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Аналитические и теоретико-числовые свойства двумерных сигма-функций

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Аннотация

Обзор посвящен классическим и современным задачам, связанным с целой функцией $\sigma(\mathbf{u};\lambda)$, которая определяется семейством неособых алгебраических кривых рода 2, где $\mathbf{u}=(u_1,u_3),\ \lambda=(\lambda_4,\lambda_6,\lambda_8,\lambda_{10}).$ Эта функция является аналогом сигмафункции Вейерштрасса $\sigma(u;g_2,g_3)$ семейства эллиптических кривых. Логарифмические производные порядка 2 и выше функции $\sigma(\mathbf{u};\lambda)$ порождают поле гиперэллиптических функций от $\mathbf{u}=(u_1,u_3)$ на якобианах кривых с фиксированным значением вектора параметров λ . Мы рассматриваем три ряда Гурвица $\sigma(\mathbf{u};\lambda)=\sum_{m,n\geqslant 0}a_{m,n}(\lambda)\frac{u_1^mu_3^n}{m!n!},$ $\sigma(\mathbf{u};\lambda)=\sum_{k\geqslant 0}\xi_k(u_1;\lambda)\frac{u_3^k}{k!}$ и $\sigma(\mathbf{u};\lambda)=\sum_{k\geqslant 0}\mu_k(u_3;\lambda)\frac{u_1^k}{k!}$. Обзор посвящен теоретикочисловым свойствам функций $a_{m,n}(\lambda),\xi_k(u_1;\lambda)$ и $\mu_k(u_3;\lambda)$. Он включает самые последние результаты, доказательства которых использует тот фундаментальный факт, что функция $\sigma(\mathbf{u};\lambda)$ определяется системой четырех уравнений теплопроводности в неголономном репере шестимерного пространства.

Ключевые слова: абелевы функции, двумерные сигма-функции, целочисленность Гурвица, обобщенные числа Бернулли — Гурвица, уравнение теплопроводности в неголономном репере.

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Analytical and number-theoretical properties of the two-dimensional sigma function

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Abstract

This survey is devoted to the classical and modern problems related to the entire function $\sigma(\mathbf{u};\lambda)$, defined by a family of nonsingular algebraic curves of genus 2, where $\mathbf{u}=(u_1,u_3)$ and $\lambda=(\lambda_4,\lambda_6,\lambda_8,\lambda_{10})$. It is an analogue of the Weierstrass sigma function $\sigma(u;g_2,g_3)$ of a family of elliptic curves. Logarithmic derivatives of order 2 and higher of the function $\sigma(\mathbf{u};\lambda)$ generate fields of hyperelliptic functions of $\mathbf{u}=(u_1,u_3)$ on the Jacobians of curves with a fixed parameter vector λ . We consider three Hurwitz series $\sigma(\mathbf{u};\lambda)=\sum_{m,n\geqslant 0}a_{m,n}(\lambda)\frac{u_1^mu_3^n}{m!n!}$, $\sigma(\mathbf{u};\lambda)=\sum_{k\geqslant 0}\xi_k(u_1;\lambda)\frac{u_3^k}{k!}$ and $\sigma(\mathbf{u};\lambda)=\sum_{k\geqslant 0}\mu_k(u_3;\lambda)\frac{u_1^k}{k!}$. The survey is devoted to the number-theoretic properties of the functions $a_{m,n}(\lambda)$, $\xi_k(u_1;\lambda)$ and $\mu_k(u_3;\lambda)$. It includes the latest results, which proofs use the fundamental fact that the function $\sigma(\mathbf{u};\lambda)$ is determined by the system of four heat equations in a nonholonomic frame of six-dimensional space.

Keywords: Abelian functions, two-dimensional sigma functions, Hurwitz integrality, generalized Bernoulli—Hurwitz number, heat equation in a nonholonomic frame.

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Dedicated to the outstanding mathematician Academician Vladimir Petrovich Platonov in connection with his anniversary.

1. Introduction

Deep results on the number-theoretic properties of fields of hyperelliptic functions were obtained in the papers of V. P. Platonov, where he gave answers to long-standing questions. The fields of meromorphic functions on the Jacobian of curves of genus 2 occupy one of the main places in these papers (see [40], [41] and [42]). Abelian functions, including meromorphic functions on the Jacobians of algebraic curves, were a central topic of the 19th century mathematics. In this review, we mainly discuss the results obtained due to a new direction in the study of fields of

Abelian functions. This direction arose in the mid-seventies of the last century in response to the discovery that Abelian functions provide a solution to a number of challenging problems of modern theoretical and mathematical physics. The elliptic sigma function, which was defined and investigated by Weierstrass, is important in many fields in mathematics and physics. This function is closely related to the theory of the elliptic curves. In [28] and [29], F. Klein posed the problem of the construction of multi-dimensional sigma functions associated with the hyperelliptic curves. He obtained important results in this direction. Many years later, F. Klein wrote a paper and a survey in which he acknowledged that the theory of his sigma functions is still far from complete (see [30] and [31]). The theory of the hyperelliptic sigma functions was developed by H. F. Baker in [5], [6], [7], and [8]. Recently, by a series of work of V. M. Buchstaber, V. Z. Enolskii, and D. V. Leykin, the theory of the hyperelliptic sigma functions was developed significantly and they were generalized to the large family of algebraic curves called (n,s) curves, which include the hyperelliptic curves as special cases (see [12], [13], [14], [15], [17]). After the publications of Buchstaber, Enolskii, and Leykin, many papers appeared on the theory and applications of multi-dimensional sigma functions. Our survey is devoted to the sigma functions of curves of genus 2. The focus of our attention is the number-theoretic aspects of the results on these functions. Throughout the present survey, we denote the sets of positive integers, integers, rational and complex numbers by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and \mathbb{C} , respectively.

Let V be a hyperelliptic curve of genus g defined by

$$y^{2} = x^{2g+1} + \lambda_{4}x^{2g-1} + \lambda_{6}x^{2g-2} + \dots + \lambda_{4q}x + \lambda_{4q+2}, \quad \lambda_{i} \in \mathbb{C}.$$
 (1)

The sigma function $\sigma(\mathbf{u}; \lambda)$, where $\mathbf{u} = (u_1, u_3, \dots, u_{2g-1})$ and $\lambda = (\lambda_4, \dots, \lambda_{4g+2})$, associated with V, is an entire function in $\mathbf{u} \in \mathbb{C}^g$. It is shown that the coefficients of the power series expansion of $\sigma(\mathbf{u})$ around $\mathbf{u} = \mathbf{0}$ are polynomials of the coefficients $\lambda_4, \dots, \lambda_{4g+2}$ over the rationals ([14], [15], [17], [33]). Let R be an integral domain with characteristic $0, u_1, u_3, \dots, u_{2g-1}$ be indeterminates, and

$$R\langle\langle u_1, u_3 \dots, u_{2g-1} \rangle\rangle = \left\{ \sum_{i_1, i_3, \dots, i_{2g-1} \geqslant 0} a_{i_1, i_3, \dots, i_{2g-1}} \frac{u_1^{i_1} u_3^{i_3} \cdots u_{2g-1}^{i_{2g-1}}}{i_1! i_3! \cdots i_{2g-1}!} \, \middle| \, a_{i_1, i_3, \dots, i_{2g-1}} \in R \right\}.$$

If a power series belongs to $R\langle\langle u_1,u_3\ldots,u_{2g-1}\rangle\rangle$, then it is said to be *Hurwitz integral* over R. In [39], Y. Ônishi proved that the power series expansion of $\sigma(\mathbf{u})$ around $\mathbf{u}=\mathbf{0}$ is Hurwitz integral over the ring $\mathbb{Z}[\lambda_4,\ldots,\lambda_{4g+2}]$ by using the expression of the sigma function in terms of the tau function of KP-hierarchy given in [34]. In [37], in the case of g=1, the Hurwitz integrality of the sigma function is proved in a different way from [39] and relationships with number theory are discussed. In [21], in the case of g=1, it is conjectured that the power series expansion of the sigma function is Hurwitz integral over $\mathbb{Z}[2\lambda_4,24\lambda_6]$. The focus of our survey is on the above fundamental fact, i.e., the power series expansion of the sigma function around the origin is Hurwitz integral over $\mathbb{Z}[\lambda_4,\ldots,\lambda_{4g+2}]$. In this survey, we will discuss in detail expansions of the sigma functions of curves of genus 1 and 2, including the Ônishi's proof for Hurwitz integrality (see Sections 2.2 and 2.3).

Weierstrass [45] showed that the elliptic sigma function $\sigma(u; \lambda_4, \lambda_6)$ satisfies the following system of equations

$$4\lambda_4 \sigma_{\lambda_4} + 6\lambda_6 \sigma_{\lambda_6} - u\sigma_u + \sigma = 0,$$

$$6\lambda_6 \sigma_{\lambda_4} - \frac{4}{3}\lambda_4^2 \sigma_{\lambda_6} - \frac{1}{2}\sigma_{uu} + \frac{1}{6}\lambda_4 u^2 \sigma = 0.$$

The second equation of this system is the heat equation or, equivalently, the Schrödinger equation of type $\ell_2 \sigma = \frac{1}{2} H_2 \sigma$, where $\ell_2 = 6 \lambda_6 \partial/\partial \lambda_4 - (4/3) \lambda_4^2 \partial/\partial \lambda_6$ and $H_2 = \frac{\partial^2}{\partial u^2} - \frac{1}{3} \lambda_4 u^2$. Weierstrass gave

recurrence relations of the coefficients of series expansion of the elliptic sigma function. Buchstaber and Leykin succeeded in generalizing the theory of the heat equations to the sigma functions of higher genus curves ([18], [19], and [20]). In [24], the detailed proof of their theory is given. In [19] and [24], the recurrence relations of the coefficients of series expansion of the two-dimensional sigma function are given based on the heat equations. In [25], the theory of the heat equations is constructed for the elliptic curves defined by the most general Weierstrass equation. In [23], for g = 1, 2, it is shown that the holomorphic solution of the heat equations around $(\mathbf{u}_0, \mathbf{0}) \in \mathbb{C}^{3g}$ for some $\mathbf{u}_0 \in \mathbb{C}^g$ is the sigma function up to a multiplicative constant. We consider the case of g = 2. For $\lambda = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$, we set

$$\sigma(\mathbf{u};\lambda) = \sum_{k\geqslant 0} \xi_k(u_1;\lambda) \frac{u_3^k}{k!}, \qquad \sigma(\mathbf{u};\lambda) = \sum_{k\geqslant 0} \mu_k(u_3;\lambda) \frac{u_1^k}{k!}.$$

In Section 3, we will derive the differential equations satisfied by ξ_k and μ_k from the heat equations. From these results, we will prove that two-dimensional sigma function is Hurwitz integral over $\mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, 2\lambda_{10}]$ (Corollary 2).

For $(x, y) \in V$, let

$$du_1 = -\frac{x}{2y}dx, \quad du_3 = -\frac{1}{2y}dx,$$

which are a basis of the vector space of holomorphic one forms on V. We have two ultra-elliptic integrals $\int_{\infty}^{P} du_1$ and $\int_{\infty}^{P} du_3$ obtained with the help of two holomorphic differentials du_1 and du_3 . We take a point $P_* \in V$ and an open neighborhood U_* of this point P_* such that U_* is homeomorphic to an open disk in \mathbb{C} . Let us fix a path γ_* on the curve V from ∞ to the point P_* . We consider the holomorphic mappings defined by the ultra-elliptic integrals

$$I_3: U_* \to \mathbb{C}, \quad P = (x,y) \mapsto \int_{-\infty}^{P} du_3,$$

$$I_1: U_* \to \mathbb{C}, \quad P = (x, y) \mapsto \int_{\infty}^{P} du_1,$$

where as the path of integration we choose the composition of the fixed path γ_* from ∞ to the point P_* and some path in the neighborhood U_* from P_* to the point P. When we consider the map I_3 , we assume $P_* \neq \infty$. When we consider the map I_1 , we assume $P_* \neq (0, \pm \sqrt{\lambda_{10}})$. If U_* is sufficiently small, then the maps I_1 and I_3 are biholomorphisms. In [4], the inversion problems of the maps I_1 and I_3 are considered. In Section 4, we will summarize the results of [4]. Proposition 18 in the present survey gives the recurrence formula of the coefficients of series expansion of the solution of the inversion problem with respect to I_1 in the case of $P_* = \infty$. This result is not included in [4].

The classical Bernoulli numbers B_n are defined by the generating function

$$\frac{u}{e^u - 1} = \sum_{n=0}^{\infty} B_n \frac{u^n}{n!}.$$
 (2)

Bernoulli numbers B_n are important in many areas of mathematics, including number theory and algebraic topology. Many beautiful properties for the Bernoulli numbers B_n are known. For example, the von Staudt-Clausen theorem states

$$B_{2n} + \sum_{(p-1)|2n} \frac{1}{p} \in \mathbb{Z},$$

where the summation is over all primes p such that p-1 divides 2n. Let $a \ge 1$ be an integer, p be a prime, and m, n be even positive integers such that $m, n \ge a+1$, m and n are not divisible by p-1, and $n \equiv m \mod (p-1)p^{a-1}$. Then the Kummer's congruence states

$$\frac{B_n}{n} \equiv \frac{B_m}{m} \mod p^a.$$

Let us introduce the universal logarithmic series

$$\alpha(u) = u + \sum_{n \geqslant 1} \alpha_n \frac{u^{n+1}}{n+1} \tag{3}$$

over the grading ring $A = \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$, deg $\alpha_n = -2n$, and the universal exponential series

$$\beta(t) = t + \sum_{n \ge 1} \beta_n \frac{t^{n+1}}{(n+1)!} \tag{4}$$

over the grading ring $B = \mathbb{Z}[\beta_1, \beta_2, \ldots]$, deg $\beta_n = -2n$. Set deg $u = \deg t = 2$. Then $\alpha(u)$ and $\beta(t)$ are homogeneous series of degree 2. Imposing condition $\alpha(\beta(t)) = t$ that equivalent to condition $\beta(\alpha(u)) = u$, we obtain an isomorphism of rings preserving grading

$$\xi \colon \hat{A} = \mathbb{Z}[\hat{\alpha}_1, \hat{\alpha}_2, \ldots] \longrightarrow \hat{B} = \mathbb{Z}[\hat{\beta}_1, \hat{\beta}_2, \ldots],$$

where $\hat{\alpha}_n = \frac{\alpha_n}{n+1}$ and $\hat{\beta}_n = \frac{\beta_n}{(n+1)!}$. Thus, we obtain the polynomials

$$\hat{\beta}_n = \hat{\beta}_n(\hat{\alpha}_1, \dots, \hat{\alpha}_n) \in \hat{A}, n = 1, 2, \dots,$$

which coefficients are integers satisfying the relations

$$\beta(\alpha(u) + \alpha(v)) \in \mathcal{A}[[u, v]],$$

where $\mathcal{A} = \mathbb{Z}[a_1, a_2, \ldots] \subset \hat{A}$, and

$$\alpha_n \in \mathcal{A}, \qquad \beta_n = (n+1)! \ \hat{\beta}_n \left(\frac{\alpha_1}{2}, \dots, \frac{\alpha_n}{n+1} \right) \in \mathcal{A}.$$

These relations play an important role in describing the coefficient ring of the universal formal group (see [32], [16]) and in the algebraic-topological applications of the formal group in the theory of complex cobordisms (see [36], [43]).

The polynomials $B_n = B_n(\hat{\alpha}_1, \dots, \hat{\alpha}_n) \in \hat{A}$ which generating series is given by the Hurwitz exponential series over the ring \hat{A}

$$\sum_{n\geqslant 0} B_n \frac{t^n}{n!} = \frac{t}{\beta(t)} \tag{5}$$

are called universal Bernoulli numbers.

The classic Bernoulli numbers are obtained by substituting $\alpha_n = (-1)^n$. For example,

$$B_1 = \hat{\alpha}_1, \quad B_2 = 2(\hat{\alpha}_2 - \hat{\alpha}_1^2), \quad B_3 = 3!(\hat{\alpha}_3 - 3\hat{\alpha}_1\hat{\alpha}_2 + 2\hat{\alpha}_1^3).$$

Substituting $\hat{\alpha}_1 = -\frac{1}{2}$, $\hat{\alpha}_2 = \frac{1}{3}$, $\hat{\alpha}_3 = -\frac{1}{4}$, we obtain numbers $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$.

Classical Bernoulli numbers entered into algebraic geometry and algebraic topology due to the fact that the generating series (2) defines the Hirzebruch genus, which associates to any smooth complex manifold an integer equal to the Todd genus of this manifold (see [26]). The generating series (5) of universal Bernoulli numbers defines the universal Todd genus, which associates to any

smooth complex manifold an integer polynomial (see details in [9]). In [22], F. Clarke generalized the von Staudt-Clausen theorem for the classical Bernoulli numbers to the universal Bernoulli numbers. The Kummer's congruence for the classical Bernoulli numbers was generalized to the universal Bernoulli numbers ([1], [2], [3], [38]).

For a hyperelliptic curve of genus g defined by equation (1), in a neighborhood of point (∞, ∞) , we can choose a local coordinate u such that the functions x(u) and y(u) can be expanded around u = 0 as

$$x(u) = \frac{1}{u^2} + \frac{c_{-1}}{u} + \sum_{n=2}^{\infty} \frac{C_n}{n} \frac{u^{n-2}}{(n-2)!},$$
$$y(u) = \frac{1}{u^{2g+1}} + \frac{d_{-2g}}{u^{2g}} + \dots + \frac{d_{-1}}{u} + \sum_{n=2g+1}^{\infty} \frac{D_n}{n} \frac{u^{n-2g-1}}{(n-2g-1)!}.$$

Then C_n and D_n are called generalized Bernoulli-Hurwitz numbers. In [38], the von Staudt-Clausen theorem and the Kummer's congruence for the classical Bernoulli numbers are extended to the generalized Bernoulli-Hurwitz numbers in the case of the curves $y^2 = x^{2g+1} - 1$ and $y^2 = x^{2g+1} - x$. We will extend the methods of [38] to the curve $y^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10}$ and show some number-theoretical properties for the generalized Bernoulli-Hurwitz numbers associated with this curve (Theorem 11). These results will give the precise information on the series expansion of the solution of the inversion problem of the ultra-elliptic integrals.

2. Preliminaries

2.1. The sigma function

For a positive integer g, we set

$$\Delta = \{(\lambda_4, \lambda_6, \dots, \lambda_{4q+2}) \in \mathbb{C}^{2g} \mid f_q(x) \text{ has a multiple root}\},$$

where

$$f_a(x) = x^{2g+1} + \lambda_4 x^{2g-1} + \lambda_6 x^{2g-2} + \dots + \lambda_{4a} x + \lambda_{4a+2}$$

and $\mathcal{B} = \mathbb{C}^{2g} \setminus \Delta$. We consider the non-singular hyperelliptic curve of genus g

$$V = \{(x, y) \in \mathbb{C}^2 \mid y^2 = f_q(x)\},\tag{6}$$

where $(\lambda_4, \lambda_6, \dots, \lambda_{4g+2}) \in \mathcal{B}$. In this paragraph we recall the definition of the sigma-function for the curve V (see [15]) and give facts about it which will be used later on. For $(x, y) \in V$, let

$$du_{2i-1} = -\frac{x^{g-i}}{2y}dx, \quad 1 \leqslant i \leqslant g,$$

which are a basis of the vector space of holomorphic one forms on V, and $d\mathbf{u} = {}^{t}(du_1, du_3, \dots, du_{2g-1})$. Further, let

$$dr_{2i-1} = \frac{1}{2y} \sum_{k=g-i+1}^{g+i-1} (-1)^{g+i-k} (k+i-g) \lambda_{2g+2i-2k-2} x^k dx, \quad 1 \leqslant i \leqslant g,$$
 (7)

which are meromorphic one forms on V with a pole only at ∞ . In (7) we set $\lambda_0 = 1$ and $\lambda_2 = 0$. For g = 1, we have

$$du_1 = -\frac{1}{2y}dx, dr_1 = -\frac{x}{2y}dx,$$

for g = 2, we have

$$du_1 = -\frac{x}{2y}dx$$
, $du_3 = -\frac{1}{2y}dx$, $dr_1 = -\frac{x^2}{2y}dx$, $dr_3 = \frac{-\lambda_4 x - 3x^3}{2y}dx$.

Let $\{\alpha_i, \beta_i\}_{i=1}^g$ be a canonical basis in the one-dimensional homology group of the curve V. We define the matrices of periods by

$$2\omega_1 = \left(\int_{\alpha_j} du_i\right), \ 2\omega_2 = \left(\int_{\beta_j} du_i\right), \ -2\eta_1 = \left(\int_{\alpha_j} dr_i\right), \ -2\eta_2 = \left(\int_{\beta_j} dr_i\right).$$

The matrix of normalized periods has the form $\tau = \omega_1^{-1}\omega_2$. Let $\delta = \tau \delta' + \delta''$, $\delta', \delta'' \in \mathbb{R}^g$, be the vectors of Riemann's constants with respect to the choice $(\{\alpha_i, \beta_i\}, \infty)$ and $\delta := {}^t({}^t\delta', {}^t\delta'')$. Then we have $\delta' = {}^t(\frac{1}{2}, \ldots, \frac{1}{2})$ and $\delta'' = {}^t(\frac{g}{2}, \frac{g-1}{2}, \ldots, \frac{1}{2})$. If g is even, we define $A_0 = (2g-1, 2g-5, \ldots, 7, 3)$ and c_0 by the sign of the permutation

$$c_0 = \operatorname{sgn} \begin{pmatrix} 0 & 2 & \cdots & g-4 & g-2 & g-1 & g-3 & \cdots & 3 & 1 \\ g-1 & g-2 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 \end{pmatrix}.$$

If g is odd, we define $A_0 = (2g - 1, 2g - 5, \dots, 5, 1)$ and c_0 by the sign of the permutation

$$c_0 = \operatorname{sgn} \begin{pmatrix} 0 & 2 & \cdots & g-3 & g-1 & g-2 & g-4 & \cdots & 3 & 1 \\ g-1 & g-2 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 \end{pmatrix}.$$

We consider the Riemann's theta-function with the characteristic δ , which is defined by

$$\theta[\delta](\mathbf{u},\tau) = \sum_{n \in \mathbb{Z}^g} \exp\{\pi\sqrt{-1} t(n+\delta')\tau(n+\delta') + 2\pi\sqrt{-1} t(n+\delta')(\mathbf{u}+\delta'')\},$$

where $\mathbf{u} = {}^t(u_1, u_3, \dots, u_{2g-1}) \in \mathbb{C}^g$. We set $\partial_{u_i} = \partial/\partial u_i$. For a non-empty subset $I = \{i_1, \dots, i_k\} \subset \{1, 3, \dots, 2g-1\}$, we set

$$\partial_I = \partial_{u_{i_1}} \cdots \partial_{u_{i_k}}.$$

It is known that $\partial_{A_0}\theta[\delta](\mathbf{0},\tau)\neq 0$ ([35]). The sigma-function $\sigma(\mathbf{u})$ is defined by (cf. [15], [35])

$$\sigma(\mathbf{u}) = \exp\left(\frac{1}{2}^t \mathbf{u} \eta_1 \omega_1^{-1} \mathbf{u}\right) \frac{\theta[\delta] ((2\omega_1)^{-1} \mathbf{u}, \tau)}{c_0 \partial_{A_0} \theta[\delta] (\mathbf{0}, \tau)},$$

which is an entire function on \mathbb{C}^g . We set $\wp_{i,j} = -\partial_{u_i}\partial_{u_j}\log\sigma$, $\sigma_i = \partial_{u_i}\sigma$, and $\sigma_{i,j} = \partial_{u_i}\partial_{u_j}\sigma$. We define the period lattice $\Lambda = \{2\omega_1 m_1 + 2\omega_2 m_2 \mid m_1, m_2 \in \mathbb{Z}^g\}$ and set $W = \{\mathbf{u} \in \mathbb{C}^g \mid \sigma(\mathbf{u}) = 0\}$.

PROPOSITION 1. ([15] Theorem 1.1 and [33] p.193) For $m_1, m_2 \in \mathbb{Z}^g$, let $\Omega = 2\omega_1 m_1 + 2\omega_2 m_2$, and let

$$A = (-1)^{2(t\delta' m_1 - t\delta'' m_2) + tm_1 m_2} \exp(t(2\eta_1 m_1 + 2\eta_2 m_2)(\mathbf{u} + \omega_1 m_1 + \omega_2 m_2)).$$

Then

- (i) $\sigma(\mathbf{u} + \Omega) = A\sigma(\mathbf{u})$, where $\mathbf{u} \in \mathbb{C}^g$.
- (ii) $\sigma_i(\mathbf{u} + \Omega) = A\sigma_i(\mathbf{u}), i = 1, 3, \dots, 2g 1, where \mathbf{u} \in W.$

Proposition 1 (i) implies that $\mathbf{u} + \Omega \in W$ for any $\mathbf{u} \in W$ and $\Omega \in \Lambda$. The surface

$$(\sigma) := \{ \mathbf{u} \in \mathbb{C}^g / \Lambda \mid \sigma(\mathbf{u}) = 0 \}$$

is called the sigma divisor. We set $\deg u_{2k-1} = -(2k-1)$ and $\deg \lambda_{2i} = 2i$, where $1 \leqslant k \leqslant g$ and $2 \leqslant i \leqslant 2g+1$. A sequence of non-negative integers $\mu = (\mu_1, \mu_2, \dots, \mu_l)$ such that $\mu_1 \geqslant \mu_2 \geqslant \dots \geqslant \mu_l$ is called a partition. Let $S_{\mu_g}(\mathbf{u})$ be the Schur function associated with the partition $\mu_g = (g, g-1, \dots, 1)$ and set $|\mu_g| = g + (g-1) + \dots + 1$ (cf. [33] Section 4).

THEOREM 1. ([14] Theorem 6.3, [15] Theorem 7.7, [17], [33] Theorem 3, [35] Theorem 13) The sigma function $\sigma(\mathbf{u})$ does not depend on the choice of $\{\alpha_i, \beta_i\}_{i=1}^g$ and has the series expansion of the form

$$\sigma(\mathbf{u}) = S_{\mu_g}(\mathbf{u}) + \sum_{i_1 + 3i_3 + \dots + (2g-1)i_{2g-1} > |\mu_g|} \alpha_{i_1, i_3, \dots, i_{2g-1}}^{(g)} u_1^{i_1} u_3^{i_3} \dots u_{2g-1}^{i_{2g-1}}, \tag{8}$$

where the coefficient $\alpha^{(g)}_{i_1,i_3,\dots,i_{2g-1}}$ is a homogeneous polynomial in $\mathbb{Q}[\lambda_4,\lambda_6,\dots,\lambda_{4g+2}]$ of degree $i_1+3i_3+\dots+(2g-1)i_{2g-1}-|\mu_g|$ if $\alpha^{(g)}_{i_1,i_3,\dots,i_{2g-1}}\neq 0$.

For g=1, the sigma function $\sigma(u)$ is an entire odd function on \mathbb{C} and it is given by the series

$$\sigma(u) = u + \sum_{i>4} \alpha_i^{(1)} u^{i+1},$$

where the coefficient $\alpha_i^{(1)}$ is a homogeneous polynomial in $\mathbb{Q}[\lambda_4, \lambda_6]$ of degree i if $\alpha_i^{(1)} \neq 0$. For g = 2, the sigma function $\sigma(\mathbf{u}) = \sigma(u_1, u_3)$ is an entire odd function on \mathbb{C}^2 and it is given by the series

$$\sigma(u_1, u_3) = \frac{1}{3}u_1^3 - u_3 + \sum_{i+3j \ge 7} \alpha_{i,j}^{(2)} u_1^i u_3^j, \tag{9}$$

where the coefficient $\alpha_{i,j}^{(2)}$ is a homogeneous polynomial in $\mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree i + 3j - 3 if $\alpha_{i,j}^{(2)} \neq 0$.

THEOREM 2. ([10, 12, 18, 19, 20, 24])

(i) For g=1, the sigma function $\sigma(u; \lambda_4, \lambda_6)$ satisfies the following system of equations:

$$4\lambda_4 \sigma_{\lambda_4} + 6\lambda_6 \sigma_{\lambda_6} - u\sigma_u + \sigma = 0,$$

$$6\lambda_6 \sigma_{\lambda_4} - \frac{4}{3}\lambda_4^2 \sigma_{\lambda_6} - \frac{1}{2}\sigma_{uu} + \frac{1}{6}\lambda_4 u^2 \sigma = 0.$$

(ii) For g=2, the sigma function $\sigma(u_1,u_3;\lambda_4,\lambda_6,\lambda_8,\lambda_{10})$ satisfies the following system of equations:

$$Q_i \sigma = 0$$
, where $Q_i = \ell_i - H_i$, $i = 0, 2, 4, 6$,

$${}^t(\ell_0, \ell_2, \ell_4, \ell_6) = T^{t}(\partial_{\lambda_4}, \partial_{\lambda_6}, \partial_{\lambda_8}, \partial_{\lambda_{10}}),$$

$$T = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ 6\lambda_6 & 8\lambda_8 - \frac{12}{5}\lambda_4^2 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & -\frac{4}{5}\lambda_4\lambda_8 \\ 8\lambda_8 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & 4\lambda_4\lambda_8 - \frac{12}{5}\lambda_6^2 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 \\ 10\lambda_{10} & 4\lambda_1\lambda_2 & 6\lambda_1\lambda_3 & 4\lambda_2\lambda_4 & 8\lambda_2^2 \end{pmatrix},$$

$$\begin{split} H_0 &= u_1 \partial_{u_1} + 3 u_3 \partial_{u_3} - 3, \\ H_2 &= \frac{1}{2} \partial_{u_1}^2 - \frac{4}{5} \lambda_4 u_3 \partial_{u_1} + u_1 \partial_{u_3} - \frac{3}{10} \lambda_4 u_1^2 + \frac{1}{10} (15 \lambda_8 - 4 \lambda_4^2) u_3^2, \\ H_4 &= \partial_{u_1} \partial_{u_3} - \frac{6}{5} \lambda_6 u_3 \partial_{u_1} + \lambda_4 u_3 \partial_{u_3} - \frac{1}{5} \lambda_6 u_1^2 + \lambda_8 u_1 u_3 + \frac{1}{10} (30 \lambda_{10} - 6 \lambda_4 \lambda_6) u_3^2 - \lambda_4, \\ H_6 &= \frac{1}{2} \partial_{u_3}^2 - \frac{3}{5} \lambda_8 u_3 \partial_{u_1} - \frac{1}{10} \lambda_8 u_1^2 + 2 \lambda_{10} u_1 u_3 - \frac{3}{10} \lambda_8 \lambda_4 u_3^2 - \frac{1}{2} \lambda_6. \end{split}$$

2.2. Hurwitz integrality of the expansion of the elliptic sigma function

In [39], Hurwitz integrality of the expansion of the sigma functions is proved. In this subsection, we will explain the proof of [39] for g = 1.

In this subsection, we assume g = 1. For simplicity, we denote u_1 and du_1 by u and du, respectively. For an integral domain R with characteristic 0 and a variable u, let

$$R\langle\langle u\rangle\rangle = \left\{\sum_{i=0}^{\infty} \alpha_i \frac{u^i}{i!} \mid \alpha_i \in R\right\}.$$

For n < 0 let $p_n(u) = 0$ and for $n \ge 0$ let

$$p_n(u) = \frac{u^n}{n!}.$$

For an arbitrary partition $\mu = (\mu_1, \mu_2, \dots, \mu_l)$, we define the Schur polynomial $s_{\mu}(u)$ by

$$s_{\mu}(u) = \det (p_{\mu_i - i + j}(u))_{1 \le i, j \le l}.$$

Lemma 1. We have $s_{\mu}(u) \in \mathbb{Z}\langle\langle u \rangle\rangle$.

PROOF. For $i, j \ge 0$, we have

$$\frac{u^i}{i!}\frac{u^j}{j!} = \frac{(i+j)!}{i!j!}\frac{u^{i+j}}{(i+j)!} = \binom{i+j}{i}\frac{u^{i+j}}{(i+j)!}.$$

Since $\binom{i+j}{i} \in \mathbb{Z}$, we obtain the statement of the lemma. \square

$$t = \frac{x}{y}, \quad s = \frac{1}{x}.$$

Then t is a local parameter of V around ∞ . We have

$$x = \frac{1}{s}, \quad y = \frac{1}{st}.\tag{10}$$

Denote by $\mathbb{Z}_{\geq r}$ the set of integers that are not less than r.

Lemma 2. We have the following expansion of s in terms of t around t = 0

$$s = t^2 \left(1 + \sum_{i=4}^{\infty} \beta_i t^i \right),$$

where β_i is a homogeneous polynomial in $\mathbb{Z}[\lambda_4, \lambda_6]$ of degree i. In particular, we have

$$s = t^2 + \lambda_4 t^6 + \lambda_6 t^8 + \cdots.$$

PROOF. By substituting (10) into $y^2 = x^3 + \lambda_4 x + \lambda_6$, we have

$$s = t^2 + \lambda_4 s^2 t^2 + \lambda_6 s^3 t^2. \tag{11}$$

The expansion of s with respect to t around t = 0 takes the following form

$$s = t^2 \sum_{i=0}^{\infty} \beta_i t^i,$$

where $\beta_i \in \mathbb{C}$. By substituting the above expansion into (11), we have

$$\sum_{i=0}^{\infty} \beta_i t^i = 1 + \lambda_4 t^4 \left(\sum_{i=0}^{\infty} \beta_i t^i \right)^2 + \lambda_6 t^6 \left(\sum_{i=0}^{\infty} \beta_i t^i \right)^3.$$

By comparing the coefficients, we obtain $\beta_0 = 1$, $\beta_4 = \lambda_4$, $\beta_6 = \lambda_6$, and $\beta_n = 0$ for n = 1, 2, 3, 5. For $n \ge 6$, we have

$$\beta_n = \lambda_4 \sum_{(i_1, i_2) \in I_1} \beta_{i_1} \beta_{i_2} + \lambda_6 \sum_{(i_1, i_2, i_3) \in I_2} \beta_{i_1} \beta_{i_2} \beta_{i_3},$$

where $I_1 = \{(i_1, i_2) \in \mathbb{Z}_{\geq 0}^2 \mid i_1 + i_2 = n - 4\}$ and $I_2 = \{(i_1, i_2, i_3) \in \mathbb{Z}_{\geq 0}^3 \mid i_1 + i_2 + i_3 = n - 6\}$. Therefore we obtain the statement of the lemma. \square

From Lemma 2 and (10), we have the expansions

$$x = \frac{1}{t^2} \left(1 + \sum_{i=4}^{\infty} a_i t^i \right), \quad y = \frac{1}{t^3} \left(1 + \sum_{i=4}^{\infty} a_i t^i \right), \tag{12}$$

where a_i is a homogeneous polynomial in $\mathbb{Z}[\lambda_4, \lambda_6]$ of degree i. We enumerate the monomials $x^m y^n$, where m is a non-negative integer and n = 0, 1, according as the order of a pole at ∞ and denote them by φ_j , $j \ge 1$. In particular we have $\varphi_1 = 1$. We set $e_i = t^{i+1}$. We expand $t\varphi_j$ around ∞ with respect to t. Let

$$t\varphi_j = \sum_i \xi_{i,j} e_i,$$

where $\xi_{i,j} \in \mathbb{Z}[\lambda_4, \lambda_6]$. For a partition $\mu = (\mu_1, \mu_2, \dots,)$, we define

$$\xi_{\mu} = \det(\xi_{m_i,j})_{i,j \in \mathbb{N}},$$

where $m_i = \mu_i - i$ and the infinite determinant is well defined. Then we have $\xi_{\mu} \in \mathbb{Z}[\lambda_4, \lambda_6]$. We define the tau function $\tau(u)$ by

$$\tau(u) = \sum_{\mu} \xi_{\mu} s_{\mu}(u),$$

where the sum is over all partitions. From Lemma 1, we obtain the following proposition.

Proposition 2. We have $\tau(u) \in \mathbb{Z}[\lambda_4, \lambda_6] \langle \langle u \rangle \rangle$.

Lemma 3. The expansion of du around ∞ with respect to t takes the form

$$du = \left(\sum_{j=1}^{\infty} b_j t^{j-1}\right) dt,$$

where $b_1 = 1$ and $b_2 = b_3 = b_4 = 0$.

PROOF. From (12), we have

$$du = -\frac{dx}{2y} = \frac{1 - \sum_{i=4}^{\infty} a_i \frac{i-2}{2} t^i}{1 + \sum_{i=4}^{\infty} a_i t^i} dt = (1 + O(t^4)) dt.$$

We take the algebraic bilinear form

$$\omega(P,Q) = \frac{x_1 x_2 (x_1 + x_2) + \lambda_4 (x_1 + x_2) + 2y_1 y_2 + 2\lambda_6}{4y_1 y_2 (x_1 - x_2)^2} dx_1 dx_2, \tag{13}$$

where $P = (x_1, y_1), Q = (x_2, y_2) \in V$. We can expand $\omega(P, Q)$ around $\infty \times \infty$ as

$$\omega(P,Q) = \left(\frac{1}{(t_1 - t_2)^2} + \sum_{i,j \ge 1} q_{ij} t_1^{i-1} t_2^{j-1}\right) dt_1 dt_2, \tag{14}$$

where $q_{ij} \in \mathbb{C}$ and t_1, t_2 are copies of the local parameter t.

Lemma 4. We have $q_{11} = 0$.

PROOF. From (13) and (14), we have

$$\{x_1x_2(x_1+x_2)+\lambda_4(x_1+x_2)+2y_1y_2+2\lambda_6\}dx_1dx_2$$

$$=4y_1y_2(x_1-x_2)^2\left(\frac{1}{(t_1-t_2)^2}+\sum_{i,j\geqslant 1}q_{ij}t_1^{i-1}t_2^{j-1}\right)dt_1dt_2.$$

By substituting the expansions of x_1, x_2, y_1, y_2 into the above equation and multiplying the both sides of this equation by $t_1^7 t_2^7 (t_1 - t_2)^2$, we obtain

$$(t_1 - t_2)^2 (-2 + 2a_4t_1^4 + \cdots)(-2 + 2a_4t_2^4 + \cdots) \times \left[f(t_1)f(t_2) \{ t_2^2 f(t_1) + t_1^2 f(t_2) \} + \lambda_4 t_1^2 t_2^2 \{ t_2^2 f(t_1) + t_1^2 f(t_2) \} + 2t_1 t_2 f(t_1) f(t_2) + 2\lambda_6 t_1^4 t_2^4 \right]$$

$$= 4f(t_1)f(t_2) \{ t_2^2 f(t_1) - t_1^2 f(t_2) \}^2 \left\{ 1 + q_{11}(t_1 - t_2)^2 + \sum_{i+j \geqslant 3} \widetilde{q}_{ij} t_1^i t_2^j \right\},$$

where f(t) is defined by $x = t^{-2}f(t)$ and $\widetilde{q}_{ij} \in \mathbb{C}$. By comparing the coefficient of t_2^6 in the above equation, we obtain $q_{11} = 0$. \square

We define c_i by the following relation

$$\sqrt{\frac{du}{dt}} = \exp\left(\sum_{i=1}^{\infty} \frac{c_i}{i} t^i\right).$$

Lemma 5. We have $c_1 = 0$.

PROOF. From Lemma 3, we have the following expansion

$$\sqrt{\frac{du}{dt}} = 1 + O(t^4).$$

On the other hand, we have

$$\exp\left(\sum_{i=1}^{\infty} \frac{c_i}{i} t^i\right) = 1 + c_1 t + O(t^2).$$

Thus we have $c_1 = 0$. \square

THEOREM 3. ([34], Theorem 4) We have

$$\tau(u) = \exp\left(-c_1 u + \frac{1}{2}q_{11}u^2\right)\sigma(b_1 u).$$

From Theorem 3, Lemma 3, Lemma 4, and Lemma 5, we have

$$\sigma(u) = \tau(u). \tag{15}$$

From Proposition 2 and (15), we obtain the following theorem.

THEOREM 4. ([39]) We have

$$\sigma(u) \in \mathbb{Z}[\lambda_4, \lambda_6]\langle\langle u \rangle\rangle.$$

2.3. Hurwitz integrality of the expansion of the two-dimensional sigma function

In this subsection, we will explain the proof of [39] for Hurwitz integrality of the expansion of the sigma function for g = 2.

In this subsection, we assume g = 2. For an integral domain R with characteristic 0 and variables u_1, u_3 , let

$$R\langle\langle u_1, u_3\rangle\rangle = \left\{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{i,j} \frac{u_1^i u_3^j}{i!j!} \mid \alpha_{i,j} \in R\right\}.$$

For n < 0 let $p_n(u_1, u_3) = 0$ and for $n \ge 0$ let

$$p_n(u_1, u_3) = \sum \frac{u_1^i u_3^j}{i! j!},$$

where the summation is over all $(i, j) \in \mathbb{Z}_{\geq 0}^2$ satisfying i + 3j = n. For an arbitrary partition $\mu = (\mu_1, \mu_2, \dots, \mu_l)$, we define the Schur polynomial $s_{\mu}(u_1, u_3)$ by

$$s_{\mu}(u_1, u_3) = \det (p_{\mu_i - i + j}(u_1, u_3))_{1 \le i, j \le l}.$$

LEMMA 6. We have $s_{\mu}(u_1, u_3) \in \mathbb{Z}\langle\langle u_1, u_3\rangle\rangle$.

PROOF. For $i, j, k, \ell \ge 0$, we have

$$\frac{u_1^i u_3^j}{i!j!} \frac{u_1^k u_3^\ell}{k!\ell!} = \frac{(i+k)!(j+\ell)!}{i!j!k!\ell!} \frac{u_1^{i+k} u_3^{j+\ell}}{(i+k)!(j+\ell)!} = \binom{i+k}{i} \binom{j+\ell}{j} \frac{u_1^{i+k} u_3^{j+\ell}}{(i+k)!(j+\ell)!}.$$

Since $\binom{i+k}{i}\binom{j+\ell}{j} \in \mathbb{Z}$, we obtain the statement of the lemma. \square

$$t = \frac{x^2}{y}, \quad s = \frac{1}{x}.$$

Then t is a local parameter of V around ∞ . We have

$$x = \frac{1}{s}, \quad y = \frac{1}{s^2t}.$$
 (16)

Lemma 7. We have the following expansion of s in terms of t around t=0

$$s = t^2 \left(1 + \sum_{i=4}^{\infty} \gamma_i t^i \right),$$

where γ_i is a homogeneous polynomial in $\mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree i. In particular, we have

$$s = t^{2} + \lambda_{4}t^{6} + \lambda_{6}t^{8} + (2\lambda_{4}^{2} + \lambda_{8})t^{10} + (5\lambda_{4}\lambda_{6} + \lambda_{10})t^{12} + \cdots$$

PROOF. By substituting (16) into $y^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10}$, we have

$$s = t^2 + \lambda_4 s^2 t^2 + \lambda_6 s^3 t^2 + \lambda_8 s^4 t^2 + \lambda_{10} s^5 t^2.$$
 (17)

The expansion of s with respect to t around t = 0 takes the following form

$$s = t^2 \sum_{i=0}^{\infty} \gamma_i t^i,$$

where $\gamma_i \in \mathbb{C}$. By substituting the above expansion into (17), we have

$$\sum_{i=0}^{\infty} \gamma_i t^i = 1 + \lambda_4 t^4 \left(\sum_{i=0}^{\infty} \gamma_i t^i \right)^2 + \lambda_6 t^6 \left(\sum_{i=0}^{\infty} \gamma_i t^i \right)^3 + \lambda_8 t^8 \left(\sum_{i=0}^{\infty} \gamma_i t^i \right)^4 + \lambda_{10} t^{10} \left(\sum_{i=0}^{\infty} \gamma_i t^i \right)^5.$$

By comparing the coefficients, we obtain $\gamma_0 = 1$, $\gamma_4 = \lambda_4$, $\gamma_6 = \lambda_6$, $\gamma_8 = 2\lambda_4^2 + \lambda_8$, $\gamma_{10} = 5\lambda_4\lambda_6 + \lambda_{10}$ and $\gamma_n = 0$ for n = 1, 2, 3, 5, 7, 9. For $n \ge 10$, we have

$$\gamma_{n} = \lambda_{4} \sum_{(i_{1},i_{2})\in I_{1}} \gamma_{i_{1}} \gamma_{i_{2}} + \lambda_{6} \sum_{(i_{1},i_{2},i_{3})\in I_{2}} \gamma_{i_{1}} \gamma_{i_{2}} \gamma_{i_{3}} + \lambda_{8} \sum_{(i_{1},i_{2},i_{3},i_{4})\in I_{3}} \gamma_{i_{1}} \gamma_{i_{2}} \gamma_{i_{3}} \gamma_{i_{4}}
+ \lambda_{10} \sum_{(i_{1},i_{2},i_{3},i_{4},i_{5})\in I_{4}} \gamma_{i_{1}} \gamma_{i_{2}} \gamma_{i_{3}} \gamma_{i_{4}} \gamma_{i_{5}},$$

where $I_1 = \{(i_1, i_2) \in \mathbb{Z}_{\geq 0}^2 \mid i_1 + i_2 = n - 4\}$, $I_2 = \{(i_1, i_2, i_3) \in \mathbb{Z}_{\geq 0}^3 \mid i_1 + i_2 + i_3 = n - 6\}$, $I_3 = \{(i_1, i_2, i_3, i_4) \in \mathbb{Z}_{\geq 0}^4 \mid i_1 + i_2 + i_3 + i_4 = n - 8\}$, and $I_4 = \{(i_1, i_2, i_3, i_4, i_5) \in \mathbb{Z}_{\geq 0}^5 \mid i_1 + i_2 + i_3 + i_4 + i_5 = n - 10\}$. Therefore we obtain the statement of the lemma. \square

From Lemma 7, we have the expansions

$$x = \frac{1}{t^2} \left(1 + \sum_{i=4}^{\infty} d_i^{(1)} t^i \right), \quad y = \frac{1}{t^5} \left(1 + \sum_{i=4}^{\infty} d_i^{(2)} t^i \right),$$

where $d_i^{(1)}, d_i^{(2)}$ are homogeneous polynomials in $\mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree i. We enumerate the monomials $x^m y^n$, where m is a non-negative integer and n = 0, 1, according as the order of a pole at ∞ and denote them by φ_j , $j \ge 1$. In particular we have $\varphi_1 = 1$. We set $e_i = t^{i+1}$. We expand $t^2 \varphi_j$ around ∞ with respect to t. Let

$$t^2 \varphi_j = \sum \xi_{i,j} e_i,$$

where $\xi_{i,j} \in \mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$. For a partition $\mu = (\mu_1, \mu_2, \dots,)$, we define

$$\xi_{\mu} = \det(\xi_{m_i,j})_{i,j \in \mathbb{N}},$$

where $m_i = \mu_i - i$ and the infinite determinant is well defined. Then we have $\xi_{\mu} \in \mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$. We define the tau function $\tau(u_1, u_3)$ by

$$\tau(u_1, u_3) = \sum_{\mu} \xi_{\mu} s_{\mu}(u_1, u_3),$$

where the sum is over all partitions. From Lemma 6, we obtain the following proposition.

PROPOSITION 3. We have $\tau(u_1, u_3) \in \mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}] \langle \langle u_1, u_3 \rangle \rangle$.

The expansions of du_i around ∞ with respect to t take the following form

$$du_i = \sum_{j=1}^{\infty} b_{ij} t^{j-1} dt.$$

Lemma 8. We have $b_{11} = 1, b_{13} = 0, b_{31} = 0, b_{33} = 1$.

PROOF. We have

$$du_1 = -\frac{x}{2y}dx = -\frac{t^{-2}(1 + \sum_{i=4}^{\infty} d_i^{(1)}t^i)}{2t^{-5}(1 + \sum_{i=4}^{\infty} d_i^{(2)}t^i)} \left(-2t^{-3} + \sum_{i=4}^{\infty} (i-2)d_i^{(1)}t^{i-3}\right) = (1 + O(t^4))dt.$$

Therefore we obtain $b_{11} = 1$ and $b_{13} = 0$. We have

$$du_3 = -\frac{dx}{2y} = -\frac{-2t^{-3} + \sum_{i=4}^{\infty} (i-2)d_i^{(1)}t^{i-3}}{2t^{-5}(1 + \sum_{i=4}^{\infty} d_i^{(2)}t^i)} = (t^2 + O(t^6))dt.$$

Therefore we obtain $b_{31} = 0$ and $b_{33} = 1$. \square

We define c_i by the following relation

$$\sqrt{\frac{du_3}{dt}} = t \exp\left(\sum_{i=1}^{\infty} \frac{c_i}{i} t^i\right).$$

Lemma 9. We have $c_1 = c_2 = c_3 = 0$.

PROOF. We have the following expansion

$$\frac{du_3}{dt} = t^2(1 + O(t^4)).$$

Therefore we have the following expansion

$$\sqrt{\frac{du_3}{dt}} = t(1 + O(t^4)).$$

On the other hand, we have

$$\exp\left(\sum_{i=1}^{\infty} \frac{c_i}{i} t^i\right) = 1 + c_1 t + \left(\frac{c_2}{2} + \frac{c_1^2}{2}\right) t^2 + \left(\frac{c_3}{3} + \frac{c_1 c_2}{2} + \frac{c_1^3}{6}\right) t^3 + \cdots$$

Thus we have $c_1 = c_2 = c_3 = 0$. \Box

We take the algebraic bilinear form

$$\omega(P,Q) = \frac{x_1^2 x_2^2 (x_1 + x_2) + \lambda_4 x_1 x_2 (x_1 + x_2) + 2\lambda_6 x_1 x_2 + \lambda_8 (x_1 + x_2) + 2y_1 y_2 + 2\lambda_{10}}{4y_1 y_2 (x_1 - x_2)^2} dx_1 dx_2,$$
(18)

where $P = (x_1, y_1), Q = (x_2, y_2) \in V$ ([15], p.217). We can expand $\omega(P, Q)$ around $\infty \times \infty$ as

$$\omega(P,Q) = \left(\frac{1}{(t_1 - t_2)^2} + \sum_{i,j \ge 1} q_{ij} t_1^{i-1} t_2^{j-1}\right) dt_1 dt_2, \tag{19}$$

where $q_{ij} \in \mathbb{C}$ and t_1, t_2 are copies of the local parameter t.

Lemma 10. We have $q_{11} = 0$, $q_{13} = q_{31} = \lambda_4$, $q_{51} = q_{15} = 2\lambda_6$, $q_{33} = 3\lambda_6$.

PROOF. We define f(t) by $s = t^2 f(t)$. From Lemma 7, we have

$$f(t) = 1 + \lambda_4 t^4 + \lambda_6 t^6 + \cdots$$

From (18) and (19), we obtain

$$AB - C = D \cdot (q_{11} + q_{31}t_1^2 + q_{13}t_2^2 + q_{51}t_1^4 + q_{15}t_2^4 + q_{33}t_1^2t_2^2 + \cdots),$$

where

$$A = t_1^2 f(t_1) + t_2^2 f(t_2) + \lambda_4 t_1^2 t_2^2 f(t_1) f(t_2) (t_1^2 f(t_1) + t_2^2 f(t_2)) + 2\lambda_6 t_1^4 t_2^4 f(t_1)^2 f(t_2)^2 + \lambda_8 t_1^4 t_2^4 f(t_1)^2 f(t_2)^2 (t_1^2 f(t_1) + t_2^2 f(t_2)) + 2t_1 t_2 f(t_1) f(t_2) + 2\lambda_{10} t_1^6 t_2^6 f(t_1)^3 f(t_2)^3 B = (1 + 3\lambda_4 t_1^4 + 4\lambda_6 t_1^6 + \cdots) (1 + 3\lambda_4 t_2^4 + 4\lambda_6 t_2^6 + \cdots) C = f(t_1) f(t_2) (t_1 + t_2)^2 \{1 + \lambda_4 (t_1^4 + t_1^2 t_2^2 + t_2^4) + \lambda_6 (t_1^6 + t_1^4 t_2^2 + t_1^2 t_2^4 + t_2^6) + \cdots \}^2 D = f(t_1) f(t_2) (t_1^2 - t_2^2)^2 \{1 + \lambda_4 (t_1^4 + t_1^2 t_2^2 + t_2^4) + \lambda_6 (t_1^6 + t_1^4 t_2^2 + t_1^2 t_2^4 + t_2^6) + \cdots \}^2.$$

By comparing the coefficient of t_1^4 , we obtain $q_{11}=0$. By comparing the coefficient of t_1^6 , we obtain $q_{31}=q_{13}=\lambda_4$. By comparing the coefficient of t_1^8 , we obtain $q_{51}=q_{15}=2\lambda_6$. By comparing the coefficient of $t_1^6t_2^2$, we obtain

$$q_{33} - 2q_{51} = -\lambda_6.$$

Therefore we obtain $q_{33} = 3\lambda_6$. \square

THEOREM 5. ([34], Theorem 4) We have

$$\tau(u_1, u_3) = \exp\left(-c_1u_1 - c_3u_3 + \frac{1}{2}q_{11}u_1^2 + \frac{1}{2}q_{33}u_3^2 + q_{13}u_1u_3\right)\sigma(b_{11}u_1 + b_{13}u_3, b_{31}u_1 + b_{33}u_3).$$

From Theorem 5, Lemma 8, Lemma 9, and Lemma 10, we have

$$\sigma(u_1, u_3) = \exp\left(-3\lambda_6 \frac{u_3^2}{2} - \lambda_4 u_1 u_3\right) \tau(u_1, u_3). \tag{20}$$

LEMMA 11. For any non-negative integer n, we have $\varepsilon_n := \frac{(2n)!}{2^n n!} \in \mathbb{Z}$.

PROOF. We have $\varepsilon_1 = 1 \in \mathbb{Z}$. Assume $\varepsilon_n \in \mathbb{Z}$. Then we have

$$\varepsilon_{n+1} = \frac{(2n+2)!}{2^{n+1}(n+1)!} = \frac{(2n+2)(2n+1)}{2(n+1)} \varepsilon_n = (2n+1)\varepsilon_n \in \mathbb{Z}.$$

By mathematical induction, we obtain the statement. \Box

Lemma 12. We have

$$\exp\left(-3\lambda_6 \frac{u_3^2}{2} - \lambda_4 u_1 u_3\right) \in \mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}] \langle \langle u_1, u_3 \rangle \rangle.$$

PROOF. We have

$$\exp\left(-3\lambda_6 \frac{u_3^2}{2} - \lambda_4 u_1 u_3\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-3\lambda_6 \frac{u_3^2}{2} - \lambda_4 u_1 u_3\right)^n.$$

From Lemma 11, for any $k, \ell \in \mathbb{Z}_{\geqslant 0}$, we have

$$\frac{1}{(k+\ell)!} {k+\ell \choose k} \left(-3\lambda_6 \frac{u_3^2}{2} \right)^k (-\lambda_4 u_1 u_3)^{\ell}
= (-3\lambda_6)^k (-\lambda_4)^{\ell} \frac{u_1^{\ell}}{\ell!} \frac{u_3^{2k+\ell}}{(2k+\ell)!} \frac{(2k+\ell)!}{2^k k!} \in \mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}] \langle \langle u_1, u_3 \rangle \rangle.$$

Therefore we obtain the statement. \Box

From Proposition 3, Lemma 12, and (20), we obtain the following theorem.

Theorem 6. (39) We have

$$\sigma(u_1, u_3) \in \mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}] \langle \langle u_1, u_3 \rangle \rangle.$$

2.4. Universal Bernoulli numbers

In this subsection, we will describe the definition of the universal Bernoulli numbers and their properties according to [38].

Let f_1, f_2, \ldots be infinitely many indeterminates. We consider the power series

$$u = u(z) = z + \sum_{n=1}^{\infty} f_n \frac{z^{n+1}}{n+1}$$

and its formal inverse series

$$z = z(u) = u - f_1 \frac{u^2}{2!} + (3f_1^2 - 2f_2) \frac{u^3}{3!} + \cdots,$$

namely, the series such that u(z(u)) = u. Then we define $\hat{B}_n \in \mathbb{Q}[f_1, f_2, \dots]$ by

$$\frac{u}{z(u)} = \sum_{n=0}^{\infty} \hat{B}_n \frac{u^n}{n!}$$

and call them the universal Bernoulli numbers. We have $\hat{B}_0 = 1$. If we set $\deg f_i = i$, then \hat{B}_n is homogeneous of degree n if $\hat{B}_n \neq 0$. Let \mathcal{S} be the set of finite sequences $U = (U_1, U_2, \dots)$ of non-negative integers. For $U = (U_1, U_2, \dots) \in \mathcal{S}$, we use the notations $U! = U_1!U_2! \cdots$, $\Lambda^U = 2^{U_1}3^{U_2}4^{U_3} \cdots$, $f^U = f_1^{U_1}f_2^{U_2} \cdots$, $\gamma_U = \Lambda^U U!$, $w(U) = \sum_j jU_j$, and $d(U) = \sum_j U_j$.

Proposition 4. ([38] Proposition 2.8) We have the expression

$$\frac{\hat{B}_n}{n} = \sum_{w(U)=n} \tau_U f^U,$$

where

$$\tau_U = (-1)^{d(U)-1} \frac{(w(U) + d(U) - 2)!}{\gamma_U}.$$
(21)

For a rational number α , we denote by $\lfloor \alpha \rfloor$ the largest integer which does not exceed α . If p is a prime and the p-part of given rational number r is p^e , then we write $e = \operatorname{ord}_p r$. If τ is a polynomial (possibly in several variables) with rational coefficients, then we denote by $\operatorname{ord}_p \tau$ the least number of $\operatorname{ord}_p r$ for all the coefficients r of τ . For a prime number p and an integer a, let $a|_p = a/p^{ord_p a}$. For positive integers a, b and a prime number p, we have

$$\operatorname{ord}_p(a+b)! - \operatorname{ord}_p a! - \operatorname{ord}_p b! = \operatorname{ord}_p \frac{(a+b)!}{a!b!}.$$

Since $(a+b)!/(a!b!) \in \mathbb{Z}$, we have

$$\operatorname{ord}_{p}(a+b)! \geqslant \operatorname{ord}_{p}a! + \operatorname{ord}_{p}b!. \tag{22}$$

For a positive integer a and a prime number p

$$\operatorname{ord}_{p}(a!) = \sum_{\nu=1}^{\infty} \left\lfloor \frac{a}{p^{\nu}} \right\rfloor. \tag{23}$$

If positive integers a and b are relatively prime, we denote it by $a \perp b$.

LEMMA 13. ([38] Proposition 3.11) Let p be an odd prime and $U=(U_1,U_2,\ldots,)$ be an element of S. If $p \geqslant 5$, we assume $U_1=U_2=U_{p-1}=U_{2p-1}=0$ and $d(U)\neq 0$. If p=3, we assume $U_1=U_2=U_5=U_8=0$ and $d(U)\neq 0$. Then τ_U defined in (21) satisfies

$$ord_p(\tau_U) \geqslant \left| \frac{w(U) + d(U) - 2}{2p} \right|.$$

PROOF. For the sake to be complete and self-contained, we give a proof of this lemma. By using (22) and (23), we have

$$\begin{aligned} \operatorname{ord}_{p}(\tau_{U}) &= \operatorname{ord}_{p}(w(U) + d(U) - 2)! - \operatorname{ord}_{p}(\gamma_{U}) \\ &= \operatorname{ord}_{p}\left(\sum_{j\geqslant 1} jU_{j} + \sum_{j\geqslant 1} U_{j} - 2\right)! - \sum_{\varepsilon,k\geqslant 1,\; \varepsilon\perp p} kU_{\varepsilon p^{k}-1} - \sum_{j\geqslant 1} \operatorname{ord}_{p}(U_{j}!) \\ &\geqslant \operatorname{ord}_{p}\left(\sum_{j\geqslant 1} jU_{j} - 2\right)! - \sum_{\varepsilon,k\geqslant 1,\; \varepsilon\perp p} kU_{\varepsilon p^{k}-1} \\ &= \operatorname{ord}_{p}\left(-2 + \sum_{p\nmid j+1} jU_{j} + \sum_{\varepsilon,k\geqslant 1,\; \varepsilon\perp p} (\varepsilon p^{k} - 1)U_{\varepsilon p^{k}-1}\right)! - \sum_{\varepsilon,k\geqslant 1,\; \varepsilon\perp p} kU_{\varepsilon p^{k}-1} \\ &= \sum_{\nu=1}^{\infty} \left\lfloor \frac{1}{p^{\nu}} \left(-2 + \sum_{p\nmid j+1} jU_{j} + \sum_{\varepsilon,k\geqslant 1,\; \varepsilon\perp p} (\varepsilon p^{k} - 1)U_{\varepsilon p^{k}-1}\right) \right\rfloor - \sum_{\varepsilon,k\geqslant 1,\; \varepsilon\perp p} kU_{\varepsilon p^{k}-1} \\ &\geqslant \left\lfloor \frac{1}{p} \left(-2 + \sum_{p\nmid j+1} jU_{j} + \sum_{\varepsilon,k\geqslant 1,\; \varepsilon\perp p} (\varepsilon p^{k} - 1)U_{\varepsilon p^{k}-1}\right) \right\rfloor - \sum_{\varepsilon,k\geqslant 1,\; \varepsilon\perp p} kU_{\varepsilon p^{k}-1} \\ &= \left\lfloor \frac{1}{p} \left(-2 + \sum_{p\nmid j+1} jU_{j} + \sum_{\varepsilon,k\geqslant 1,\; \varepsilon\perp p} (\varepsilon p^{k} - kp - 1)U_{\varepsilon p^{k}-1}\right) \right\rfloor \\ &= \left\lfloor \frac{1}{2p} \left(-4 + \sum_{p\nmid j+1} 2jU_{j} + \sum_{\varepsilon,k\geqslant 1,\; \varepsilon\perp p} 2(\varepsilon p^{k} - kp - 1)U_{\varepsilon p^{k}-1}\right) \right\rfloor. \end{aligned}$$

By the assumption of the lemma, there exists a positive integer i such that $i \ge 3$ and $U_i \ne 0$. For $j \ge 3$, we have $2j - (j+1) \ge 2$. If $p \ge 5$, let $T_p = \{(1,1),(2,1)\}$. If p = 3, let $T_p = \{(1,1),(2,1),(1,2)\}$. By the assumption of the lemma, we have $U_{\varepsilon p^k-1} = 0$ for any $(\varepsilon,k) \in T_p$. For any integers $\varepsilon \ge 1$ and $k \ge 1$ such that $(\varepsilon,k) \notin T_p$ and $\varepsilon \perp p$, we can check

$$2(\varepsilon p^k - kp - 1) - \varepsilon p^k \geqslant 2.$$

Therefore we have

$$\operatorname{ord}_{p}(\tau_{U}) \geq \left[\frac{1}{2p}\left(-2 + \sum_{p\nmid j+1}(j+1)U_{j} + \sum_{\varepsilon,k\geqslant 1,\ \varepsilon\perp p}\varepsilon p^{k}U_{\varepsilon p^{k}-1}\right)\right] = \left\lfloor\frac{w(U) + d(U) - 2}{2p}\right\rfloor.$$

LEMMA 14. Let $U = (U_1, U_2, ...,)$ be an element of S such that $U_i = 0$ for any odd integer i, $U_2 = 0$, and $d(U) \neq 0$. Then we have

$$ord_2(\tau_U) \geqslant \left| \frac{w(U) + d(U) - 2}{4} \right|.$$

PROOF. We can prove this lemma in the same way as [38] Proposition 3.11. For the sake to be complete and self-contained, we give a proof of this lemma. By using (22), (23), and $\operatorname{ord}_2(\Lambda^U) = 0$, we have

$$\operatorname{ord}_{2}(\tau_{U}) = \operatorname{ord}_{2}(w(U) + d(U) - 2)! - \operatorname{ord}_{2}(\gamma_{U}) = \operatorname{ord}_{2}\left(\sum_{j \geqslant 4} jU_{j} + \sum_{j \geqslant 4} U_{j} - 2\right)! - \operatorname{ord}_{2}(U!)$$

$$\geqslant \operatorname{ord}_{2}\left(\sum_{j \geqslant 4} jU_{j} - 2\right)! = \sum_{\nu=1}^{\infty} \left\lfloor \frac{\sum_{j \geqslant 4} jU_{j} - 2}{2^{\nu}} \right\rfloor \geqslant \left\lfloor \frac{\sum_{j \geqslant 4} jU_{j} - 2}{2} \right\rfloor$$

$$= \left\lfloor \frac{\sum_{j \geqslant 4} 2jU_{j} - 4}{4} \right\rfloor.$$

Since $j \ge 4$, we have $2j - (j + 1) \ge 3$. Therefore we have

$$\left| \frac{\sum_{j \geqslant 4} 2jU_j - 4}{4} \right| \geqslant \left| \frac{\sum_{j \geqslant 4} (j+1)U_j - 2}{4} \right| = \left| \frac{w(U) + d(U) - 2}{4} \right|.$$

In [22], F. Clarke showed the following results, which are a generalization of the von Staudt-Clausen theorem for the classical Bernoulli numbers to the universal Bernoulli numbers, cf. the paper of Onishi [38]. These results were used in the proof of our Theorem 11 in the present survey.

Theorem 7. (i) We have

$$\hat{B}_1 = \frac{1}{2}f_1, \quad \frac{\hat{B}_2}{2} = -\frac{1}{4}f_1^2 + \frac{1}{3}f_2,$$

(ii) If $n \equiv 0 \mod 4$, then we have

$$\frac{\hat{B}_n}{n} \equiv \sum_{n=a(p-1), \ p \ : \ prime} \frac{a|_p^{-1} \mod p^{1+ord_p a}}{p^{1+ord_p a}} f_{p-1}^a \mod \mathbb{Z}[f_1, f_2, \dots].$$

(iii) If $n \equiv 2 \mod 4$ and $n \neq 2$, then we have

$$\frac{\hat{B}_n}{n} \equiv \frac{f_1^{n-6} f_3^2}{2} - \frac{n f_1^n}{8} + \sum_{n=a(p-1), \ p : odd \ prime} \frac{a|_p^{-1} \mod p^{1+ord_p a}}{p^{1+ord_p a}} f_{p-1}^a \mod \mathbb{Z}[f_1, f_2, \dots].$$

(iv) If $n \equiv 1, 3 \mod 4$ and $n \neq 1$, then we have

$$\frac{\hat{B}_n}{n} \equiv \frac{f_1^n + f_1^{n-3} f_3}{2} \mod \mathbb{Z}[f_1, f_2, \dots].$$

In (ii) and (iii), $a|_p^{-1} \mod p^{1+ord_pa}$ denotes an integer δ such that $(a|_p)\delta \equiv 1 \mod p^{1+ord_pa}$.

3. Differential equations for the coefficients of the expansions of the two-dimensional sigma function

In this section, we assume g=2.

3.1. The coefficients of u_3

Let $\lambda = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$. We set

$$\sigma(u_1, u_3; \lambda) = \sum_{k \ge 0} \xi_k(u_1; \lambda) \frac{u_3^k}{k!}.$$

PROPOSITION 5. For $k \ge 0$, the functions ξ_0, ξ_1, \ldots satisfy the following hierarchy of systems

$$u_1 \xi_k' + 3(k-1)\xi_k = 4\lambda_4 \xi_{k,\lambda_4} + 6\lambda_6 \xi_{k,\lambda_6} + 8\lambda_8 \xi_{k,\lambda_8} + 10\lambda_{10} \xi_{k,\lambda_{10}}, \tag{24}$$

$$u_1 \xi_{k+1} = -\frac{1}{2} \xi_k'' + \frac{4k}{5} \lambda_4 \xi_{k-1}' + \frac{3}{10} \lambda_4 u_1^2 \xi_k - \frac{k(k-1)}{10} (15\lambda_8 - 4\lambda_4^2) \xi_{k-2}$$

$$+6\lambda_6 \xi_{k,\lambda_4} + (8\lambda_8 - \frac{12}{5} \lambda_4^2) \xi_{k,\lambda_6} + (10\lambda_{10} - \frac{8}{5} \lambda_4 \lambda_6) \xi_{k,\lambda_8} - \frac{4}{5} \lambda_4 \lambda_8 \xi_{k,\lambda_{10}},$$
(25)

$$\xi'_{k+1} = \frac{6k}{5}\lambda_6 \xi'_{k-1} - k\lambda_4 \xi_k + \frac{\lambda_6}{5}u_1^2 \xi_k - k\lambda_8 u_1 \xi_{k-1} - \frac{k(k-1)}{10} (30\lambda_{10} - 6\lambda_4 \lambda_6) \xi_{k-2} + \lambda_4 \xi_k + 8\lambda_8 \xi_{k,\lambda_4} + (10\lambda_{10} - \frac{8}{5}\lambda_4 \lambda_6) \xi_{k,\lambda_6} + (4\lambda_4 \lambda_8 - \frac{12}{5}\lambda_6^2) \xi_{k,\lambda_8} + (6\lambda_4 \lambda_{10} - \frac{6}{5}\lambda_6 \lambda_8) \xi_{k,\lambda_{10}},$$
 (26)

$$\xi_{k+2} = \frac{6k}{5} \lambda_8 \xi'_{k-1} + \frac{\lambda_8}{5} u_1^2 \xi_k - 4k \lambda_{10} u_1 \xi_{k-1} + \frac{3k(k-1)}{5} \lambda_8 \lambda_4 \xi_{k-2} + \lambda_6 \xi_k$$

$$+20\lambda_{10} \xi_{k,\lambda_4} - \frac{8}{5} \lambda_4 \lambda_8 \xi_{k,\lambda_6} + (12\lambda_4 \lambda_{10} - \frac{12}{5} \lambda_6 \lambda_8) \xi_{k,\lambda_8} + (8\lambda_6 \lambda_{10} - \frac{16}{5} \lambda_8^2) \xi_{k,\lambda_{10}},$$
(27)

where the prime denotes the derivation with respect to u_1 and $\xi_{k,\lambda_{2j}}$ denotes the derivation of ξ_k with respect to λ_{2j} .

PROOF. We substitute the expressions

$$\sigma_{1} = \sum_{k \geqslant 0} \xi'_{k}(u_{1}) \frac{u_{3}^{k}}{k!}, \quad \sigma_{3} = \sum_{k \geqslant 0} \xi_{k+1}(u_{1}) \frac{u_{3}^{k}}{k!}, \quad \sigma_{11} = \sum_{k \geqslant 0} \xi''_{k}(u_{1}) \frac{u_{3}^{k}}{k!},$$

$$\sigma_{13} = \sum_{k \geqslant 0} \xi'_{k+1}(u_{1}) \frac{u_{3}^{k}}{k!}, \quad \sigma_{33} = \sum_{k \geqslant 0} \xi_{k+2}(u_{1}) \frac{u_{3}^{k}}{k!}.$$

into the equations $Q_i\sigma=0$ for i=0,2,4,6 in Theorem 2 (ii) and compare the coefficients of $u_3^k/k!$. Then, from $Q_0\sigma=0, Q_2\sigma=0, Q_4\sigma=0, Q_6\sigma=0$, we obtain (24), (25), (26), (27), respectively. \square

LEMMA 15. The fact that ξ_k satisfies the differential equation (24) means that ξ_k is homogeneous in u_1 and λ_{2j} , $j=2,\ldots,5$, with degree 3k-3.

Proof. We set

$$\xi_k(u_1;\lambda) = \sum_{i_1,j_4,j_6,j_8,j_{10}\geqslant 0} a^{(k)}_{i_1,j_4,j_6,j_8,j_{10}} u^{i_1}_1 \lambda^{j_4}_4 \lambda^{j_6}_6 \lambda^{j_8}_8 \lambda^{j_{10}}_{10}, \qquad a^{(k)}_{i_1,j_4,j_6,j_8,j_{10}} \in \mathbb{C}.$$

By substituting the above expression into (24) and comparing the coefficient of $u_1^{i_1} \lambda_4^{j_4} \lambda_6^{j_6} \lambda_8^{j_8} \lambda_{10}^{j_{10}}$, we have

$$a_{i_1,j_4,j_6,j_8,j_{10}}^{(k)}(i_1+3k-3) = a_{i_1,j_4,j_6,j_8,j_{10}}^{(k)}(4j_4+6j_6+8j_8+10j_{10}).$$

If
$$a_{i_1,j_4,j_6,j_8,j_{10}}^{(k)} \neq 0$$
, we have $4j_4+6j_6+8j_8+10j_{10}-i_1=3k-3$. \square

Proposition 6. The function ξ_0 satisfies the following differential equation

$$\begin{split} \frac{1}{2}\xi_0'' - \frac{1}{2}u_1\xi_0''' - \frac{7}{10}\lambda_4u_1^2\xi_0 - \frac{\lambda_6}{5}u_1^4\xi_0 + \frac{3}{10}\lambda_4u_1^3\xi_0' - 8\lambda_8u_1^2\xi_{0,\lambda_4} \\ - (10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6)u_1^2\xi_{0,\lambda_6} - (4\lambda_4\lambda_8 - \frac{12}{5}\lambda_6^2)u_1^2\xi_{0,\lambda_8} - (6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8)u_1^2\xi_{0,\lambda_{10}} \\ + 6\lambda_6u_1\xi_{0,\lambda_4}' + (8\lambda_8 - \frac{12}{5}\lambda_4^2)u_1\xi_{0,\lambda_6}' + (10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6)u_1\xi_{0,\lambda_8}' - \frac{4}{5}\lambda_4\lambda_8u_1\xi_{0,\lambda_{10}}' \\ - 6\lambda_6\xi_{0,\lambda_4} - (8\lambda_8 - \frac{12}{5}\lambda_4^2)\xi_{0,\lambda_6} - (10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6)\xi_{0,\lambda_8} + \frac{4}{5}\lambda_4\lambda_8\xi_{0,\lambda_{10}} = 0. \end{split}$$

PROOF. From (25) with k = 0, we have

$$u_1\xi_1 = -\frac{1}{2}\xi_0'' + \frac{3}{10}\lambda_4 u_1^2 \xi_0 + 6\lambda_6 \xi_{0,\lambda_4} + (8\lambda_8 - \frac{12}{5}\lambda_4^2)\xi_{0,\lambda_6} + (10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6)\xi_{0,\lambda_8} - \frac{4}{5}\lambda_4\lambda_8 \xi_{0,\lambda_{10}}.$$
(28)

From (26) with k = 0, ξ'_1 can be expressed in terms of ξ_0 and its derivatives. We take the derivative with respect to u_1 of (28) and substitute into it the expression for ξ'_1 . As a result, we obtain

$$\xi_{1} = -\frac{1}{2}\xi_{0}^{\prime\prime\prime} - \frac{2}{5}\lambda_{4}u_{1}\xi_{0} - \frac{\lambda_{6}}{5}u_{1}^{3}\xi_{0} + \frac{3}{10}\lambda_{4}u_{1}^{2}\xi_{0}^{\prime} - 8\lambda_{8}u_{1}\xi_{0,\lambda_{4}} - (10\lambda_{10} - \frac{8}{5}\lambda_{4}\lambda_{6})u_{1}\xi_{0,\lambda_{6}}$$

$$-(4\lambda_{4}\lambda_{8} - \frac{12}{5}\lambda_{6}^{2})u_{1}\xi_{0,\lambda_{8}} - (6\lambda_{4}\lambda_{10} - \frac{6}{5}\lambda_{6}\lambda_{8})u_{1}\xi_{0,\lambda_{10}} + 6\lambda_{6}\xi_{0,\lambda_{4}}^{\prime} + (8\lambda_{8} - \frac{12}{5}\lambda_{4}^{2})\xi_{0,\lambda_{6}}^{\prime}$$

$$+(10\lambda_{10} - \frac{8}{5}\lambda_{4}\lambda_{6})\xi_{0,\lambda_{8}}^{\prime} - \frac{4}{5}\lambda_{4}\lambda_{8}\xi_{0,\lambda_{10}}^{\prime}.$$

We substitute the above equation into (28) and finally obtain the statement of the proposition. \Box We set

$$\xi_0 = \sum_{\ell \geqslant 0} p_\ell(\lambda) \frac{u_1^\ell}{\ell!}.\tag{29}$$

Lemma 16. If ℓ is even, then $p_{\ell}=0$. We have $p_1=0$ and $p_3\in\mathbb{C}$.

PROOF. From Lemma 15 and (24) with k=0, we obtain the statement of the lemma. \Box

Proposition 7. For $\ell \geqslant 2$, the following recurrence relation holds :

$$\begin{split} p_{\ell+2} &= \frac{\lambda_4 \ell (3\ell-13)}{5} p_{\ell-2} - \frac{2\lambda_6}{5} \ell (\ell-2)(\ell-3) p_{\ell-4} - 16\lambda_8 \ell p_{\ell-2,\lambda_4} \\ &- (20\lambda_{10} - \frac{16}{5}\lambda_4 \lambda_6) \ell p_{\ell-2,\lambda_6} - (8\lambda_4 \lambda_8 - \frac{24}{5}\lambda_6^2) \ell p_{\ell-2,\lambda_8} - (12\lambda_4 \lambda_{10} - \frac{12}{5}\lambda_6 \lambda_8) \ell p_{\ell-2,\lambda_{10}} \\ &+ 12\lambda_6 p_{\ell,\lambda_4} + (16\lambda_8 - \frac{24}{5}\lambda_4^2) p_{\ell,\lambda_6} + (20\lambda_{10} - \frac{16}{5}\lambda_4 \lambda_6) p_{\ell,\lambda_8} - \frac{8}{5}\lambda_4 \lambda_8 p_{\ell,\lambda_{10}}, \end{split}$$

where $p_{i,\lambda_{2j}}$ denotes the derivative of p_i with respect to λ_{2j} .

PROOF. By substituting (29) into the differential equation in Proposition 6 and comparing the coefficients of $u_1^{\ell}/\ell!$, we obtain the statement of the proposition. \Box

COROLLARY 1. The sigma function $\sigma(u_1, u_3; \lambda)$ is uniquely determined by the differential equations $Q_i \sigma = 0$, i = 0, 2, 4, up to a multiplicative constant.

PROOF. From Lemma 16 and Proposition 7, we find that all the coefficients p_{ℓ} are determined from p_3 . Note that Lemma 16 and Proposition 7 follow from (24), (25), and (26). By (25), all the functions ξ_k are determined from ξ_0 . As mentioned in the proof of Proposition 5, (24), (25), (26) follow from $Q_0\sigma = 0$, $Q_2\sigma = 0$, $Q_4\sigma = 0$. Therefore we obtain the statement of this corollary. \square

REMARK 1. It is known that the sigma function $\sigma(u_1, u_3; \lambda)$ is uniquely determined by the differential equations $Q_i \sigma = 0$, i = 0, 2, 4, 6, up to a multiplicative constant ([18], [19], [24]). In [11], the following expression is proved:

$$10Q_6 = 5[Q_2, Q_4] - 8\lambda_6 Q_0 + 8\lambda_4 Q_2,$$

where $[Q_2, Q_4]$ is the commutator of Q_2 and Q_4 . From this result, in [11], it is shown that the sigma function $\sigma(u_1, u_3; \lambda)$ is uniquely determined by the differential equations $Q_i \sigma = 0$, i = 0, 2, 4, up to a multiplicative constant.

COROLLARY 2. We have

$$\sigma(u_1, u_3) \in \mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, 2\lambda_{10}] \langle \langle u_1, u_3 \rangle \rangle.$$

PROOF. From Lemma 16 and (9), we have $p_0 = p_1 = p_2 = 0$ and $p_3 = 2$. From Proposition 7, we can show $p_{\ell} \in \mathbb{Z}[1/5, \lambda_4, \lambda_6, \lambda_8, 2\lambda_{10}]$ for any ℓ by mathematical induction. Therefore we have $\xi_0 \in \mathbb{Z}[1/5, \lambda_4, \lambda_6, \lambda_8, 2\lambda_{10}]\langle\langle u_1 \rangle\rangle$. From (26) with k = 0, we can show $\xi'_1 \in \mathbb{Z}[1/5, \lambda_4, \lambda_6, \lambda_8, 2\lambda_{10}]\langle\langle u_1 \rangle\rangle$. Since $\xi_1 = -1 + O(u_1)$, we have $\xi_1 \in \mathbb{Z}[1/5, \lambda_4, \lambda_6, \lambda_8, 2\lambda_{10}]\langle\langle u_1 \rangle\rangle$. From (27), we can show $\xi_k \in \mathbb{Z}[1/5, \lambda_4, \lambda_6, \lambda_8, 2\lambda_{10}]\langle\langle u_1 \rangle\rangle$ for any k by mathematical induction. Therefore we have

$$\sigma(u_1, u_3) \in \mathbb{Z}[1/5, \lambda_4, \lambda_6, \lambda_8, 2\lambda_{10}] \langle \langle u_1, u_3 \rangle \rangle.$$

From Theorem 6, we obtain the statement of the corollary. \Box

3.2. Expansions of ξ_k

In this subsection we will calculate the expansions of ξ_k . From (9), we have $p_3 = 2$. The initial terms of ξ_i , i = 0, 1, 2, 3, 4, are as follows.

terms of
$$\xi_i$$
, $i = 0, 1, 2, 3, 4$, are as follows.
$$\xi_0 = 2\frac{u_1^3}{3!} + 2^2\lambda_4\frac{u_1^7}{7!} - 2^6\lambda_6\frac{u_1^9}{9!} + 2^3(51\lambda_4^2 - 200\lambda_8)\frac{u_1^{11}}{11!} + 2^7(67\lambda_4\lambda_6 - 140\lambda_{10})\frac{u_1^{13}}{13!} + \cdots,$$

$$\xi_1 = -1 + 2\lambda_4\frac{u_1^4}{4!} + 2^3\lambda_6\frac{u_1^6}{6!} + 2^2(\lambda_4^2 + 8\lambda_8)\frac{u_1^8}{8!} + 2^5(3\lambda_4\lambda_6 - 20\lambda_{10})\frac{u_1^{10}}{10!} + 2^3(304\lambda_6^2 + 51\lambda_4^3 - 184\lambda_4\lambda_8)\frac{u_1^{12}}{12!} + 2^5(1256\lambda_6\lambda_8 + 237\lambda_4^2\lambda_6 - 1240\lambda_4\lambda_{10})\frac{u_1^{14}}{14!} + \cdots,$$

$$\xi_2 = 2\lambda_6\frac{u_1^3}{3!} + 2^3\lambda_8\frac{u_2^5}{5!} + 2^2(20\lambda_{10} + \lambda_4\lambda_6)\frac{u_1^7}{7!} + 2^5(5\lambda_4\lambda_8 - 2\lambda_6^2)\frac{u_1^9}{9!} + 2^3(51\lambda_4^2\lambda_6 + 104\lambda_6\lambda_8 - 360\lambda_4\lambda_{10})\frac{u_1^{11}}{11!} + 2^5(232\lambda_8^2 - 31\lambda_4^2\lambda_8 + 268\lambda_4\lambda_6^2 + 320\lambda_6\lambda_{10})\frac{u_1^{13}}{13!} + \cdots,$$

$$\xi_3 = -\lambda_6 + 2\lambda_8\frac{u_1^2}{2!} + 2(\lambda_4\lambda_6 + 4\lambda_{10})\frac{u_1^4}{4!} + 2^2(\lambda_4\lambda_8 + 2\lambda_6^2)\frac{u_1^6}{6!} + 2^2(\lambda_4^2\lambda_6 + 104\lambda_4\lambda_{10} - 8\lambda_6\lambda_8)\frac{u_1^8}{8!} + 2^3(88\lambda_8^2 + 51\lambda_4^2\lambda_8 + 12\lambda_4\lambda_6^2 - 160\lambda_6\lambda_{10})\frac{u_1^{10}}{10!} + (7104\lambda_4\lambda_6\lambda_8 + 8960\lambda_8\lambda_{10} + 2432\lambda_6^3 + 408\lambda_4^3\lambda_6 - 12768\lambda_4^2\lambda_{10})\frac{u_1^{12}}{12!} + \cdots,$$

$$\xi_4 = 2^3\lambda_{10}\frac{u_1}{1!} + 2(\lambda_6^2 + 2\lambda_4\lambda_8)\frac{u_1^3}{3!} + 2^4(\lambda_6\lambda_8 + \lambda_4\lambda_{10})\frac{u_1^5}{5!} + 2^2(2\lambda_4^2\lambda_8 + \lambda_4\lambda_6^2 + 88\lambda_6\lambda_{10} - 28\lambda_8^2)\frac{u_1^7}{7!} + 2^5(6\lambda_4\lambda_6\lambda_8 + 16\lambda_8\lambda_{10} - 2\lambda_6^3 + 51\lambda_4^2\lambda_{10})\frac{u_1^9}{9!}$$

 $+2^{3}(744\lambda_{4}\lambda_{8}^{2}+408\lambda_{6}^{2}\lambda_{8}+102\lambda_{4}^{3}\lambda_{8}+51\lambda_{4}^{2}\lambda_{6}^{2}-1008\lambda_{4}\lambda_{6}\lambda_{10}-160\lambda_{10}^{2})\frac{u_{11}^{11}}{111}$

 $+2^{6}(960\lambda_{6}\lambda_{8}^{2}+237\lambda_{4}^{2}\lambda_{6}\lambda_{8}+1072\lambda_{4}\lambda_{8}\lambda_{10}+134\lambda_{4}\lambda_{6}^{3}+168\lambda_{6}^{2}\lambda_{10}-849\lambda_{4}^{3}\lambda_{10})\frac{u_{1}^{13}}{12!}+\cdots$

3.3. The coefficients of u_1

We set

$$\sigma(u_1, u_3; \lambda) = \sum_{k \geqslant 0} \mu_k(u_3; \lambda) \frac{u_1^k}{k!}.$$
(30)

Proposition 8. For $k \ge 0$, the functions μ_0, μ_1, \ldots satisfy the following hierarchy of systems

$$(k-3)\mu_k + 3u_3\mu_k' = 4\lambda_4\mu_{k,\lambda_4} + 6\lambda_6\mu_{k,\lambda_6} + 8\lambda_8\mu_{k,\lambda_8} + 10\lambda_{10}\mu_{k,\lambda_{10}},$$
(31)

$$\mu_{k+2} = \frac{8}{5} \lambda_4 u_3 \mu_{k+1} - 2k \mu'_{k-1} + \frac{3k(k-1)}{5} \lambda_4 \mu_{k-2} - \frac{15\lambda_8 - 4\lambda_4^2}{5} u_3^2 \mu_k$$

$$+ 12\lambda_6 \mu_{k,\lambda_4} + (16\lambda_8 - \frac{24}{5}\lambda_4^2) \mu_{k,\lambda_6} + (20\lambda_{10} - \frac{16}{5}\lambda_4 \lambda_6) \mu_{k,\lambda_8} - \frac{8}{5}\lambda_4 \lambda_8 \mu_{k,\lambda_{10}},$$
(32)

$$\mu'_{k+1} - \frac{6}{5}\lambda_6 u_3 \mu_{k+1} = -\lambda_4 u_3 \mu'_k + \frac{k(k-1)}{5}\lambda_6 \mu_{k-2} - k\lambda_8 u_3 \mu_{k-1} - (3\lambda_{10} - \frac{3}{5}\lambda_4 \lambda_6) u_3^2 \mu_k$$

$$+\lambda_4 \mu_k + 8\lambda_8 \mu_{k,\lambda_4} + (10\lambda_{10} - \frac{8}{5}\lambda_4 \lambda_6) \mu_{k,\lambda_6} + (4\lambda_4 \lambda_8 - \frac{12}{5}\lambda_6^2) \mu_{k,\lambda_8}$$

$$+(6\lambda_4 \lambda_{10} - \frac{6}{5}\lambda_6 \lambda_8) \mu_{k,\lambda_{10}},$$

$$(33)$$

$$\lambda_8 u_3 \mu_{k+1} = \frac{5}{6} \mu_k'' - \frac{k(k-1)}{6} \lambda_8 \mu_{k-2} + \frac{10}{3} k \lambda_{10} u_3 \mu_{k-1} - \frac{\lambda_4 \lambda_8}{2} u_3^2 \mu_k - \frac{5}{6} \lambda_6 \mu_k$$

$$- \frac{50}{3} \lambda_{10} \mu_{k,\lambda_4} + \frac{4}{3} \lambda_4 \lambda_8 \mu_{k,\lambda_6} - (10\lambda_4 \lambda_{10} - 2\lambda_6 \lambda_8) \mu_{k,\lambda_8} - (\frac{20}{3} \lambda_6 \lambda_{10} - \frac{8}{3} \lambda_8^2) \mu_{k,\lambda_{10}}, \tag{34}$$

where the prime denotes the derivation with respect to u_3 and $\mu_{k,\lambda_{2j}}$ denotes the derivation of μ_k with respect to λ_{2j} .

PROOF. We substitute the expression (30) into the equations $Q_i\sigma = 0$ for i = 0, 2, 4, 6 in Theorem 2 (ii) and compare the coefficients of $u_1^k/k!$. Then, from $Q_0\sigma = 0$, $Q_2\sigma = 0$, $Q_4\sigma = 0$, $Q_6\sigma = 0$, we obtain (31), (32), (33), (34), respectively. \square

LEMMA 17. The fact that μ_k satisfies the differential equation (31) means that μ_k is homogeneous in u_3 and λ_{2j} , $j=2,\ldots,5$, with degree k-3.

Proof. In the same way as Lemma 15, we obtain the statement of the lemma. \Box

Proposition 9. The function μ_0 satisfies the following differential equation

$$\mu_0'' + \frac{9}{5}\lambda_4\lambda_8u_3^2\mu_0 - \lambda_6\mu_0 - \frac{6}{5}\lambda_6^2u_3^2\mu_0 - \frac{18}{5}\lambda_8\lambda_{10}u_3^4\mu_0 - u_3\mu_0''' + \frac{6}{5}\lambda_6u_3^2\mu_0'' - \frac{3}{5}\lambda_4\lambda_8u_3^3\mu_0'$$

$$+\lambda_6u_3\mu_0' - 20\lambda_{10}\mu_{0,\lambda_4} + \frac{8}{5}\lambda_4\lambda_8\mu_{0,\lambda_6} + \left(\frac{12}{5}\lambda_6\lambda_8 - 12\lambda_4\lambda_{10}\right)\mu_{0,\lambda_8} + \left(\frac{16}{5}\lambda_8^2 - 8\lambda_6\lambda_{10}\right)\mu_{0,\lambda_{10}}$$

$$+20\lambda_{10}u_3\mu_{0,\lambda_4}' - \frac{8}{5}\lambda_4\lambda_8u_3\mu_{0,\lambda_6}' + \left(12\lambda_4\lambda_{10} - \frac{12}{5}\lambda_6\lambda_8\right)u_3\mu_{0,\lambda_8}' + \left(8\lambda_6\lambda_{10} - \frac{16}{5}\lambda_8^2\right)u_3\mu_{0,\lambda_{10}}'$$

$$+ \left(\frac{48}{5}\lambda_8^2 - 24\lambda_6\lambda_{10}\right)u_3^2\mu_{0,\lambda_4} + 12\lambda_8\lambda_{10}u_3^2\mu_{0,\lambda_6} + \left(\frac{24}{5}\lambda_4\lambda_8^2 - \frac{72}{5}\lambda_4\lambda_6\lambda_{10}\right)u_3^2\mu_{0,\lambda_8}$$

$$+ \left(\frac{36}{5}\lambda_4\lambda_8\lambda_{10} - \frac{48}{5}\lambda_6^2\lambda_{10} + \frac{12}{5}\lambda_6\lambda_8^2\right)u_3^2\mu_{0,\lambda_{10}} = 0$$

PROOF. By substituting (34) for k=0 into (33) for k=0, we obtain the statement of the proposition. \square

We set

$$\mu_0 = \sum_{\ell > 0} q_\ell(\lambda) \frac{u_3^\ell}{\ell!}.\tag{35}$$

Lemma 18. If ℓ is even, then $q_{\ell}=0$. We have $q_1\in\mathbb{C}$ and $q_3=\lambda_6q_1$.

PROOF. From Lemma 17 and (31), we obtain $q_{\ell} = 0$ for any non-negative even integer ℓ and $q_1 \in \mathbb{C}$. Further, we find that the coefficient of u_3^0 in μ_1 is equal to 0. By comparing the coefficient of u_3 in the equation (34) for k = 0, we obtain $q_3 = \lambda_6 q_1$. \square

Proposition 10. For $\ell \geq 2$, the following recurrence relation holds:

$$\begin{array}{ll} q_{\ell+2} & = & (\frac{6}{5}\ell+1)\lambda_6q_\ell+\ell(3-\frac{3}{5}\ell)\lambda_4\lambda_8q_{\ell-2}-\frac{6}{5}\lambda_6^2\ell q_{\ell-2}-\frac{18\ell(\ell-2)(\ell-3)}{5}\lambda_8\lambda_{10}q_{\ell-4} \\ & + & 20\lambda_{10}q_{\ell,\lambda_4}-\frac{8}{5}\lambda_4\lambda_8q_{\ell,\lambda_6}+(12\lambda_4\lambda_{10}-\frac{12}{5}\lambda_6\lambda_8)q_{\ell,\lambda_8}+(8\lambda_6\lambda_{10}-\frac{16}{5}\lambda_8^2)q_{\ell,\lambda_{10}} \\ & + & (\frac{48}{5}\lambda_8^2-24\lambda_6\lambda_{10})\ell q_{\ell-2,\lambda_4}+12\lambda_8\lambda_{10}\ell q_{\ell-2,\lambda_6}+(\frac{24}{5}\lambda_4\lambda_8^2-\frac{72}{5}\lambda_4\lambda_6\lambda_{10})\ell q_{\ell-2,\lambda_8} \\ & + & (\frac{36}{5}\lambda_4\lambda_8\lambda_{10}-\frac{48}{5}\lambda_6^2\lambda_{10}+\frac{12}{5}\lambda_6\lambda_8^2)\ell q_{\ell-2,\lambda_{10}}, \end{array}$$

where $q_{i,\lambda_{2j}}$ denotes the derivative of q_i with respect to λ_{2j} .

PROOF. By substituting (35) into the differential equation in Proposition 9 and comparing the coefficients of $u_3^{\ell}/\ell!$, we obtain the statement of the proposition. \Box

COROLLARY 3. Let $\lambda_8 \neq 0$. Then the sigma function $\sigma(u_1, u_3; \lambda)$ is uniquely determined by the differential equations $Q_i \sigma = 0$, i = 0, 4, 6, up to a multiplicative constant.

PROOF. From Lemma 18 and Proposition 10, we find that all the coefficients q_{ℓ} are determined from q_1 . Note that Lemma 18 and Proposition 10 follow from (31), (33), and (34). By (34), all the functions μ_k are determined from μ_0 . As mentioned in the proof of Proposition 8, (31), (33), (34) follow from $Q_0\sigma = 0$, $Q_4\sigma = 0$, $Q_6\sigma = 0$. Therefore we obtain the statement of this corollary. \square

Remark 2. In [11], the following expression is proved:

$$6\lambda_8 Q_2 = 5[Q_4, Q_6] + 10\lambda_{10}Q_0 + 6\lambda_6 Q_4 - 10\lambda_4 Q_6.$$

From this result, the statement of Corollary 3 can be also proved.

3.4. Expansions of μ_k

In this subsection we will calculate the expansions of μ_k . From (9), we have $q_1 = -1$. The initial terms of μ_i , i = 0, 1, 2, 3, are as follows.

$$\mu_{0} = -u_{3} - \lambda_{6} \frac{u_{3}^{3}}{3!} - (\lambda_{6}^{2} + 2\lambda_{4}\lambda_{8}) \frac{u_{3}^{5}}{5!} + (8\lambda_{8}\lambda_{10} - 6\lambda_{4}\lambda_{6}\lambda_{8} - \lambda_{6}^{3} - 24\lambda_{4}^{2}\lambda_{10}) \frac{u_{3}^{7}}{7!} + \cdots,$$

$$\mu_{1} = 8\lambda_{10} \frac{u_{3}^{4}}{4!} + (88\lambda_{6}\lambda_{10} - 16\lambda_{8}^{2}) \frac{u_{3}^{6}}{6!} + (816\lambda_{6}^{2}\lambda_{10} - 192\lambda_{6}\lambda_{8}^{2} - 160\lambda_{4}\lambda_{8}\lambda_{10}) \frac{u_{3}^{8}}{8!} + \cdots,$$

$$\mu_{2} = 2\lambda_{8} \frac{u_{3}^{3}}{3!} + (24\lambda_{4}\lambda_{10} + 4\lambda_{6}\lambda_{8}) \frac{u_{3}^{5}}{5!} + (160\lambda_{10}^{2} + 264\lambda_{4}\lambda_{6}\lambda_{10} + 6\lambda_{6}^{2}\lambda_{8} - 36\lambda_{4}\lambda_{8}^{2}) \frac{u_{3}^{7}}{7!} + \cdots,$$

$$\mu_{3} = 2 + 2\lambda_{6} \frac{u_{3}^{2}}{2!} + (2\lambda_{6}^{2} + 4\lambda_{4}\lambda_{8}) \frac{u_{3}^{4}}{4!} + (12\lambda_{4}\lambda_{6}\lambda_{8} + 32\lambda_{8}\lambda_{10} + 2\lambda_{6}^{3} + 48\lambda_{4}^{2}\lambda_{10}) \frac{u_{3}^{6}}{6!} + \cdots.$$

4. The ultra-elliptic integrals

In [4], the inversion problem of the ultra-elliptic integrals is considered. In this section, we will summarize the main results in [4]. Proposition 18 is not described in [4].

In this section, we assume g = 2. Let us take a point $P_* \in V$ and an open neighborhood U_* of this point that is homeomorphic to an open disk in \mathbb{C} . We fix a path γ_* on the curve V from ∞ to P_* . Let us consider the holomorphic mappings

$$I_1: U_* \to \mathbb{C}, \quad P = (x, y) \mapsto \int_{-\infty}^{P} du_1,$$

$$I_3: U_* \to \mathbb{C}, \quad P = (x, y) \mapsto \int_{-\infty}^{P} du_3,$$

where as the path of integration we choose the composition of the path γ_* from ∞ to the point P_* and any path in the neighborhood U_* from P_* to the point P. We consider the meromorphic function on \mathbb{C}^2

$$f = -\frac{\sigma_3}{\sigma_1}.$$

We assume $P_* \neq \infty$. If we take the open neighborhood U_* sufficiently small, then I_3 is injective. Let $\varphi(u)$ be the implicit function defined by $\sigma(\varphi(u), u) = 0$ around $(I_1(P_*), I_3(P_*))$. We define the function $F(u) = f(\varphi(u), u)$.

PROPOSITION 11. ([4]) Set $u = I_3(P)$, where $P = (x, y) \in U_*$. Then x = F(u) and y = -F'(u)/2, where F' is the derivative of F with respect to u.

THEOREM 8. ([4]) The function F(u) satisfies the following ordinary differential equations:

$$(F'/2)^2 = F^5 + \lambda_4 F^3 + \lambda_6 F^2 + \lambda_8 F + \lambda_{10}, \tag{36}$$

$$F'' = 10F^4 + 6\lambda_4 F^2 + 4\lambda_6 F + 2\lambda_8. \tag{37}$$

From Proposition 11 and Theorem 8, one can obtain the series expansion of F(u). Since the function F(u) is holomorphic in a neighborhood of $u^* = I_3(P_*)$, the expansion in the neighborhood of this point has the form

$$F(u) = \sum_{n=0}^{\infty} \tilde{p}_{3n+2} (u - u^*)^n, \quad \tilde{p}_{3n+2} \in \mathbb{C}.$$
 (38)

PROPOSITION 12. ([4]) Set $P_* = (x_*, y_*)$. Then in the expansion (38) we have $\tilde{p}_2 = x_*$ and $\tilde{p}_5 = -2y_*$.

We set $\deg \tilde{p}_2 = 2$ and $\deg \tilde{p}_5 = 5$.

PROPOSITION 13. ([4]) The coefficients \tilde{p}_{3n+2} in the expansion (38) are determined from the following recurrence relations:

- $\tilde{p}_8 = 5\tilde{p}_2^4 + 3\lambda_4\tilde{p}_2^2 + 2\lambda_6\tilde{p}_2 + \lambda_8$,
- $(n+2)(n+1)\tilde{p}_{3n+8} = 10\sum_{(n_1,n_2,n_3,n_4)\in S_1} \tilde{p}_{3n_1+2} \tilde{p}_{3n_2+2} \tilde{p}_{3n_3+2} \tilde{p}_{3n_4+2} + 6\lambda_4 \sum_{(n_1,n_2)\in S_2} \tilde{p}_{3n_1+2} \tilde{p}_{3n_2+2} + 4\lambda_6 \tilde{p}_{3n+2}, \quad n \geqslant 1, \text{ where}$

$$S_1 = \{(n_1, n_2, n_3, n_4) \in \mathbb{Z}_{\geq 0}^4 \mid n_1 + n_2 + n_3 + n_4 = n\}, S_2 = \{(n_1, n_2) \in \mathbb{Z}_{\geq 0}^2 \mid n_1 + n_2 = n\},$$

and the coefficient \tilde{p}_{3n+2} is a homogeneous polynomial in $\mathbb{Q}[\tilde{p}_2, \tilde{p}_5, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree 3n+2, if $\tilde{p}_{3n+2} \neq 0$.

We assume $P_* \neq (0, \pm \sqrt{\lambda_{10}})$. If we take the open neighborhood U_* sufficiently small, then I_1 is injective. Let $\eta(u)$ be the implicit function defined by $\sigma(u, \eta(u)) = 0$ around $(I_1(P_*), I_3(P_*))$. Let us define the function $G(u) = f(u, \eta(u))$.

PROPOSITION 14. ([4]) For $P = (x, y) \in U_*$ let $u = I_1(P)$. Then we have x = G(u) and y = -G(u)G'(u)/2, where G' is the derivative of G with respect to u.

Theorem 9. ([4]) The function G(u) satisfies the following ordinary differential equations:

$$(GG'/2)^2 = G^5 + \lambda_4 G^3 + \lambda_6 G^2 + \lambda_8 G + \lambda_{10}, \tag{39}$$

$$G^{4}(G''' - 12GG') - 4\lambda_{8}GG' - 12\lambda_{10}G' = 0.$$
(40)

Let us assume that $P_* \neq (0, \pm \sqrt{\lambda_{10}})$ and $P_* \neq \infty$. Using Proposition 14 and Theorem 9, one can obtain the series expansion of the function G(u). Since the function G(u) is holomorphic in the neighborhood of the point $u^* = I_1(P_*)$, this expansion in the neighborhood of this point has the form

$$G(u) = \sum_{n=0}^{\infty} \tilde{q}_{n+2} (u - u^*)^n, \quad \tilde{q}_{n+2} \in \mathbb{C}.$$
(41)

Proposition 15. ([4]) Let $P_* = (x_*, y_*)$. Then we have $\tilde{q}_2 = x_*$ and $\tilde{q}_3 = -2y_*/x_*$.

Let us set deg $\tilde{q}_2 = 2$ and deg $\tilde{q}_3 = 3$.

PROPOSITION 16. ([4]) The coefficients \tilde{q}_{n+2} are determined from the following recurrence relations:

•
$$\tilde{q}_4 = \tilde{q}_2^{-3}(3\tilde{q}_2^5 + \lambda_4\tilde{q}_2^3 - \lambda_8\tilde{q}_2 - 2\lambda_{10})$$

•
$$\tilde{q}_2^3(n+2)(n+1)\tilde{q}_{n+4} = -\sum_{k=0}^{n-1} \left\{ (k+2)(k+1)\tilde{q}_{k+4} \sum_{(n_1,n_2,n_3)\in T_1^{(k)}} \tilde{q}_{n_1+2}\,\tilde{q}_{n_2+2}\,\tilde{q}_{n_3+2} \right\} +$$

$$+6\sum_{\substack{(n_1,n_2,n_3,n_4,n_5)\in T_2}} \tilde{q}_{n_1+2} \,\tilde{q}_{n_2+2} \,\tilde{q}_{n_3+2} \,\tilde{q}_{n_4+2} \,\tilde{q}_{n_5+2} +$$

$$+2\lambda_4 \sum_{(n_1,n_2,n_3)\in T_3} \tilde{q}_{n_1+2} \; \tilde{q}_{n_2+2} \; \tilde{q}_{n_3+2} - 2\lambda_8 \tilde{q}_{n+2}, \quad n \geqslant 1, \; \text{where}$$

 $T_1^{(k)} = \{(n_1, n_2, n_3) \in \mathbb{Z}_{\geqslant 0}^3 \mid n_1 + n_2 + n_3 = n - k\}, T_2 = \{(n_1, n_2, n_3, n_4, n_5) \in \mathbb{Z}_{\geqslant 0}^5 \mid n_1 + n_2 + n_3 + n_4 + n_5 = n\}, T_3 = \{(n_1, n_2, n_3) \in \mathbb{Z}_{\geqslant 0}^3 \mid n_1 + n_2 + n_3 = n\}, \text{ and the coefficient } \tilde{q}_{n+2} \text{ is a homogeneous polynomial in } \mathbb{Q}[\tilde{q}_2, \tilde{q}_3, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}] \text{ of degree } n + 2 \text{ if } \tilde{q}_{n+2} \neq 0.$

Let us take $P_* = \infty$ and the path γ_* defined by the function $R : [0,1] \to V$ such that $R(r) = \infty$ for any point $r \in [0,1]$. Then we have $(I_1(P_*), I_3(P_*)) = (0,0)$.

Proposition 17. ([4]) In a neighborhood of the point u = 0, the function G(u) is given by a series

$$G(u) = \frac{1}{u^2} - \frac{\lambda_4}{5}u^2 - \frac{\lambda_6}{7}u^4 + \left(\frac{\lambda_4^2}{75} - \frac{\lambda_8}{9}\right)u^6 + \left(\frac{3}{385}\lambda_4\lambda_6 - \frac{\lambda_{10}}{11}\right)u^8 + \sum_{n \ge 10} \tau_{n+2}u^n,$$

where the coefficient τ_{n+2} is a homogeneous polynomial in $\mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n+2 if $\tau_{n+2} \neq 0$.

PROPOSITION 18. The coefficients τ_n for $n \ge 12$ are determined from the following recurrence formula:

$$(n+1)\tau_{n} = \sum_{\substack{(n_{1},n_{2},n_{3},n_{4}) \in T_{1}}} \frac{n_{3}-2}{2} \frac{n_{4}-2}{2} \tau_{n_{1}} \tau_{n_{2}} \tau_{n_{3}} \tau_{n_{4}}$$

$$- \sum_{\substack{(n_{1},n_{2},n_{3},n_{4},n_{5}) \in T_{2}}} \tau_{n_{1}} \tau_{n_{2}} \tau_{n_{3}} \tau_{n_{4}} \tau_{n_{5}} - \lambda_{4} \sum_{\substack{(n_{1},n_{2},n_{3}) \in T_{3}}} \tau_{n_{1}} \tau_{n_{2}} \tau_{n_{3}}$$

$$- \lambda_{6} \sum_{\substack{(n_{1},n_{2}) \in T_{4}}} \tau_{n_{1}} \tau_{n_{2}} - \lambda_{8} \tau_{n-8},$$

where $T_1 = \{(n_1, n_2, n_3, n_4) \in \mathbb{Z}_{\geq 0}^4 \mid n_1 + \dots + n_4 = n, 0 \leq n_1, \dots, n_4 < n\}, T_2 = \{(n_1, n_2, n_3, n_4, n_5) \in \mathbb{Z}_{\geq 0}^5 \mid n_1 + \dots + n_5 = n, 0 \leq n_1, \dots, n_5 < n\}, T_3 = \{(n_1, n_2, n_3) \in \mathbb{Z}_{\geq 0}^3 \mid n_1 + \dots + n_2 + n_3 = n - 4\}, and <math>T_4 = \{(n_1, n_2) \in \mathbb{Z}_{\geq 0}^2 \mid n_1 + n_2 = n - 6\}.$

PROOF. Let

$$G(u) = \frac{1}{u^2} + \sum_{n=0}^{\infty} \tau_{n+2} u^n.$$
 (42)

Then we have

$$-\frac{G'(u)}{2} = \frac{1}{u^3} - \sum_{n=0}^{\infty} \frac{n+1}{2} \tau_{n+3} u^n,$$
$$u^2 G(u) = \sum_{n=0}^{\infty} \tau_n u^n, \quad u^3 \frac{G'(u)}{2} = \sum_{n=0}^{\infty} \frac{n-2}{2} \tau_n u^n,$$

where $\tau_0 = 1$ and $\tau_1 = 0$. By substituting (42) into (39) and multiplying the both sides by u^{10} , we obtain

$$\left(\sum_{n=0}^{\infty} \tau_n u^n\right)^2 \left(\sum_{n=0}^{\infty} \frac{n-2}{2} \tau_n u^n\right)^2 = \left(\sum_{n=0}^{\infty} \tau_n u^n\right)^5 + \lambda_4 u^4 \left(\sum_{n=0}^{\infty} \tau_n u^n\right)^3 + \lambda_6 u^6 \left(\sum_{n=0}^{\infty} \tau_n u^n\right)^2 + \lambda_8 u^8 \sum_{n=0}^{\infty} \tau_n u^n + \lambda_{10} u^{10}.$$

By comparing the coefficient of u^n for $n \ge 12$, we obtain the statement of the proposition. \square

Proposition 19. ([4]) There exists the formula

$$G(u) = \wp(u) + g(u),$$

where g(u) is a holomorphic function that in a neighborhood of the point u=0 is given by a series

$$g(u) = -\frac{\lambda_8}{9}u^6 - \frac{\lambda_{10}}{11}u^8 + \sum_{n \ge 10} \tilde{\tau}_{n+2}u^n.$$

Here the coefficient $\tilde{\tau}_{n+2}$ is a homogeneous polynomial in $\mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n+2 if $\tilde{\tau}_{n+2} \neq 0$.

Denote by $G_d(u)$ the formal Laurent series obtained from G(u) by substitution $\lambda_8 = \lambda_{10} = 0$ in the series expansion of this function in a neighborhood of the point u = 0.

COROLLARY 4. ([4]) We have $G_d(u) = \wp(u)$.

5. Number-theoretical properties of the generalized Bernoulli-Hurwitz numbers for the curve of genus 2

In this section, we assume g=2. Let V be a hyperelliptic curve of genus 2 defined by

$$y^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10}. (43)$$

We take an open neighborhood U_* of ∞ such that U_* is homeomorphic to an open disk in \mathbb{C} . We consider the map

$$I_1: U_* \to \mathbb{C}, \quad P = (x, y) \mapsto \int_{\infty}^{P} -\frac{x}{2y} dx,$$

where as the path of integration we take any path in U_* from ∞ to P. For $P = (x, y) \in U_*$, let $u = I_1(P)$. If U_* is sufficiently small, then the map I_1 is biholomorphism. Therefore, we can regard x and y as functions of u. From Proposition 14, we have x(u) = G(u) and y(u) = -G(u)G'(u)/2. From Proposition 17, the function x(u) can be expanded around u = 0 as

$$x(u) = \frac{1}{u^2} + \sum_{n=4}^{\infty} \frac{C_n}{n} \frac{u^{n-2}}{(n-2)!},$$
(44)

where the coefficient C_n is a homogeneous polynomial in $\mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n if $C_n \neq 0$.

Lemma 19. We have

$$y(u) = -\frac{1}{4}(x(u)^2)',$$

where $(x(u)^2)'$ denotes the derivative of $x(u)^2$ with respect to u.

PROOF. From Proposition 14, we obtain the statement of the lemma. □

From (44) and Lemma 19, we find that y(u) can be expanded around u=0 as

$$y(u) = \frac{1}{u^5} + \sum_{n=6}^{\infty} \frac{D_n}{n} \frac{u^{n-5}}{(n-5)!},$$
(45)

where the coefficient D_n is a homogeneous polynomial in $\mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n if $D_n \neq 0$. Then C_n and D_n are called *generalized Bernoulli-Hurwitz numbers*. In particular, we find that $C_n = D_n = 0$ for any odd integer n. In [38], the hyperelliptic curve of genus 2 defined by $y^2 = x^5 - 1$ is considered and the following formulae are proved.

THEOREM 10. ([38]) For the curve $y^2 = x^5 - 1$, we have

$$\frac{C_{10n}}{10n} \equiv -\sum_{p : prime, \ p \equiv 1 \ mod \ 10, \ 10n = a(p-1)} \frac{a|_p^{-1} \ mod \ p^{1+ord_p a}}{p^{1+ord_p a}} A_p^a \quad mod \ \mathbb{Z},$$

$$\frac{D_{10n}}{10n} \equiv -\sum_{\substack{p: prime, p=1 \text{ mod } 10, 10p=a(p-1)}} \frac{(4!a)|_p^{-1} \mod p^{1+ord_pa}}{p^{1+ord_pa}} A_p^a \mod \mathbb{Z},$$

where
$$A_p = (-1)^{(p-1)/10} \cdot \binom{(p-1)/2}{(p-1)/10}$$
.

In this section, we will generalize the method of [38] to the curve V defined by (43) and derive some number-theoretical properties of the generalized Bernoulli-Hurwitz numbers for the curve V.

Proposition 20. It is possible to take a local parameter z of V around ∞ such that

$$x = \frac{1}{z^2}, \quad y = \frac{1}{z^5} (1 + \sum_{n=4}^{\infty} a_n z^n),$$

where a_n is a homogeneous polynomial in $\mathbb{Z}[\frac{1}{2}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n if $a_n \neq 0$.

PROOF. It is possible to take a local parameter z_1 such that

$$x = \frac{1}{z_1^2}.$$

The expansion of y around ∞ with respect to z_1 takes the following form

$$y = \frac{\alpha}{z_1^5} (1 + O(z_1)), \quad \alpha \in \mathbb{C}.$$

By substituting the above expansions into (43), multiplying the both sides by z_1^{10} , and comparing the coefficient of z_1^0 , we obtain

$$\alpha^2 = 1$$
.

If $\alpha = 1$, then we set $z = z_1$. If $\alpha = -1$, then we set $z = -z_1$. Then we have

$$x = \frac{1}{z^2}, \quad y = \frac{1}{z^5} (1 + \sum_{n=1}^{\infty} a_n z^n),$$

where $a_n \in \mathbb{C}$. By substituting the above expressions into (43), we obtain

$$\left(1 + \sum_{n=1}^{\infty} a_n z^n\right)^2 = 1 + \lambda_4 z^4 + \lambda_6 z^6 + \lambda_8 z^8 + \lambda_{10} z^{10}.$$

From the above equation, we can find that a_n is a homogeneous polynomial in $\mathbb{Z}[\frac{1}{2}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n if $a_n \neq 0$ recursively. \square

We can regard u(z) as a function defined around z=0.

Proposition 21. The function u(z) is expanded around z=0 as

$$u(z) = z + \sum_{n=1}^{\infty} f_n \frac{z^{n+1}}{n+1},$$
(46)

where $f_1 = f_2 = f_3 = 0$ and f_n is a homogeneous polynomial in $\mathbb{Z}[\frac{1}{2}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n if $f_n \neq 0$.

PROOF. From Proposition 20, we have

$$u = \int_0^z -\frac{z^{-2} \cdot (-2)z^{-3}}{2z^{-5}(1 + \sum_{n=4}^\infty a_n z^n)} dz = \int_0^z \left(1 + \sum_{n=4}^\infty a_n z^n\right)^{-1} dz.$$

From Proposition 20, we obtain the statement of the proposition. \Box

For positive integers n and k such that $n \ge k$, we use the notation

$$(n)_k = n(n-1)\cdots(n-k+1).$$

We consider the inverse mapping z(u) of u(z). The expansions of z^{-k} , where k = 1, 2, 3, 4, have the following forms:

$$\frac{1}{z^k} = \frac{1}{u^k} + \sum_{n=4}^{\infty} \frac{C_n^{(k)}}{(n)_k} \frac{u^{n-k}}{(n-k)!}, \quad k = 1, 2, 3, 4,$$
(47)

where $C_n^{(k)}$ is a homogeneous polynomial in $\mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n if $C_n^{(k)} \neq 0$.

From Theorem 7, we obtain

$$\frac{C_n^{(1)}}{n} = \sum_{\substack{n=a(p-1), \ p \ge 5 \ : \ prime}} \frac{z_1(p,n)}{p^{1+ord_p a}} f_{p-1}^a + \widetilde{f}_n, \tag{48}$$

where $z_1(p,n) \in \mathbb{Z}$ and $\widetilde{f}_n \in \mathbb{Z}[\frac{1}{2}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$

LEMMA 20. For k = 1, 2, 3, we have

$$k\sum_{n=4}^{\infty} \frac{C_n^{(k+1)}}{(n)_{k+1}} \frac{u^{n-k}}{(n-k)!} + \sum_{n=4}^{\infty} \frac{C_n^{(k)}}{(n)_k} \frac{u^{n-k}}{(n-k)!} = k\sum_{n=4}^{\infty} f_n \frac{z^{n-k}}{n-k}.$$

PROOF. For k = 1, 2, 3, we have

$$\int_0^u \left(\frac{1}{z^{k+1}} - \frac{1}{u^{k+1}} \right) du = \sum_{n=4}^\infty \frac{C_n^{(k+1)}}{(n)_{k+1}} \frac{u^{n-k}}{(n-k)!}.$$

By differentiating the both sides of (46) with respect to u, we obtain

$$1 = \frac{dz}{du} + \sum_{n=4}^{\infty} f_n z^n \frac{dz}{du}.$$

By dividing the both sides of the above equation by z^{k+1} , we have

$$\frac{1}{z^{k+1}} = \frac{1}{z^{k+1}} \frac{dz}{du} + \sum_{n=4}^{\infty} f_n z^{n-k-1} \frac{dz}{du}.$$

Therefore we have

$$\int_0^u \left(\frac{1}{z^{k+1}} - \frac{1}{u^{k+1}} \right) du = -\frac{1}{k} \frac{1}{z^k} + \sum_{n=4}^\infty f_n \frac{z^{n-k}}{n-k} + \frac{1}{k} \frac{1}{u^k} = -\frac{1}{k} \sum_{n=4}^\infty \frac{C_n^{(k)}}{(n)_k} \frac{u^{n-k}}{(n-k)!} + \sum_{n=4}^\infty f_n \frac{z^{n-k}}{n-k}.$$

Thus we have

$$k\sum_{n=4}^{\infty}\frac{C_n^{(k+1)}}{(n)_{k+1}}\frac{u^{n-k}}{(n-k)!}+\sum_{n=4}^{\infty}\frac{C_n^{(k)}}{(n)_k}\frac{u^{n-k}}{(n-k)!}=k\sum_{n=4}^{\infty}f_n\frac{z^{n-k}}{n-k}.$$

Lemma 21. ([27], [38]) Let R_1 and R_2 be two integral domains with characteristic 0 satisfying $R_1 \subset R_2$. We consider a formal power series of z

$$h(z) = \sum_{n=0}^{\infty} \alpha_n \frac{z^n}{n!}, \quad \alpha_n \in R_2.$$

If $\alpha_0, \ldots, \alpha_{n-1}$ belong to R_1 , and there is a polynomial F of n variables over R_1 such that

$$h^{(n)}(z) = F(h(z), h'(z), \dots, h^{(n-1)}(z)),$$

where $h^{(n)}(z)$ is the n-th derivative of h(z) with respect to z, then we have $h(z) \in R_1(\langle z \rangle)$.

Lemma 22. ([27], [38]) Let R be an integral domain with characteristic 0 and

$$h(z) = z + O(z^2) \in R\langle\langle z \rangle\rangle.$$

Then for any positive integer m,

$$\frac{h(z)^m}{m!}$$

also belongs to $R\langle\langle z\rangle\rangle$.

LEMMA 23. ([27], [38], [44]) Let R be an integral domain with characteristic 0 and

$$w(z) = z + O(z^2) \in R\langle\langle z \rangle\rangle.$$

Then, the formal inverse series $z(w) = w + O(w^2)$ belongs to $R(\langle w \rangle)$.

We set $\deg u = -1$.

PROPOSITION 22. We have $z(u) = u + O(u^2) \in \mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}] \langle \langle u \rangle \rangle$ and z(u) is homogeneous of degree -1 with respect to $u, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}$.

PROOF. We have $x(u) = z(u)^{-2}$. Therefore we have $x'(u) = -2z(u)^{-3}z'(u)$. From Lemma 19, we obtain $y(u) = z(u)^{-5}z'(u)$. From (43), we obtain

$$(z')^2 = 1 + \lambda_4 z^4 + \lambda_6 z^6 + \lambda_8 z^8 + \lambda_{10} z^{10}.$$

By differentiating the both sides of the above equation with respect to u and dividing by 2z', we obtain

$$z'' = 2\lambda_4 z^3 + 3\lambda_6 z^5 + 4\lambda_8 z^7 + 5\lambda_{10} z^9.$$
(49)

We define the polynomial $F(Z_1, Z_2)$ over $\mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ by

$$F(Z_1, Z_2) = 2\lambda_4 Z_1^3 + 3\lambda_6 Z_1^5 + 4\lambda_8 Z_1^7 + 5\lambda_{10} Z_1^9.$$

From (49), we have z'' = F(z, z'). Since the function z(u) is expanded around u = 0 as

$$z(u) = u + O(u^2),$$

we have z(0) = 0 and z'(0) = 1. From Lemma 21, we have $z(u) \in \mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]\langle\langle u \rangle\rangle$. \square

Lemma 24. For $n \ge 4$, we have the following relations.

(i)
$$\frac{C_n^{(2)}}{(n)_2} + \frac{C_n^{(1)}}{n} \in \mathbb{Z}[1/2, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$$

(ii)
$$2\frac{C_n^{(3)}}{(n)_3} + \frac{C_n^{(2)}}{(n)_2} \in \mathbb{Z}[1/2, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$$

(iii)
$$3\frac{C_n^{(4)}}{(n)_4} + \frac{C_n^{(3)}}{(n)_3} \in 3 \mathbb{Z}[1/2, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$$

(iv)
$$\frac{C_n^{(1)}}{n} + 6\frac{C_n^{(4)}}{(n)_4} \in \mathbb{Z}[1/2, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$$

PROOF. In Lemma 20, we set k = 1. Then we have

$$\sum_{n=4}^{\infty} \frac{C_n^{(2)}}{(n)_2} \frac{u^{n-1}}{(n-1)!} + \sum_{n=4}^{\infty} \frac{C_n^{(1)}}{n} \frac{u^{n-1}}{(n-1)!} = \sum_{n=4}^{\infty} f_n(n-2)! \frac{z^{n-1}}{(n-1)!}.$$
 (50)

From Lemma 22 and Proposition 22, we have

$$\frac{z^{n-1}}{(n-1)!} \in \mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}] \langle \langle u \rangle \rangle.$$

By comparing the coefficient of $\frac{u^{n-1}}{(n-1)!}$ in (50) and using $f_n \in \mathbb{Z}[\frac{1}{2}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$, we obtain

$$\frac{C_n^{(2)}}{(n)_2} + \frac{C_n^{(1)}}{n} \in \mathbb{Z}[1/2, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}].$$

In Lemma 20, we set k=2. Then we have

$$2\sum_{n=4}^{\infty} \frac{C_n^{(3)}}{(n)_3} \frac{u^{n-2}}{(n-2)!} + \sum_{n=4}^{\infty} \frac{C_n^{(2)}}{(n)_2} \frac{u^{n-2}}{(n-2)!} = 2\sum_{n=4}^{\infty} f_n(n-3)! \frac{z^{n-2}}{(n-2)!}.$$
 (51)

By comparing the coefficient of $\frac{u^{n-2}}{(n-2)!}$ in (51) and using $f_n \in \mathbb{Z}[\frac{1}{2}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$, we obtain

$$2\frac{C_n^{(3)}}{(n)_3} + \frac{C_n^{(2)}}{(n)_2} \in \mathbb{Z}[1/2, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}].$$

In Lemma 20, we set k = 3. Then we have

$$3\sum_{n=4}^{\infty} \frac{C_n^{(4)}}{(n)_4} \frac{u^{n-3}}{(n-3)!} + \sum_{n=4}^{\infty} \frac{C_n^{(3)}}{(n)_3} \frac{u^{n-3}}{(n-3)!} = 3\sum_{n=4}^{\infty} f_n(n-4)! \frac{z^{n-3}}{(n-3)!}.$$
 (52)

By comparing the coefficient of $\frac{u^{n-3}}{(n-3)!}$ in (52) and using $f_n \in \mathbb{Z}[\frac{1}{2}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$, we obtain

$$3\frac{C_n^{(4)}}{(n)_4} + \frac{C_n^{(3)}}{(n)_3} \in 3 \ \mathbb{Z}[1/2, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}].$$

From (i), (ii), and (iii), we obtain (iv). \Box

Lemma 25. It is possible to take a local parameter s of V around ∞ such that

$$x = \frac{1}{s^2} (1 + \sum_{n=1}^{\infty} \alpha_n s^n), \quad y = \frac{1}{s^5},$$

where α_n is a homogeneous polynomial in $\mathbb{Z}[\frac{1}{5}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n if $\alpha_n \neq 0$.

PROOF. It is possible to take a local parameter s_1 of V around ∞ such that

$$y = \frac{1}{s_1^5}.$$

The expansion of x around ∞ with respect to s_1 takes the following form

$$x = \frac{\alpha}{s_1^2} (1 + O(s_1)), \quad \alpha \in \mathbb{C}.$$

By substituting the above expansions into (43), multiplying the both sides by s_1^{10} , and comparing the coefficient of s_1^0 , we obtain

$$1 = \alpha^5$$
.

There exists $\beta \in \mathbb{C}$ such that $\beta^5 = 1$ and $\beta^2 = \alpha$. Let $s = \beta^{-1}s_1$. Then we have

$$x = \frac{1}{s^2} (1 + \sum_{n=1}^{\infty} \alpha_n s^n), \quad y = \frac{1}{s^5},$$

where $\alpha_n \in \mathbb{C}$. By substituting the above expressions into (43) and multiplying the both sides by s^{10} , we obtain

$$1 = (1 + \sum_{n=1}^{\infty} \alpha_n s^n)^5 + \lambda_4 s^4 (1 + \sum_{n=1}^{\infty} \alpha_n s^n)^3 + \lambda_6 s^6 (1 + \sum_{n=1}^{\infty} \alpha_n s^n)^2 + \lambda_8 s^8 (1 + \sum_{n=1}^{\infty} \alpha_n s^n) + \lambda_{10} s^{10}.$$

From the above equation, we can find that α_n is a homogeneous polynomial in $\mathbb{Z}[\frac{1}{5}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n if $\alpha_n \neq 0$ recursively. \square

We can regard u(s) as a function defined around s = 0.

Lemma 26. The function u(s) is expanded around s = 0 as

$$u(s) = s + \sum_{n=1}^{\infty} g_n \frac{s^{n+1}}{n+1},\tag{53}$$

where $g_1 = g_2 = g_3 = g_4 = 0$ and g_n is a homogeneous polynomial in $\mathbb{Z}[\frac{1}{5}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n if $g_n \neq 0$.

PROOF. We have

$$x(s) = s^{-2} + \sum_{n=4}^{\infty} \alpha_n s^{n-2}, \quad x'(s) = -2s^{-3} + \sum_{n=4}^{\infty} (n-2)\alpha_n s^{n-3}.$$

Since $\alpha_n \in \mathbb{Z}[\frac{1}{5}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ is homogeneous of degree n if $\alpha_n \neq 0$, we have $\alpha_n = 0$ if n is odd. Therefore, all the coefficients of the expansion of x'(s) are included in $2\mathbb{Z}[\frac{1}{5}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$. We have

$$u(s) = \int_0^s -\frac{(s^{-2} + \sum_{n=4}^\infty \alpha_n s^{n-2})(-2s^{-3} + \sum_{n=4}^\infty (n-2)\alpha_n s^{n-3})}{2s^{-5}} ds$$

$$= \int_0^s s^5 (s^{-2} + \sum_{n=4}^\infty \alpha_n s^{n-2})(s^{-3} - \sum_{n=4}^\infty \frac{n-2}{2} \alpha_n s^{n-3}) ds = s + \sum_{n=1}^\infty g_n \frac{s^{n+1}}{n+1}, \tag{54}$$

where $g_1 = g_2 = g_3 = 0$ and g_n is a homogeneous polynomial in $\mathbb{Z}[\frac{1}{5}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n if $g_n \neq 0$. We find that the coefficient of s^{-1} in

$$(s^{-2} + \sum_{n=4}^{\infty} \alpha_n s^{n-2})(s^{-3} - \sum_{n=4}^{\infty} \frac{n-2}{2} \alpha_n s^{n-3})$$

is equal to 0. From (54), we have $g_4 = 0$. \square

We consider the inverse mapping s(u) of u(s).

PROPOSITION 23. We have $s(u) \in \mathbb{Z}[\frac{1}{5}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}] \langle \langle u \rangle \rangle$ and s(u) is homogeneous of degree -1 with respect to $u, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}$.

PROOF. From Lemma 26, we have $u(s) = s + O(s^2) \in \mathbb{Z}[\frac{1}{5}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}] \langle \langle s \rangle \rangle$. Therefore, from Lemma 23, we have $s(u) = u + O(u^2) \in \mathbb{Z}[\frac{1}{5}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}] \langle \langle u \rangle \rangle$. From Lemma 26, we can find that s(u) is homogeneous of degree -1 with respect to $u, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}$. \square

Therefore the expansions of s^{-k} , where k = 1, 2, 3, 4, 5, have the following forms:

$$\frac{1}{s^k} = \frac{1}{u^k} + \sum_{n=6}^{\infty} \frac{D_n^{(k)}}{(n)_k} \frac{u^{n-k}}{(n-k)!}, \quad k = 1, 2, 3, 4, 5,$$

where $D_n^{(k)}$ is a homogeneous polynomial in $\mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ of degree n if $D_n^{(k)} \neq 0$. From Theorem 7, we obtain

$$\frac{D_n^{(1)}}{n} = \sum_{n=a(p-1), \ p\geqslant 7 \ : \ prime} \frac{z_2(p,n)}{p^{1+ord_p a}} g_{p-1}^a + \widetilde{g}_n, \tag{55}$$

where $z_2(p,n) \in \mathbb{Z}$ and $\widetilde{g}_n \in \mathbb{Z}[\frac{1}{5}, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$.

Lemma 27. For any integer $n \ge 6$, we have

$$ord_2\left(\frac{D_n^{(1)}}{n}\right) \geqslant \left\lfloor \frac{n-1}{4} \right\rfloor.$$

PROOF. From Proposition 4, we have

$$\frac{D_n^{(1)}}{n} = \sum_{w(U)=n} \tau_U g^U.$$

Note that $g_2=0$ and $g_i=0$ for any odd integer i. Therefore, from Lemma 14, we obtain the statement of the lemma. \square

Lemma 28. For k = 1, 2, 3, 4, we have

$$k\sum_{n=6}^{\infty} \frac{D_n^{(k+1)}}{(n)_{k+1}} \frac{u^{n-k}}{(n-k)!} + \sum_{n=6}^{\infty} \frac{D_n^{(k)}}{(n)_k} \frac{u^{n-k}}{(n-k)!} = k\sum_{n=6}^{\infty} g_n \frac{s^{n-k}}{n-k}.$$

PROOF. We can prove this lemma in the same way as Lemma 20. \square

Lemma 29. For $n \ge 6$, we have the following relations.

(i)
$$\frac{D_n^{(2)}}{(n)_2} + \frac{D_n^{(1)}}{n} \in 2^3 \cdot 3 \mathbb{Z}[1/5, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$$

(ii)
$$2\frac{D_n^{(3)}}{(n)_3} + \frac{D_n^{(2)}}{(n)_2} \in 2^2 \cdot 3 \mathbb{Z}[1/5, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$$

(iii)
$$3\frac{D_n^{(4)}}{(n)_4} + \frac{D_n^{(3)}}{(n)_3} \in 2 \cdot 3 \ \mathbb{Z}[1/5, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$$

(iv)
$$4\frac{D_n^{(5)}}{(n)_5} + \frac{D_n^{(4)}}{(n)_4} \in 2^2 \mathbb{Z}[1/5, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$$

(v)
$$\frac{D_n^{(1)}}{n} - 24 \frac{D_n^{(5)}}{(n)_5} \in 2^2 \cdot 3 \ \mathbb{Z}[1/5, \lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$$

Proof. We can prove this lemma in the same way as Lemma 24. \Box

Lemma 30. The function x(u) satisfies the following differential equation

$$x'' = 6x^2 + 2\lambda_4 - 2\lambda_8 x^{-2} - 4\lambda_{10} x^{-3}.$$

PROOF. From y(u) = -x(u)x'(u)/2 and (43), we have

$$\left(-\frac{xx'}{2}\right)^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10}.$$

By multiplying the above equation by x^{-2} and differentiating this equation with respect to u, we obtain the statement of the lemma. \square

Lemma 31. The first terms of C_n/n and D_n/n are as follows:

$$\frac{C_4}{4} = -\frac{2}{5}\lambda_4, \quad \frac{C_6}{6} = -\frac{2^3 \cdot 3}{7}\lambda_6, \quad \frac{C_8}{8} = \frac{2^4 \cdot 3}{5}\lambda_4^2 - 2^4 \cdot 5\lambda_8,
\frac{C_{10}}{10} = \frac{2^7 \cdot 3^3}{11}\lambda_4\lambda_6 - \frac{2^7 \cdot 3^2 \cdot 5 \cdot 7}{11}\lambda_{10},
\frac{D_6}{6} = \frac{1}{7}\lambda_6, \quad \frac{D_8}{8} = \frac{2^2}{3}\lambda_8 - \frac{2}{5}\lambda_4^2, \quad \frac{D_{10}}{10} = \frac{2^3 \cdot 3^2 \cdot 5}{11}\lambda_{10} - \frac{2^4 \cdot 3^2}{11}\lambda_4\lambda_6.$$

Proof. From Proposition 17 and Lemma 19, we obtain the statement of the lemma. □

Theorem 11. (i) For any $n \ge 4$, we have

$$ord_2\left(\frac{C_n}{n}\right) \geqslant 1, \quad ord_3\left(\frac{C_n}{n}\right) \geqslant 0.$$

(ii) For any $n \ge 6$, we have

$$ord_2\left(\frac{D_n}{n}\right) \geqslant -1, \quad ord_3\left(\frac{D_n}{n}\right) \geqslant -1.$$

Let $p \geqslant 5$ be a prime.

(iii) If $p-1 \nmid n$, then we have

$$ord_p\left(\frac{C_n}{n}\right) \geqslant 0, \quad ord_p\left(\frac{D_n}{n}\right) \geqslant 0.$$

(iv) If $p-1 \mid n$, then we have

$$ord_p\left(\frac{C_n}{n}\right) \geqslant -1 - ord_p a, \quad ord_p\left(\frac{D_n}{n}\right) \geqslant -1 - ord_p a.$$

where n = a(p-1).

PROOF. For $n \geqslant 4$, we have

$$\frac{C_n}{n} = \frac{C_n^{(2)}}{(n)_2}.$$

From (48) and Lemma 24 (i), we obtain

$$\operatorname{ord}_3\left(\frac{C_n}{n}\right) \geqslant 0.$$

From (48) and Lemma 24 (i), we find that if $p \ge 5$ and $p-1 \nmid n$, then we have

$$\operatorname{ord}_p\left(\frac{C_n}{n}\right) \geqslant 0,$$

if $p \ge 5$ and $p-1 \mid n$, then we have

$$\operatorname{ord}_p\left(\frac{C_n}{n}\right) \geqslant -1 - \operatorname{ord}_p a,$$

where n = a(p-1). From $x^2 = 1/z^4$ and Lemma 19, for $n \ge 6$, we have

$$\frac{D_n}{n} = -\frac{1}{4} \frac{C_n^{(4)}}{(n)_4}. (56)$$

From Lemma 24 (iv), (48), and (56), we obtain if $p \ge 5$ and $p-1 \nmid n$,

$$\operatorname{ord}_p\left(\frac{D_n}{n}\right) \geqslant 0,$$

if $p \geqslant 5$ and $p-1 \mid n$,

$$\operatorname{ord}_p\left(\frac{D_n}{n}\right) \geqslant -1 - \operatorname{ord}_p a,$$

where n = a(p-1). Since $y = 1/s^5$, we have

$$\frac{D_n}{n} = \frac{D_n^{(5)}}{(n)_5}. (57)$$

From Lemma 29 (v), (55), and (57), we obtain

$$\operatorname{ord}_3\left(\frac{D_n}{n}\right) \geqslant -1.$$

From Lemma 27, we have

$$\operatorname{ord}_2\left(\frac{D_n^{(1)}}{n}\right) \geqslant 2, \quad \text{for } n \geqslant 10.$$

Therefore, from Lemma 29 (v), (57), and Lemma 31, we have

$$\operatorname{ord}_2\left(\frac{D_n}{n}\right) \geqslant -1, \quad \text{for } n \geqslant 6.$$

From (56), we have

$$\operatorname{ord}_2\left(\frac{C_n^{(4)}}{(n)_4}\right) \geqslant 1, \quad \text{for } n \geqslant 6.$$

From the above equation, $x^{-1} = z^2 \in \mathbb{Z}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]\langle\langle u \rangle\rangle$, Lemma 30, and Lemma 31, we obtain

$$\operatorname{ord}_2\left(\frac{C_n}{n}\right) \geqslant 1, \quad \text{for } n \geqslant 4.$$

REMARK 3. Theorem 11 gives the precise information on the series expansion of the solution of the inversion problem of the ultra-elliptic integrals given in Proposition 17.

We assume $\lambda_4 = \lambda_6 = \lambda_8 = 0$ and consider the curve defined by $y^2 = x^5 + \lambda_{10}$.

Lemma 32. For any integer $m \ge 1$, we have

$$\operatorname{ord}_{3}\left(\frac{C_{10m}^{(1)}}{10m}\right) \geqslant \left\lfloor \frac{10m-1}{6} \right\rfloor, \quad \operatorname{ord}_{5}\left(\frac{C_{10m}^{(1)}}{10m}\right) \geqslant \left\lfloor \frac{10m-1}{10} \right\rfloor, \quad \operatorname{ord}_{7}\left(\frac{C_{10m}^{(1)}}{10m}\right) \geqslant \left\lfloor \frac{10m-1}{14} \right\rfloor.$$

PROOF. From Proposition 4 and Lemma 13, we obtain the statement of the lemma. □

Lemma 33. ([38]) For $m \ge 1$, we have the following relations.

(i)
$$\frac{C_{10m}^{(2)}}{(10m)_2} + \frac{C_{10m}^{(1)}}{10m} \in 3^2 \cdot 5 \cdot 7 \ \mathbb{Z}[1/2, \lambda_{10}]$$

(ii)
$$2\frac{C_{10m}^{(3)}}{(10m)_3} + \frac{C_{10m}^{(2)}}{(10m)_2} \in 3^2 \cdot 5 \cdot 7 \ \mathbb{Z}[1/2, \lambda_{10}]$$

(iii)
$$3\frac{C_{10m}^{(4)}}{(10m)_4} + \frac{C_{10m}^{(3)}}{(10m)_3} \in 3^3 \cdot 5 \mathbb{Z}[1/2, \lambda_{10}]$$

(iv)
$$6 \frac{C_{10m}^{(4)}}{(10m)_4} + \frac{C_{10m}^{(1)}}{10m} \in 3^2 \cdot 5 \ \mathbb{Z}[1/2, \lambda_{10}]$$

PROOF. Since $\lambda_4 = \lambda_6 = \lambda_8 = 0$, we have $f_n = 0$ for $1 \le n \le 9$ in (46). We can prove this lemma in the same way as Lemma 24. \square

Lemma 34. ([38]) For $m \ge 1$, we have the following relations.

(i)
$$\frac{D_{10m}^{(2)}}{(10m)_2} + \frac{D_{10m}^{(1)}}{10m} \in 2^7 \cdot 3^2 \cdot 7 \ \mathbb{Z}[1/5, \lambda_{10}]$$

(ii)
$$2\frac{D_{10m}^{(3)}}{(10m)_3} + \frac{D_{10m}^{(2)}}{(10m)_2} \in 2^5 \cdot 3^2 \cdot 7 \ \mathbb{Z}[1/5, \lambda_{10}]$$

(iii)
$$3\frac{D_{10m}^{(4)}}{(10m)_4} + \frac{D_{10m}^{(3)}}{(10m)_3} \in 2^4 \cdot 3^3 \mathbb{Z}[1/5, \lambda_{10}]$$

$$(iv) \ 4 \frac{D_{10m}^{(5)}}{(10m)_5} + \frac{D_{10m}^{(4)}}{(10m)_4} \in 2^5 \cdot 3 \ \mathbb{Z}[1/5, \lambda_{10}]$$

$$(v) -24 \frac{D_{10m}^{(5)}}{(10m)_5} + \frac{D_{10m}^{(1)}}{10m} \in 2^5 \cdot 3^2 \ \mathbb{Z}[1/5, \lambda_{10}]$$

PROOF. Since $\lambda_4 = \lambda_6 = \lambda_8 = 0$, we have $g_n = 0$ for $1 \le n \le 9$ in (53). We can prove this lemma in the same way as Lemma 24. \square

For the curve $y^2 = x^5 + \lambda_{10}$, we obtain the following theorem.

Theorem 12. For $m \ge 1$, we have

$$\operatorname{ord}_2\left(\frac{C_{10m}}{10m}\right)\geqslant 2,\quad \operatorname{ord}_3\left(\frac{C_{10m}}{10m}\right)\geqslant 2,\quad \operatorname{ord}_5\left(\frac{C_{10m}}{10m}\right)\geqslant 1,\quad \operatorname{ord}_7\left(\frac{C_{10m}}{10m}\right)\geqslant 1.$$

$$ord_2\left(\frac{D_{10m}}{10m}\right)\geqslant 2, \quad ord_3\left(\frac{D_{10m}}{10m}\right)\geqslant 1, \quad ord_5\left(\frac{D_{10m}}{10m}\right)\geqslant 1, \quad ord_7\left(\frac{D_{10m}}{10m}\right)\geqslant 0.$$

PROOF. From Lemma 32, Lemma 33 (i), and Lemma 31, for $m \ge 1$, we have

$$\operatorname{ord}_{3}\left(\frac{C_{10m}}{10m}\right) \geqslant 2, \quad \operatorname{ord}_{5}\left(\frac{C_{10m}}{10m}\right) \geqslant 1, \quad \operatorname{ord}_{7}\left(\frac{C_{10m}}{10m}\right) \geqslant 1.$$

From Lemma 33 (iv), for $m \ge 1$, we have

$$\operatorname{ord}_{3}\left(\frac{C_{10m}^{(4)}}{(10m)_{4}}\right) \geqslant 1, \quad \operatorname{ord}_{5}\left(\frac{C_{10m}^{(4)}}{(10m)_{4}}\right) \geqslant 1, \quad \operatorname{ord}_{7}\left(\frac{C_{10m}^{(4)}}{(10m)_{4}}\right) \geqslant 0.$$

From (56), we have

$$\operatorname{ord}_{3}\left(\frac{D_{10m}}{10m}\right) \geqslant 1, \quad \operatorname{ord}_{5}\left(\frac{D_{10m}}{10m}\right) \geqslant 1, \quad \operatorname{ord}_{7}\left(\frac{D_{10m}}{10m}\right) \geqslant 0.$$

From Lemma 27, for $m \ge 3$, we have

$$\operatorname{ord}_2\left(\frac{D_{10m}^{(1)}}{10m}\right) \geqslant 7.$$

From Lemma 34 (v), Lemma 31, and $D_{20}/20 = -2^{13} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 13 \cdot \lambda_{10}^2/11$, for $m \geqslant 1$, we have

$$\operatorname{ord}_2\left(\frac{D_{10m}}{10m}\right) \geqslant 2.$$

From (56), we have

$$\operatorname{ord}_{2}\left(\frac{C_{10m}^{(4)}}{(10m)_{4}}\right) \geqslant 4.$$

From Lemma 30, we have

$$\operatorname{ord}_2\left(\frac{C_{10m}}{10m}\right) \geqslant 2.$$

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