } (a; b) \in \mathcal{E}_2, \quad \mathcal{E}_2 \,=\, \bigl\{(1; 1), (2; 5), (5; 2), (4; 7), (7; 4), (8; 8)\bigr\},\\
&\mathcal{G} = \{4, 5\} \quad \text{for the pairs } (a; b) \in \mathcal{E}_3, \quad \mathcal{E}_3 \,=\, \bigl\{(1; 4), (4; 1), (2; 2), (5; 8), (8; 5), (7; 7)\bigr\}.
\end{aligned}$$

Also, the direct computation shows that the sets $k\mathcal{G} = \underbrace{\mathcal{G} + \ldots + \mathcal{G}}_{k \text{ times}}$ do not coincide with complete residual system $\mathbb{Z}_9$ for $3 \leqslant k \leqslant 7$. Thus, the sets $3\mathcal{G}$ have the forms

$$\mathbb{Z}_9 \setminus \bigl\{0, 2, 4, 5, 7\bigr\}, \quad \mathbb{Z}_9 \setminus \bigl\{0, 1, 4, 5, 8\bigr\}, \quad \mathbb{Z}_9 \setminus \bigl\{0, 1, 2, 7, 8\bigr\}$$

respectively, the sets $4\mathcal{G}$ have the form

$$\mathbb{Z}_9 \setminus \bigl\{1, 3, 6, 8\bigr\}, \quad \mathbb{Z}_9 \setminus \bigl\{2, 3, 6, 7\bigr\}, \quad \mathbb{Z}_9 \setminus \bigl\{3, 4, 5, 6\bigr\},$$

respectively, the sets $5\mathcal{G}$ have the form

$$\mathbb{Z}_9 \setminus \bigl\{0, 2, 7\bigr\}, \quad \mathbb{Z}_9 \setminus \bigl\{0, 4, 5\bigr\}, \quad \mathbb{Z}_9 \setminus \bigl\{0, 1, 8\bigr\},$$

respectively, the sets $6\mathcal{G}$ have the form

$$\mathbb{Z}_9 \setminus \bigl\{1, 8\bigr\}, \quad \mathbb{Z}_9 \setminus \bigl\{2, 7\bigr\}, \quad \mathbb{Z}_9 \setminus \bigl\{4, 5\bigr\},$$

respectively, and, finally, all the sets $7\mathcal{G}$ coincide with $\mathbb{Z}_9 \setminus \bigl\{0\bigr\}$. Hence, $\varkappa_3(3^2) = 0$ for the triples

$$\begin{aligned}
&(a, b, 0),\ (a, b, 2),\ (a, b, 4),\ (a, b, 5),\ (a, b, 7) \quad (a; b) \in \mathcal{E}_1,\\
&(a, b, 0),\ (a, b, 1),\ (a, b, 4),\ (a, b, 5),\ (a, b, 8) \quad (a; b) \in \mathcal{E}_2,\\
&(a, b, 0),\ (a, b, 1),\ (a, b, 2),\ (a, b, 7),\ (a, b, 8) \quad (a; b) \in \mathcal{E}_3,
\end{aligned} \tag{10}$$

$\varkappa_4(3^2)=0$ for the triples

$$\begin{aligned} &(a,b,1),\ (a,b,3),\ (a,b,6),\ (a,b,8), \quad (a;b)\in\mathcal{E}_1,\\ &(a,b,2),\ (a,b,3),\ (a,b,6),\ (a,b,7), \quad (a;b)\in\mathcal{E}_2,\\ &(a,b,3),\ (a,b,4),\ (a,b,5),\ (a,b,6), \quad (a;b)\in\mathcal{E}_3, \end{aligned} \tag{11}$$

$\varkappa_5(3^2)=0$ for the triples

$$\begin{aligned} &(a,b,0),\ (a,b,2),\ (a,b,7), \quad (a;b)\in\mathcal{E}_1,\\ &(a,b,0),\ (a,b,4),\ (a,b,5), \quad (a;b)\in\mathcal{E}_2,\\ &(a,b,0),\ (a,b,1),\ (a,b,8), \quad (a;b)\in\mathcal{E}_3, \end{aligned} \tag{12}$$

$\varkappa_6(3^2)=0$ for the triples

$$\begin{aligned} &(a,b,1),\ (a,b,8), \quad (a;b)\in\mathcal{E}_1,\\ &(a,b,2),\ (a,b,7), \quad (a;b)\in\mathcal{E}_2,\\ &(a,b,4),\ (a,b,5), \quad (a;b)\in\mathcal{E}_3, \end{aligned} \tag{13}$$

and, finally, $\varkappa_7(3^2)=0$ for the triples $(a,b,0)$, where $(a;b)\in\mathcal{E}_1\cup\mathcal{E}_2\cup\mathcal{E}_3$. It is not difficult to check that the number of such triples coincides with $18(8-k)$ in each case. This proves the lemma. □

Corollary. F*or any k, $3\leqslant k\leqslant 7$, we have*

$$\bigl|\Omega_k(3^2)\bigr| \,=\, 18(14-k).$$

Proof. This assertion easily follows from the above lemma and the Corollary of Lemma 3.1. □

Suppose now that $n\geqslant 3$ and the triple (a,b,m) with the conditions $1\leqslant a,b,m\leqslant 3^n$, $(ab,3)=1$ is congruent with some exceptional triple $(a',b',m')\in\Omega_k(3^2)$ modulo 3^2. Then the triple (a,b,m) is contained in the exceptional set $\Omega_k(3^n)$. Indeed, if the congruence

$$g(x_1)+\ldots+g(x_k)\,\equiv\, m \pmod{3^n},\quad g(x)=a\overline{x}+bx,$$

is solvable, then the congruence

$$g'(x_1)+\ldots+g'(x_k)\,\equiv\, m \pmod{3^2},\quad g'(x)=a'x^*+b'x,$$

is also solvable, and that is impossible (here $x\overline{x}\equiv 1 \pmod{3^n}$, $xx^*\equiv 1 \pmod{3^2}$, so $\overline{x}\equiv x^*$ $(\mathrm{mod}\ 3^2)$). The below lemma shows that there are no other triples in $\Omega_k(3^n)$.

Lemma 3.4. *Suppose that $n\geqslant 3$, $3\leqslant k\leqslant 7$. Then there exists an absolute constant $c_4>0$ such that the inequality*

$$\varkappa_k(3^n)\,=\,\varkappa_k(a,b,m,3^n)\,>\,c_4$$

holds for any triple (a,b,m), $1\leqslant a,b,m\leqslant 3^n$, $(ab,3)=1$, that does not coincide with some exceptional triple from the set $\Omega_k(3^2)$ modulo 3^2.

Proof. By Lemma 2.4,

$$\bigl|\varkappa_k(3^n)\,-\,\varkappa_k(3^3)\bigr|\,<\,\frac{54\cdot 3^{-k}}{1-3^{1-k/2}}\ \text{ and therefore }\ \varkappa_k(3^n)\,>\,\varkappa_k(3^3)\,-\,\frac{54\cdot 3^{-k}}{1-3^{1-k/2}}.$$

In particular,

$$\varkappa_6(3^n) > \varkappa_6(3^3) - \frac{1}{12}, \quad \varkappa_7(3^n) > \varkappa_7(3^3) - \frac{27+\sqrt{3}}{33^2} > \varkappa_7(3^3) - \frac{1}{37}. \tag{14}$$

The direct calculation with all triples (a, b, m), $1 \leqslant a, b, m \leqslant 3^3$, $(ab, 3) = 1$, that are not congruent with triples from $\Omega_k(3^k)$ modulo 3^2 $(k = 6, 7)$ shows that the least values of $\varkappa_6(3^3)$ and $\varkappa_7(3^3)$ are equal to $\dfrac{25}{192}$ and $\dfrac{13}{192}$. Thus, by (14) we conclude that

$$\varkappa_6(3^n) > \frac{25}{192} - \frac{1}{12} = \frac{3}{64}, \quad \varkappa_7(3^n) > \frac{13}{192} - \frac{1}{37} > \frac{1}{25}.$$

The calculations also shows that

$$\varkappa_5(3^n) \geqslant \frac{1}{4} \qquad \text{for } n = 3, \tag{15}$$

$$\varkappa_4(3^n) \geqslant \frac{7}{16} \qquad \text{for } n = 3, \tag{16}$$

$$\varkappa_3(3^n) \geqslant \frac{3}{4} \qquad \text{for } n = 3, 4, 5 \tag{17}$$

for any triple (a, b, m), $1 \leqslant a, b, m \leqslant 3^n$, $(ab, 3) = 1$ that does not coincide with some triple from the set $\Omega_k(3^2)$ modulo 3^2 $(3 \leqslant k \leqslant 5)$. At the same time, the inequalities of Lemmas 2.7, 2.6 and 2.5 imply that

$$\big|\varkappa_5(3^n) - \varkappa_5(3^3)\big| \leqslant \frac{35}{416} \qquad \text{for } n \geqslant 4, \tag{18}$$

$$\big|\varkappa_4(3^n) - \varkappa_4(3^3)\big| \leqslant \frac{3}{8} \qquad \text{for } n \geqslant 4, \tag{19}$$

$$\big|\varkappa_3(3^n) - \varkappa_3(3^5)\big| \leqslant \frac{9}{16} \qquad \text{for } n \geqslant 6. \tag{20}$$

Thus the relations (15)-(20) yield:

$$\varkappa_5(3^n) \geqslant \varkappa_5(3^3) - \frac{35}{416} \geqslant \frac{1}{4} - \frac{35}{416} > \frac{1}{7} \qquad \text{for } n \geqslant 4,$$

$$\varkappa_4(3^n) \geqslant \varkappa_4(3^3) - \frac{3}{8} \geqslant \frac{7}{16} - \frac{3}{8} > \frac{1}{16} \qquad \text{for } n \geqslant 4,$$

$$\varkappa_3(3^n) \geqslant \varkappa_5(3^5) - \frac{9}{16} \geqslant \frac{3}{4} - \frac{9}{16} > \frac{3}{16} \qquad \text{for } n \geqslant 6.$$

Taking $c_4 = \dfrac{1}{37}$, we arrive at the assertion of the lemma. □

Corollary. *For any* $3 \leqslant k \leqslant 7$, $n \geqslant 2$ *one has*

$$|\Omega_k(3^n)| = 3^{3(n-2)} \cdot 18(14-k) = 2 \cdot 3^{3n-4}(14-k).$$

This assertion finishes the description of the quantities $\varkappa_k(3^n)$, $k \geqslant 3$, $n \geqslant 1$.

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