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Некоммутативная теорема Бялыницкого — Бирули[1]

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Аннотация

Изучение действий алгебраических групп на алгебраических многообразиях и их координатных алгебрах является важной областью исследований в алгебраической геометрии и теории колец. Эта область связана с теорией полиномиальных отображений, ручных и диких автоморфизмов, проблемой якобиана, теорией бесконечномерных многообразий по Шафаревичу, проблемой сокращения (вместе с другими подобными вопросами), теорией локально нильпотентных дифференцирований. Одной из центральных задач теории действий алгебраических групп является проблема линеаризации, изученная в работе Т. Камбаяши и П. Расселла, утверждающая, что всякое действие тора на аффинном пространстве линейно в некоторой системе координат. Гипотеза о линеаризации была основана на хорошо известной классической теореме А. Бялыницкого — Бирули, которая гласит, что всякое эффективное регулярное действие тора максимальной размерности на аффинном пространстве над алгебраически замкнутым полем допускает линеаризацию.

Несмотря на то что гипотеза о линеаризации оказалась отрицательной в ее общем виде — контрпримеры в положительной характеристике были построены Т. Асанума — теорема Бялыницкого — Бирули остается важным результатом теории благодаря ее связи с теорией полиномиальных автоморфизмов. Недавние продвижения в последней мотивировали поиск различных некоммутативных разновидностей теоремы Бялыницкого — Бирули. В данной статье мы приведем доказательство теоремы о линеаризации эффективного действия максимального тора автоморфизмами свободной ассоциативной алгебры, являющейся таким образом свободным аналогом теоремы Бялыницкого — Бирули.

Ключевые слова: действия тора, задача линеаризации, полиномиальные автоморфизмы.

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Noncommutative Białynicki–Birula Theorem

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Abstract

The study of algebraic group actions on varieties and coordinate algebras is a major area of research in algebraic geometry and ring theory. The subject has its connections with the theory of polynomial mappings, tame and wild automorphisms, the Jacobian conjecture of O.-H. Keller, infinite-dimensional varieties according to Shafarevich, the cancellation problem (together with various cancellation-type problems), the theory of locally nilpotent derivations, among other topics. One of the central problems in the theory of algebraic group actions has been the linearization problem, formulated and studied in the work of T. Kambayashi and P. Russell, which states that any algebraic torus action on an affine space is always linear with respect to some coordinate system. The linearization conjecture was inspired by the classical and well known result of A. Białynicki–Birula, which states that every effective regular torus action of maximal dimension on the affine space over algebaically closed field is linearizable.

Although the linearization conjecture has turned out negative in its full generality, according to, among other results, the positive-characteristic counterexamples of T. Asanuma, the Białynicki–Birula has remained an important milestone of the theory thanks to its connection to the theory of polynomial automorphisms. Recent progress in the latter area has stimulated the search for various noncommutative analogues of the Białynicki–Birula theorem. In this paper, we give the proof of the linearization theorem for effective maximal torus actions by automorphisms of the free associative algebra, which is the free analogue of the Białynicki–Birula theorem.

Keywords: torus actions, linearization problem, polynomial automorphisms.

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1. Introduction

In this paper we consider algebraic torus actions on the affine space, according to Białynicki-Birula, and formulate certain noncommutative generalizations.

We begin by recalling a few basic definitions. Let $\mathbb{K}$ be an algebraically closed field.

DEFINITION 1. *An algebraic group is a variety G equipped with the structure of a group, such that the multiplication map $m : G \times G \to G : (g_1, g_2) \mapsto g_1 g_2$ and the inverse map $\iota : G \to G : g \mapsto g^{-1}$ are morphisms of varieties.*

DEFINITION 2. *A G-variety is a variety equipped with an action of the algebraic group G,*

$$\alpha : G \times X \to X : (g, x) \mapsto g \cdot x,$$

which is also a morphism of varieties. We then say that α is an algebraic G-action.

Let $\mathbb{K}$ be our ground field, which is assumed to be algebraically closed. Let $Z = \{z_1, z_2, \ldots\} = \{z_i : i \in I\}$ be a finite or a countable set of variables (where $I = \{1, 2, \ldots\}$ is an index set), and let Z^* denote the free semigroup generated by Z, $Z^+ = Z^* \backslash \{1\}$. Moreover let $F_I(\mathbb{K}) = \mathbb{K}\langle Z\rangle$ be the free associative $\mathbb{K}$-algebra and $\hat{F}_I(\mathbb{K}) = \mathbb{K}\langle\langle Z\rangle\rangle$ be the algebra of formal power series in free variables.

Denote by $\mathcal{W} = \langle Z\rangle$ the free monoid of words over the alphabet Z (with 1 as the empty word) such that $|\mathcal{W}| \geqslant 1$, for $|\mathcal{W}|$ the length of the word $\mathcal{W} \in Z^+$.

For an alphabet Z, the free associative $\mathbb{K}$-algebra on Z is

$$\mathbb{K}\langle Z\rangle := \oplus_{\mathcal{W} \in Z^*} \mathbb{K}\mathcal{W},$$

where the multiplication is $\mathbb{K}$-bilinear extension of the concatenation on words, Z^* denotes the free monoid on Z, and $\mathbb{K}\mathcal{W}$ denotes the free $\mathbb{K}$-module on one element, the word $\mathcal{W}$. Any element of $\mathbb{K}\langle Z\rangle$ can thus be written uniquely in the form

$$\sum_{k=0}^{\infty} \sum_{i_1, \ldots, i_k \in I} a_{i_1, i_2, \ldots, i_k} z_{i_1} z_{i_2} \ldots z_{i_k},$$

where the coefficients $a_{i_1, i_2, \ldots, i_k}$ are elements of the field $\mathbb{K}$ and all but finitely many of these elements are zero.

In our context, the alphabet Z is the same as the set of algebra generators, therefore the terms "monomial"and "word"will be used interchangeably.

In the sequel, we employ a (slightly ambiguous) short-hand notation for a free algebra monomial. For an element z, its powers are defined as usual. Any monomial $z_{i_1} z_{i_2} \ldots z_{i_k}$ can then be written in a reduced form with subwords $zz \ldots z$ replaced by powers.

We then write

$$z^I = z_{j_1}^{i_1} z_{j_2}^{i_2} \ldots z_{j_k}^{i_k}$$

where by I we mean an assignment of i_k to j_k in the word z^I. Sometimes we refer to I as a multi-index, although the term is not entirely accurate. If I is such a multi-index, its abosulte value $|I|$ is defined as the sum $i_1 + \cdots + i_k$.

For a field $\mathbb{K}$, let $\mathbb{K}^\times = \mathbb{K} \backslash \{0\}$ denote the multiplicative group of its non-zero elements viewed as an algebraic $\mathbb{K}$-group.

DEFINITION 3. *An n-dimensional algebraic $\mathbb{K}$-torus is a group*

$$\mathbb{T}_n \simeq (\mathbb{K}^\times)^n$$

(with obvious multiplication).

Denote by $\mathbb{A}^n$ the affine space of dimension n over $\mathbb{K}$.

Definition 4. *A (left) torus action is a morphism*

$$\sigma : \mathbb{T}_n \times \mathbb{A}^n \to \mathbb{A}^n.$$

that fulfills the usual axioms (identity and compatibility):

$$\sigma(1, x) = x, \ \ \sigma(t_1, \sigma(t_2, x)) = \sigma(t_1 t_2, x).$$

The action σ is effective *if for every $t \neq 1$ there is an element $x \in \mathbb{A}^n$ such that $\sigma(t, x) \neq x$.*

In [3], Białynicki-Birula proved the following two theorems.

Theorem 1. *Any regular action of $\mathbb{T}_n$ on $\mathbb{A}^n$ has a fixed point.*

Theorem 2. *Any effective and regular action of $\mathbb{T}_n$ on $\mathbb{A}^n$ is a representation in some coordinate system.*

The term "regular"is to be understood here as in the algebro-geometric context of regular function (Białynicki-Birula also considered birational actions). The last theorem says that any effective regular maximal torus action on the affine space is conjugate to a linear action, or, as it is sometimes called, *linearizable.*

An algebraic group action on $\mathbb{A}^n$ is the same as an action by automorphisms on the algebra

$$\mathbb{K}[x_1, \ldots, x_n]$$

of global sections of the structure sheaf. In other words, it is a homomorphism

$$\sigma : \mathbb{T}_n \to \operatorname{Aut} \mathbb{K}[x_1, \ldots, x_n].$$

An action is effective iff $\operatorname{Ker} \sigma = \{1\}$.

The polynomial algebra is a quotient of the free associative algebra

$$F_n = \mathbb{K}\langle z_1, \ldots, z_n \rangle$$

by the commutator ideal I (it is the two-sided ideal generated by all elements of the form $fg - gf$). From the standpoint of Noncommutative geometry, the algebra $\Gamma(X, \mathcal{O}_X)$ of global sections (along with the category of f.g. projective modules) contains all the relevant topological data of X, and various non-commutative algebras (PI-algebras) may be thought of as global function algebras over "noncommutative spaces". Therefore, noncommutative analogue of the Białynicki-Birula theorem is a subject of legitimate interest.

In this short note we establish the free algebra version of the Białynicki-Birula theorem. The latter is formulated as follows.

Theorem 3 (Main Theorem). *Suppose given a regular action σ of the algebraic n-torus $\mathbb{T}_n$ on the free algebra F_n. If σ is effective, then it is linearizable.*

The linearization problem, as it has become known since Kambayashi, asks whether all (effective, regular) actions of a given type of algebraic groups on the affine space of given dimension are conjugate to representations. According to Theorem 3, the linearization problem extends to the noncommutative category. Several known results concerning the (commutative) linearization problem are summarized below.

1. Any effective regular torus action on $\mathbb{A}^2$ is linearizable (Gutwirth [8]).

2. Any effective regular torus action on $\mathbb{A}^n$ has a fixed point (Białynicki-Birula [3]).

3. Any effective regular action of $\mathbb{T}_{n-1}$ on $\mathbb{A}^n$ is linearizable (Białynicki-Birula [4]).

4. Any (effective, regular) one-dimensional torus action (i.e., action of $\mathbb{K}^\times$) on $\mathbb{A}^3$ is linearizable (Koras and Russell [14]).

5. If the ground field is not algebraically closed, then a torus action on $\mathbb{A}^n$ need not be linearizable. In [1], Asanuma proved that over any field $\mathbb{K}$, if there exists a non-rectifiable closed embedding from $\mathbb{A}^m$ into $\mathbb{A}^n$, then there exist non-linearizable effective actions of $(\mathbb{K}^\times)^r$ on $\mathbb{A}^{1+n+m}$ for $1 \leqslant r \leqslant 1+m$.

6. When $\mathbb{K}$ is infinite and has positive characteristic, there are examples of non-linearizable torus actions on $\mathbb{A}^n$ (Asanuma [1]).

Remark 1. *A closed embedding $\iota : \mathbb{A}^m \to \mathbb{A}^n$ is said to be rectifiable if it is conjugate to a linear embedding by an automorphism of $\mathbb{A}^n$.*

As can be inferred from the review above, the context of the linearization problem is rather broad, even in the case of torus actions. The regulating parameters are the dimensions of the torus and the affine space. This situation is due to the fact that the general form of the linearization conjecture (i.e., the conjecture that states that any effective regular torus action on any affine space is linearizable) has a negative answer.

Transition to the noncommutative geometry presents the inquirer with an even broader context: one now may vary the dimensions as well as impose restrictions on the action in the form of preservation of the PI-identities. Caution is well advised. Some of the results are generalized in a straightforward manner — the main theorem of this paper being the typical example, others require more subtlety and effort (cf. 2 and the discussion at the end of the note). Of some note to us, given our ongoing work in deformation quantization (see, for instance, [12]) is the following instance of the linearization problem, which we formulate as a conjecture.

Conjecture 1. *For $n \geq 1$, let P_n denote the commutative Poisson algebra, i.e. the polynomial algebra*

$$\mathbb{K}[z_1, \ldots, z_{2n}]$$

equipped with the Poisson bracket defined by

$$\{z_i, z_j\} = \delta_{i,n+j} - \delta_{i+n,j}.$$

Then any effective regular action of $\mathbb{T}_n$ by automorphisms of P_n is linearizable.

It is interesting to note that the context of Conjecture 1 admits a vague analogy in the real transcendental category (with P_n replaced by an appropriate algebra of smooth functions, cf. for instance the work of Zung [16]). Although the instances of the linearization problem we consider in this note, as well as the original theorem of Białynicki-Birula, are essentially of complex algebraic nature, it may be worthwhile to search for analytic analogues of the real transcendental linearization (however whether this will give a feasible approach to Conjecture 1 is unclear, the hurdles being significant and fairly obvious).

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The main result of this note was conceived in the prior work [13] of A. K.-B., J.-T. Y. and A. E.. Theorem 3 is due to A. E. and A. K.-B.; Lemma 1 and the review of known results for the linearization problem is due to F. R., J.-T. Y. and W. Z..

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2. Proof of the Main Theorem

The proof proceeds along the lines of the original commutative case proof of Białynicki-Birula.

If σ is the effective action of Theorem 3, then for each $t \in \mathbb{T}_n$ the automorphism

$$\sigma(t) : F_n \to F_n$$

is given by the n-tuple of images of the generators $z_1, \ldots, z_n$ of the free algebra:

$$(f_1(t, z_1, \ldots, z_n), \ldots, f_n(t, z_1, \ldots, z_n)).$$

Each of the $f_1, \ldots, f_n$ is a polynomial in the free variables.

Lemma 1. *There is a translation of the free generators*

$$(z_1, \ldots, z_n) \to (z_1 - c_1, \ldots, z_n - c_n), \quad (c_i \in \mathbb{K})$$

such that (for all $t \in \mathbb{T}_n$) the polynomials $f_i(t, z_1 - c_1, \ldots, z_n - c_n)$ have zero free part.

Proof. This is a direct corollary of Theorem 1. Indeed, any action σ on the free algebra induces, by taking the canonical projection with respect to the commutator ideal I, an action $\bar{\sigma}$ on the commutative algebra $\mathbb{K}[x_1, \ldots, x_n]$. If σ is regular, then so is $\bar{\sigma}$. By Theorem 1, $\bar{\sigma}$ (or rather, its geometric counterpart) has a fixed point, therefore the images of commutative generators x_i under $\bar{\sigma}(t)$ (for every t) will be polynomials with trivial degree-zero part. Consequently, the same will hold for σ. □

We may then suppose, without loss of generality, that the polynomials f_i have the form

$$f_i(t, z_1, \ldots, z_n) = \sum_{j=1}^{n} a_{ij}(t) z_j + \sum_{j,l=1}^{n} a_{ijl}(t) z_j z_l + \sum_{k=3}^{N} \sum_{J,|J|=k} a_{i,J}(t) z^J$$

where by z^J we denote, as in the introduction, a particular monomial

$$z_{i_1}^{k_1} z_{i_2}^{k_2} \ldots$$

(a word in the alphabet $\{z_1, \ldots, z_n\}$ in the reduced notation; J is the multi-index in the sense described above); also, N is the degree of the automorphism (which is finite) and $a_{ij}, a_{ijl}, \ldots$ are polynomials in $t_1, \ldots, t_n$.

As σ_t is an automorphism, the matrix $[a_{ij}]$ that determines the linear part is non-singular. Therefore, without loss of generality we may assume it to be diagonal (just as in the commutative case [3]) of the form

$$\operatorname{diag}(t_1^{m_{11}} \ldots t_n^{m_{1n}}, \ldots, t_1^{m_{n1}} \ldots t_n^{m_{nn}}).$$

Now, just as in [3], we have the following

Lemma 2. *The power matrix $[m_{ij}]$ is non-singular.*

Proof. Consider a linear action τ defined by

$$\tau(t):(z_1,\dots,z_n)\mapsto(t_1^{m_{11}}\dots t_n^{m_{1n}}z_1,\dots,t_1^{m_{n1}}\dots t_n^{m_{nn}}z_n),\ \ (t_1,\dots,t_n)\in\mathbb{T}_n.$$

If $T_1\subset T_n$ is any one-dimensional torus, the restriction of τ to $\mathbb{T}_1$ is non-trivial. Indeed, were it to happen that for some $\mathbb{T}_1$,

$$\tau(t)z=z,\ \ t\in\mathbb{T}_1,\ \ (z=(z_1,\dots,z_n))$$

then our initial action σ, whose linear part is represented by τ, would be identity modulo terms of degree >1:

$$\sigma(t)(z_i)=z_i+\sum_{j,l}a_{ijl}(t)z_jz_l+\cdots.$$

Now, equality $\sigma(t^2)(z)=\sigma(t)(\sigma(t)(z))$ implies

$$\begin{aligned}\sigma(t)(\sigma(t)(z_i))&=\sigma(t)\left(z_i+\sum_{jl}a_{ijl}(t)z_jz_l+\cdots\right)\\&=z_i+\sum_{jl}a_{ijl}(t)z_jz_l+\sum_{jl}a_{ijl}(t)(z_j+\sum_{km}a_{jkm}(t)z_kz_m+\cdots)\\&\quad(z_l+\sum_{k'm'}a_{lk'm'}(t)z_{k'}z_{m'}+\cdots)+\cdots\\&=z_i+\sum_{jl}a_{ijl}(t^2)z_jz_l+\cdots\end{aligned}$$

which means that

$$2a_{ijl}(t)=a_{ijl}(t^2)$$

and therefore $a_{ijl}(t)=0$. The coefficients of the higher-degree terms are processed by induction (on the total degree of the monomial). Thus

$$\sigma(t)(z)=z,\ \ t\in\mathbb{T}_1$$

which is a contradiction since σ is effective. Finally, if $[m_{ij}]$ were singular, then one would easily find a one-dimensional torus such that the restriction of τ were trivial. □

Consider the action

$$\varphi(t)=\tau(t^{-1})\circ\sigma(t).$$

The images under $\varphi(t)$ are

$$(g_1(z,t),\dots,g_n(z,t)),\ \ (t=(t_1,\dots,t_n))$$

with

$$g_i(z,t)=\sum g_{i,m_1\dots m_n}(z)t_1^{m_1}\dots t_n^{m_n},\ \ m_1,\dots,m_n\in\mathbb{Z}.$$

Define $G_i(z)=g_{i,0\dots0}(z)$ and consider the map $\beta:F_n\to F_n$,

$$\beta:(z_1,\dots,z_n)\mapsto(G_1(z),\dots,G_n(z)).$$

Lemma 3. *$\beta\in\operatorname{Aut}F_n$ and*

$$\beta=\tau(t^{-1})\circ\beta\circ\sigma(t).$$

Proof. This lemma mirrors the final part in the proof in [3]. The conjugation is straightforward, since for every $s, t \in \mathbb{T}_n$ one has

$$\varphi(st) = \tau(t^{-1}s^{-1}) \circ \sigma(st) = \tau(t^{-1}) \circ \tau(s^{-1}) \circ \sigma(s) \circ \sigma(t) = \tau(t^{-1}) \circ \varphi(s) \circ \sigma(t).$$

Denote by $\hat{F}_n$ the power series completion of the free algebra F_n, and let $\hat{\sigma}$, $\hat{\tau}$ and $\hat{\beta}$ denote the endomorphisms of the power series algebra induced by corresponding morphisms of F_n. The endomorphisms $\hat{\sigma}$, $\hat{\tau}$, $\hat{\beta}$ come from (polynomial) automorphisms and therefore are invertible.

Let

$$\hat{\beta}^{-1}(z_i) \equiv B_i(z) = \sum_J b_{i,J} z^J$$

(just as before, z^J is the monomial with multi-index J). Then

$$\hat{\beta} \circ \hat{\tau}(t) \circ \hat{\beta}^{-1}(z_i) = B_i(t_1^{m_{11}} \dots t_n^{m_{1n}} G_1(z), \dots, t_1^{m_{n1}} \dots t_n^{m_{nn}} G_n(z)).$$

Now, from the conjugation property we must have

$$\hat{\beta} = \hat{\sigma}(t^{-1}) \circ \hat{\beta} \circ \hat{\tau}(t),$$

therefore $\hat{\sigma}(t) = \hat{\beta} \circ \hat{\tau}(t) \circ \hat{\beta}^{-1}$ and

$$\hat{\sigma}(t)(z_i) = \sum_J b_{i,J} (t_1^{m_{11}} \dots t_n^{m_{1n}})^{j_1} \dots (t_1^{m_{n1}} \dots t_n^{m_{nn}})^{j_n} G(z)^J;$$

here the notation $G(z)^J$ stands for a word in $G_i(z)$ with multi-index J, while the exponents $j_1, \dots, j_n$ count how many times a given index appears in J (or, equivalently, how many times a given generator z_i appears in the word z^J).

Therefore, the coefficient of $\hat{\sigma}(t)(z_i)$ at z^J has the form

$$b_{i,J} (t_1^{m_{11}} \dots t_n^{m_{1n}})^{j_1} \dots (t_1^{m_{n1}} \dots t_n^{m_{nn}})^{j_n} + S$$

with S a finite sum of monomials of the form

$$c_L (t_1^{m_{11}} \dots t_n^{m_{1n}})^{l_1} \dots (t_1^{m_{n1}} \dots t_n^{m_{nn}})^{l_n}$$

with $(j_1, \dots, j_n) \neq (l_1, \dots, l_n)$. Since the power matrix $[m_{ij}]$ is non-singular, if $b_i, J \neq 0$, we can find a $t \in \mathbb{T}_n$ such that the coefficient is not zero. Since σ is an algebraic action, the degree

$$\sup_t \deg(\hat{\sigma})$$

is a finite integer N. With the previous statement, this implies that

$$b_{i,J} = 0, \quad \text{whenever} \quad |J| > N.$$

Therefore, $B_i(z)$ are polynomials in the free variables. What remains is to notice that

$$z_i = B_i(G_1(z), \dots, G_n(z)).$$

Thus β is an automorphism. □

From Lemma 3 it follows that

$$\tau(t) = \beta^{-1} \circ \sigma(t) \circ \beta$$

which is the linearization of σ. Theorem 3 is proved.

3. Discussion

The noncommutative torus action linearization theorem that we have proved has several useful applications. In the work [13] (cf. also [7]), it is used to investigate the properties of the group $\operatorname{Aut} F_n$ of automorphisms of the free algebra. As a corollary of Theorem 3, one gets

Corollary 1. *Let θ denote the standard action of $\mathbb{T}_n$ on $K[x_1, \dots, x_n]$ — i.e., the action*

$$\theta_t : (x_1, \dots, x_n) \mapsto (t_1 x_1, \dots, t_n x_n).$$

Let $\tilde{\theta}$ denote its lifting to an action on the free associative algebra F_n. Then $\tilde{\theta}$ is also given by the standard torus action.

This statement plays a part, along with a number of results concerning the induced formal power series topology on $\operatorname{Aut} F_n$, in the establishment of the following proposition (cf. [13]).

Proposition 1. *When $n \geq 3$, any* Ind-*scheme automorphism φ of* $\operatorname{Aut}(K\langle x_1, \dots, x_n\rangle)$ *is inner.*

One could try and generalize the free algebra version of the Białynicki-Birula's theorem to other noncommutative situations. Another way of generalization lies in changing the dimension of the torus. In a complete analogy with further work of Białynicki-Birula [4], we expect the following to hold.

Conjecture 2. *Any effective action of $\mathbb{T}_{n-1}$ on F_n is linearizable.*

On the other hand, there is little reason to expect this statement to hold with further lowering of the torus dimension. In fact, even in the commutative case the conjecture that any effective toric action is linearizable, in spite of considerable effort (see [9]), proved negative (counterexamples in positive characteristic due to Asanuma, [1]).

Another direction would be to replace $\mathbb{T}$ by an arbitrary reductive algebraic group, however the commutative case also does not hold even in characteristic zero (cf. [15]).

It is also a problem of legitimate interest to obtain the proof of Conjecture 1 — i.e. to resolve the linearization problem of the regular action of the n-dimensional torus on the group $\operatorname{Sympl}(k^{2n})$ of polynomial symplectomorphisms of the $2n$-dimensional affine space (k is a field of characteristic zero). One could hope to utilize the latter result in order to obtain a description of the space of Ind-scheme automorphisms of $\operatorname{Sympl}(k^{2n})$ along the lines of [13]. This space plays a prominent role in the study of quantization of symplectomorphisms, initiated by Kanel-Belov and Kontsevich [2], where the characteristic zero isomorphism between the group of automorphisms of the n-th Poisson and Weyl algebras has been posed as the main conjecture (Kontsevich conjecture). Recently, the first, the second and the fourth named authors have proposed a proof of this conjecture [10, 11].

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. T. Asanuma. Non-linearizable k^*-actions in affine space // Invent. Math. 1999, Vol. 138. P. 281-306.

2. A. Belov-Kanel and M. Kontsevich. Automorphisms of the Weyl algebra // Lett. Math. Phys. 2005, Vol. 74, No.2. P. 181-199.

3. A. Białynicki-Birula. Remarks on the action of an algebraic torus on k^n // Bull. Acad. Polon. Sci., Ser. Sci. Math. Astro. Phys. 1966, Vol. 14. P. 177-181.

4. A. Białynicki-Birula. Remarks on the action of an algebraic torus on k^n. II // Bull. Acad. Polon. Sci. Ser. Sci. Math. Astro. Phys. 1967, Vol. 15. P. 123-125.

5. A. Białynicki-Birula. Some theorems on actions of algebraic groups // Ann. Math. Second Series 1973, Vol. 98, No. 3, P. 480-497.

6. M. Brion. Introduction to actions of algebraic groups // Les cours du CIRM 2010, Vol. 1. No. 1. P. 1-22.

7. A. Elishev. Automorphisms of polynomial algebras, quantization and Kontsevich conjecture // PhD Thesis (in Russian), 2019.

8. A. Gutwirth. The action of an algebraic torus on the affine plane // Trans. Amer. Math. Soc. 1962, Vol. 105, No.3. P. 407-414.

9. T. Kambayashi and P. Russell. On linearizing algebraic torus actions // J. Pure and Applied Algebra 1982, Vol. 23, Iss. 3. P. 243-250.

10. A. Kanel-Belov, A. Elishev, and J.-T. Yu. Augmented polynomial symplectomorphisms and quantization //arXiv:1812.02859, 2018.

11. A. Kanel-Belov, A. Elishev, and J.-T. Yu. Independence of the B-KK isomorphism of infinite prime// arXiv:1512.06533, 2015.

12. A. Kanel-Belov, S. Grigoriev, A. Elishev, J.-T. Yu and W. Zhang. Lifting of Polynomial Symplectomorphisms and Deformation Quantization // Commun. in Algebra 2018, Vol. 46, P. 3926-3938.

13. A. Kanel-Belov, J.-T. Yu, A. Elishev. On the Augmentation Topology on Automorphism Groups of Affine Spaces and Algebras // Int. J. Alg. Comp. 2018, Vol. 28, No. 08. P. 1449-1485.

14. M. Koras and P. Russell. C^*-actions on C^3: The smooth locus of the quotient is not of hyperbolic type // J. Alg. Geom. 1999, Vol. 8, No. 4. P. 603-694.

15. G. Schwarz. Exotic algebraic group actions// C. R. Acad. Sci. Paris 1989, Ser. I Vol. 309. P. 89-94.

16. N. T. Zung. Torus actions and integrable systems// arXiv math/0407455 (2004).

REFERENCES

1. Asanuma, T. 1999, “Non-linearizable k^*-actions in affine space“, *Invent. Math.*, vol. 138, pp. 281-306.

2. Belov-Kanel, A. & Kontsevich, M. 2005, “Automorphisms of the Weyl algebra“, *Lett. Math. Phys.*, vol. 74, no. 2, pp. 181-199.

3. Białynicki-Birula, A. 1966, “Remarks on the action of an algebraic torus on k^n“, *Bull. Acad. Polon. Sci., Ser. Sci. Math. Astro. Phys.*, vol. 14, pp. 177-181.

4. Białynicki-Birula, A. 1967, “Remarks on the action of an algebraic torus on k^n, II“, *Bull. Acad. Polon. Sci., Ser. Sci. Math. Astro. Phys.*, vol. 15, pp. 123-125.

5. Białynicki-Birula, A. 1973, “Some theorems on actions of algebraic groups“, *Ann. Math. Second Series*, vol. 98, no. 3, pp. 480-497.

6. Brion, M. 2010, “Introduction to actions of algebraic groups“, *Les cours du CIRM*, vol. 1, no. 1, pp. 1-22.

7. Elishev, A. 2019, “Automorphisms of polynomial algebras, quantization and Kontsevich conjecture“, PhD thesis (in Russian).

8. Gutwirth, A. 1962, “The action of an algebraic torus on the affine plane“, *Trans. Amer. Math. Soc.*, vol. 105, no. 3, pp. 407-414.

9. Kambayashi, T.& Russell, P. 1982, “On linearizing algebraic torus actions“, *J. Pure and Applied Algebra*, vol. 23, iss. 3, pp. 243-250.

10. Kanel-Belov, A., Elishev, A. & Yu, J.-T. 2018, “Augmented polynomial symplectomorphisms and quantization“, *arXiv*: 1812.02859.

11. Kanel-Belov, A., Elishev, A. & Yu, J.-T. 2015, “Independence of the B-KK isomorphism of infinite prime“, *arXiv*: 1512.06533.

12. Kanel-Belov, A., Grigoriev, S., Elishev, A., Yu, J.-T. & Zhang, W. 2018, “Lifting of Polynomial Symplectomorphisms and Deformation Quantization “, *Commun. in Algebra*, vol. 46, pp. 3926-3938.

13. Kanel-Belov, A., Yu, J.-T. & Elishev, A. 2018, “On the Augmentation Topology on Automorphism Groups of Affine Spaces and Algebras“, *Int. J. Alg. Comp.*, vol. 28, no. 08, pp. 1449-1485.

14. Koras, M.& Russell, P. 1999, “C^*-actions on C^3: The smooth locus of the quotient is not of hyperbolic type“, *J. Alg. Geom.*, vol. 8, no. 4, pp. 603-694.

15. Schwarz, G. 1989, “Exotic algebraic group actions“, *C. R. Acad. Sci. Paris Ser. I*, vol. 309, pp. 89-94.

16. Zung, N., T. 2004, “Torus actions and integrable systems“, *arXiv*, math/0407455.

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