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Классификация k-форм на $\mathbb{R}^n$ и существование ассоциированной геометрии на многообразиях[1]

Хонг Ван Ле, И. Ванжура

Ле Хонг Ван — доктор наук, профессор, Институт математики Чешской академии наук, (г. Прага, Чехия).
e-mail: hvle@math.cas.cz
Ванжура Иржи — доктор наук, профессор, Институт математики Чешской академии наук, (г. Прага, Чехия).
vanzura@math.cas.cz

Аннотация

В этой статье мы рассмотрим методы и результаты классификации k-форм (соотв. k-векторов на $\mathbb{R}^n$), понимаемых как описание пространства орбит стандартного $\mathrm{GL}(n,\mathbb{R})$-действие на $\Lambda^k\mathbb{R}^{n*}$ (соотв. на $\Lambda^k\mathbb{R}^n$). Мы обсудим существование связанной геометрии, определяемой дифференциальными формами на гладких многообразиях. Эта статья также содержит Приложение, написанное Михаил Боровым, о методах когомологии Галуа для нахождения вещественных форм комплексных орбит.

Ключевые слова: $\mathrm{GL}(n,\mathbb{R})$-орбиты в $\Lambda^k\mathbb{R}^{n*}$; θ-группа; геометрия, определяемая дифференциальными формами; когомологии Галуа

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Classification of k-forms on $\mathbb{R}^n$ and the existence of associated geometry on manifolds[2]

Hông Vân Lê, J. Vanžura

Lê Hông Vân — Doctor of Sciences, Professor, Institute of Mathematics of the Czech Academy of Sciences, (Praha, Czech Republic).
e-mail: hvle@math.cas.cz
Vanžura Jiří — Doctor of Sciences, Professor, Institute of Mathematics of the Czech Academy of Sciences, (Praha, Czech Republic).
vanzura@math.cas.cz

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Abstract

In this paper we survey methods and results of classification of k-forms (resp. k-vectors on $\mathbb{R}^n$), understood as description of the orbit space of the standard $\mathrm{GL}(n,\mathbb{R})$-action on $\Lambda^k\mathbb{R}^{n*}$ (resp. on $\Lambda^k\mathbb{R}^n$). We discuss the existence of related geometry defined by differential forms on smooth manifolds. This paper also contains an Appendix by Mikhail Borovoi on Galois cohomology methods for finding real forms of complex orbits.

Keywords: $\mathrm{GL}(n,\mathbb{R})$-orbits in $\Lambda^k\mathbb{R}^{n*}$; θ-group; geometry defined by differential forms; Galois cohomology

Bibliography: 68 titles.

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Preface

Hamiltonian systems were one of research topics of Hông Vân Lê in her undergraduate study and calibrated geometry was the topic of her Ph.D. Thesis under guidance of Professor Anatoly Timofeevich Fomenko. Hamiltonian systems are defined on symplectic manifolds and calibrated geometry is defined by closed differential forms of comass one on Riemannian manifolds. Since that time she works frequently on geometry defined by differential forms, some of her papers were written in collaboration with Jiři Vanžura, [38, 39, 40]. We dedicate this survey on algebra and geometry of k-forms on $\mathbb{R}^n$ as well as on smooth manifolds to Anatoly Timofeevich Fomenko on the occasion of his 75th birthday and we wish him good health, happiness and much success for the coming years.

1. Introduction

Differential forms are excellent tools for the study of geometry and topology of manifolds and their submanifolds as well as dynamical systems on them. Kähler manifolds, and more generally, Riemannian manifolds (M, g) with non-trivial holonomy group admit parallel differential forms and hence calibrations on (M, g) [27], [55], [40], [17]. In the study of Riemannian manifolds with non-trivial holonomy groups these parallel differential forms are extremely important [7], [29]. In their seminal paper [27] Harvey-Lawson used calibrations as powerful tool for the study of geometry of calibrated submanifolds, which are volume minimizing. Their paper opened a new field of calibrated geometry [30] where one finds more and more tools for the study of calibrated submanifolds using differential forms, see e.g., [17]. In 2000 Hitchin initiated the study of geometry defined by a differential 3-form [25], and in a subsequent paper he analyzed beautiful geometry defined by differential forms in low dimensions [26]. One starts investigation of a differential form φ^k of degree k on a manifold M^n of dimension n by finding a *normal form* of φ^k at a point $x \in M^n$ and, if possible, to find a normal form of φ^k up to certain order in a small neighborhood $U(x) \subset M^n$. Finding a normal form of φ^k at a point $x \in M^n$ is the same as finding a canonical representative of the equivalence class of $\varphi^k(x)$ in $\Lambda^k(T_x^*M^n)$, where two k-forms on T_xM^n are *equivalent* if they are in the same orbit of the standard $\mathrm{GL}(n,\mathbb{R})$-action on $\Lambda^k(T_x^*M^n) = \Lambda^k\mathbb{R}^{n*}$. We say that a manifold M^n is endowed by a differential form $\varphi \in \Omega^*(M^n)$ of type $\varphi_0 \in \Lambda^*\mathbb{R}^{n*}$, if for all $x \in M^n$ the equivalence class of $\varphi(x) \in \Lambda^*T_x^*M^n$ can be identified with the equivalent class of $\varphi_0 \in \Lambda^*\mathbb{R}^{n*}$ via a linear isomorphism $T_xM^n = \mathbb{R}^n$. Instead of investigation of a normal form of a concrete form φ^k, we may be also interested in a classification of (equivalent) k-forms on $\mathbb{R}^n$, understood as a description of the moduli space of equivalent k-forms on $\mathbb{R}^n$, which could give us insight on a normal form of φ^k and could also suggest interesting candidates for the geometry defined by differential forms.

Classification of k-forms on $\mathbb{R}^n$ is a part of algebraic invariant theory. Recall that *an invariant* of an equivalence relation on a set S, e.g., defined by orbits of an action of a group G on S, is a mapping from S to another set Q that is constant on the equivalence classes. A system of invariants is called *complete* if it separates any two equivalent classes. If a complete system of invariants consists of one element, we call this invariant complete. In the classical algebraic invariant theory one deals mainly with actions of classical or algebraic groups on some space of tensors of a fixed type over a vector space over a field $\mathbb{F}$ [23], see [48] for a survey of modern invariant theory and source of algebraic invariant theory. From a geometric point of view, the most important invariants of a form φ^k on $\mathbb{R}^n$ are *the rank of* φ^k and *the stabilizer* of φ^k under the action of $\mathrm{GL}(n,\mathbb{R})$. Recall that the rank of φ^k, denoted by $\mathrm{rk}\,\varphi^k$, is the dimension of the image of the linear operator $L_{\varphi^k} : \mathbb{R}^n \to \Lambda^{k-1}\mathbb{R}^{n*}$, $v \mapsto i_v\varphi^k$. We denote the stabilizer of φ^k by $\mathrm{St}_{\mathrm{GL}(n,\mathbb{R})}(\varphi^k)$, and in general, we denote by $\mathrm{St}_G(x)$ the stabilizer of a point x in a set S where a group G acts. A form $\varphi^k \in \Lambda^k\mathbb{R}^{n*}$ is called *non-degenerate*, or *multisymplectic*, if $\mathrm{rk}\,\varphi^k = n$. Furthermore, it is important to study the topology of the orbit $\mathrm{GL}(n,\mathbb{R})\cdot\varphi^k = \mathrm{GL}(n,\mathbb{R})/\mathrm{St}_{\mathrm{GL}(n,\mathbb{R})}(\varphi^k)$, for example, the connectedness, see Proposition 2 below, the openness, the closure of the orbit $\mathrm{GL}(n,\mathbb{R})\cdot\varphi^k \subset \Lambda^k\mathbb{R}^{n*}$. It turns out that understanding these questions helps us to understand the structure of the orbit space of $\mathrm{GL}(n,\mathbb{R})$-action on $\Lambda^k\mathbb{R}^{n*}$. These invariants of k-forms shall be highlighted in our survey.

Let us outline the plan of our paper. In the first part of Section 2 we make several observations on the duality between $\mathrm{GL}(n,\mathbb{R})$-orbits of k-forms on $\mathbb{R}^n$ and $\mathrm{GL}(n,\mathbb{R})$-orbits of k-vectors as well as the duality between $\mathrm{GL}^+(n,\mathbb{R})$-orbits of k-forms on $\mathbb{R}^n$ and $\mathrm{GL}^+(n,\mathbb{R})$-orbits of $(n-k)$-forms on $\mathbb{R}^n$. Then we recall the classification of 2-forms on $\mathbb{R}^n$ (Theorem 2) and present the Martinet's classification of $(n-2)$-forms on $\mathbb{R}^n$ (Theorem 3).

In contrast to the classification of 2-forms on $\mathbb{R}^n$, the classification of 3-forms on $\mathbb{R}^n$ depends on the dimension n. Since $\dim\Lambda^3\mathbb{R}^{n*} \geqslant \dim\mathrm{GL}(n,\mathbb{R})+1$, if $n \geqslant 9$, there are infinite numbers of inequivalent 3-forms in $\mathbb{R}^n$. Till now there is no classification of the $\mathrm{GL}(n,\mathbb{R})$-action on $\Lambda^3\mathbb{R}^{n*}$, if $n \geqslant 10$.

In the dimension $n = 9$ the classification of the $\mathrm{SL}(9,\mathbb{C})$-orbits on $\Lambda^3\mathbb{C}^9$ has been obtained by Vinberg-Elashvili [65]. In the second part of Section 2 we survey Vinberg-Elashvili's result and some further developments by Le [34] and Dietrich-Facin-de Graaf [12], which give partial information on $\mathrm{GL}(9,\mathbb{R})$-orbits on $\Lambda^3\mathbb{R}^9$. Then we review Djokovic' classification of 3-vectors in $\mathbb{R}^8$ and present a classification of 5-forms on $\mathbb{R}^8$ (Corollary 1). Djokovic's classification method combines some ideas from Vinberg-Elashvili's work and Galois cohomology method for classifying real forms of a complex orbit. Note that the classification of 3-vectors in $\mathbb{R}^8$ implies the classification of 3-forms in $\mathbb{R}^8$ (Proposition 1) as well as the classifications of 3-forms in $\mathbb{R}^n$ for $n \leqslant 7$ (Theorem 1, Remark 5). Then we review a classification of $\mathrm{GL}(8,\mathbb{C})$-action on $\Lambda^4\mathbb{C}^8$ by Antonyan [1], which is important for classification of 4-forms on $\mathbb{R}^8$. At the end of Section 2 we review a scheme of classification of 4-forms on $\mathbb{R}^8$ proposed by Lê in 2011 [34] and Dietrich-Facin-de Graaf's method of classification of 3-forms on $\mathbb{R}^8$ in [12].

In Section 3, for $k = 2, 3, 4$, we compile known results and discuss some open problems on necessary and sufficient topological conditions for the existence of a differential k-form φ of given type $\mathrm{St}_{\mathrm{GL}(n,\mathbb{R})}(\varphi(x))$ on manifolds M^n (in these cases the equivalence class of $\varphi(x)$ is defined uniquely by the type of the stablizer of $\varphi(x)$, i.e., the conjugation class of $\mathrm{St}_{\mathrm{GL}(n,\mathbb{R})}(\varphi(x))$ in $\mathrm{GL}(n,\mathbb{R})$). In dimension $n = 8$ (and hence also for $n = 6, 7$) we observe that the stabilizer $\mathrm{St}_{\mathrm{GL}(n,\mathbb{R})}(\varphi)$ of a 3-form $\varphi \in \Lambda^3\mathbb{R}^{n*}$ forms a complete system of invariants of the action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$ (Remark 6).

We include two appendices in this paper. The first appendix contains a result due to Hông Vân Lê concerning the existence of 3-form of type $\tilde{G}_2$ on a smooth 7-manifold, which has been posted in arxiv in 2007 [33]. The second appendix outlines the Galois cohomology method for classification of real forms of a complex orbit. This appendix is taken from a private note by Mikhail Borovoi with his kind permission.

Finally we would like to emphasize that our paper is not a bibliographical survey. Some important papers may have been missed if they are not directly related to the main lines of our narrative. We also don't mention in this survey the relations of geometry defined by differential forms to physics and instead refer the reader to [30], [15], [14], [60].

2. Classification of $\mathrm{GL}(n,\mathbb{R})$-orbits of k-forms on $\mathbb{R}^n$

2.1. General theorems

We begin the classification of $\mathrm{GL}(n,\mathbb{R})$-orbits on $\Lambda^k\mathbb{R}^{n*}$ with the following observation that the orbit of the standard action of $\mathrm{GL}(n,\mathbb{R})$ on $\Lambda^k\mathbb{R}^n$ can be identified with the orbit of the standard action of $\mathrm{GL}(n,\mathbb{R})$ on $\Lambda^k\mathbb{R}^{n*}$ by using an isomorphism $\mu \in Hom(\mathbb{R}^n, \mathbb{R}^{n*}) = \mathbb{R}^{n*}\otimes\mathbb{R}^{n*} \supset S^2\mathbb{R}^{n*}$. Note that there are several papers and books devoted to the classification of k-vectors on $\mathbb{R}^n$ [23, Chapter VII] [3], [11], [65]. Hence we have the following well-known fact, see e.g., [45],

Proposition 1. *There exists a bijection between the* $\mathrm{GL}(n,\mathbb{R})$*-orbits in* $\Lambda^k\mathbb{R}^n$ *and* $\mathrm{GL}(n,\mathbb{R})$*-orbits in* $\Lambda^k\mathbb{R}^{n*}$.

Next we shall compare $\mathrm{GL}^+(n,\mathbb{R})$-orbits on $\Lambda^k\mathbb{R}^n$ with $\mathrm{GL}^+(n,\mathbb{R})$-orbits on $\Lambda^{n-k}\mathbb{R}^{n*}$. We take a volume form $\Omega\in\Lambda^n\mathbb{R}^{n*}\setminus\{0\}$ and define the Poincaré isomorphism $P_\Omega : \Lambda^k\mathbb{R}^n \to \Lambda^{n-k}\mathbb{R}^{n*}, \xi\mapsto i_\xi\Omega$. Since $\mathrm{GL}^+(n,\mathbb{R})$ is a direct product of its center $Z(\mathrm{GL}^+(n,\mathbb{R})) = \mathbb{R}^+$ with its semisimple subgroup $\mathrm{SL}(n,\mathbb{R})$, for any $\lambda\in\mathbb{R}$ the group $\mathrm{GL}^+(n,\mathbb{R})$ admits a λ-twisted action on $\Lambda^k\mathbb{R}^{n*}$ defined as follows: $g_{[\lambda]}(\varphi) := (\det g)^\lambda\cdot g(\varphi)$ for $g\in\mathrm{GL}^+(n,\mathbb{R})$, $\varphi\in\Lambda^k\mathbb{R}^{n*}$, where $g(\varphi)$ denotes the standard action of g on φ.

Denote also by μ the isomorphism $\Lambda^k\mathbb{R}^n\to\Lambda^k\mathbb{R}^{n*}$ induced from a scalar product μ on $\mathbb{R}^n$.

Lemma 1. *The composition* $P_\Omega\circ\mu^{-1} : \Lambda^k\mathbb{R}^{n*}\to\Lambda^{n-k}\mathbb{R}^{n*}$ *is a* $\mathrm{GL}^+(n,\mathbb{R})$*-equivariant map where* $\mathrm{GL}^+(n,\mathbb{R})$ *acts on* $\Lambda^k\mathbb{R}^{n*}$ *by the standard action and on* $\Lambda^{n-k}\mathbb{R}^{n*}$ *by the* (-1)*-twisted action.*

Proof. Let $\varphi = \mu(X)\in\Lambda^k\mathbb{R}^{n*}$ and $g\in\mathrm{GL}^+(n,\mathbb{R})$. Then

$$P_\Omega\circ\mu^{-1}(g^*\varphi) = P_\Omega(g^{-1}\circ\mu^{-1}(\varphi)) = i_{g^{-1}\mu^{-1}(\varphi)}\Omega$$

$$= (\det g)^{-1}\cdot g(i_{\mu^{-1}(\varphi)}\Omega) = g_{[-1]}(P_\Omega\circ\mu^{-1}(\varphi)),$$

which proves the first assertion of Lemma 1. □

Proposition 2. *(1) There is a 1-1 correspondence between* $\mathrm{GL}^+(n,\mathbb{R})$*-orbits of k-forms on* $\mathbb{R}^n$ *and* $\mathrm{GL}^+(n,\mathbb{R})$*-orbits of* $(n-k)$*-forms on* $\mathbb{R}^n$*. This correspondence preserves the openness of* $\mathrm{GL}^+(n,\mathbb{R})$*-orbits (and hence the openness of* $\mathrm{GL}(n,\mathbb{R})$*-orbits).*

(2) The $\mathrm{GL}(n,\mathbb{R})$*-orbit of* $\varphi^k\in\Lambda^k\mathbb{R}^{n*}$ *has two connected components if and only if* $\mathrm{St}_{\mathrm{GL}(n,\mathbb{R})}(\varphi^k)\subset\mathrm{GL}^+(n,\mathbb{R})$*. In other cases the* $\mathrm{GL}(n,\mathbb{R})$*-orbit of* φ^k *is connected.*

(3) Assume that $\varphi^k\in\Lambda^k\mathbb{R}^{n*}$ *is degenerate. Then the* $\mathrm{GL}(n,\mathbb{R})$*-orbit of* φ^k *is connected.*

Proof. 1. The first assertion of Proposition 2 is a consequence of Lemma 1.

2. The second assertion of Proposition 2 follows from the fact that $\mathrm{GL}(n,\mathbb{R})$ has two connected components.

3. Assume that φ is degenerate. Then $W := \ker L_\varphi$ is non-empty. Let $W^\perp$ be any complement to W in $\mathbb{R}^n$ i.e., $\mathbb{R}^n = W\oplus W^\perp$. Then $\mathrm{GL}(W)\oplus Id_{W^\perp}$ is a subgroup of $St(\varphi)$. Since this subgroup has non-trivial intersection with $\mathrm{GL}^-(n,\mathbb{R})$, this implies the last assertion of Proposition 2 follows from the second one. This completes the proof of Proposition 2. □

[3] under "polyvectors"Gurevich meant both covariant and contravariant polyvectors

The following theorem due to Vinberg-Elashvili reduces a classification of (degenerate) k-forms of rank r in $\mathbb{R}^n$ to a classification of k-forms on $\mathbb{R}^r$. (Vinberg-Elashvili considered only the case $k = 3$ and the $\mathrm{SL}(n, \mathbb{C})$-action on $\Lambda^3\mathbb{C}^n$ but their argument works for any k and for $\mathrm{GL}(n, \mathbb{R})$-action on $\Lambda^k\mathbb{R}^{n*}$.)

THEOREM 1. *(cf. [65, §4.4], [53, Lemma 3.2]) There is a 1-1 correspondence between* $\mathrm{GL}(n, \mathbb{R})$*-orbits of k-forms of rank less or equal to r on $\mathbb{R}^n$ and* $\mathrm{GL}(r, \mathbb{R})$*-orbits of k-forms on $\mathbb{R}^r$.*

2.2. Classification of 2-forms and $(n-2)$-forms on $\mathbb{R}^n$

From Proposition 2 we obtain immediately the following known theorem [10], cf. [23, Theorem 34.9].

THEOREM 2. *(1) The rank of a 2-form $\varphi \in \Lambda^2\mathbb{R}^{n*}$ is a complete invariant of the standard* $\mathrm{GL}(n, \mathbb{R})$*-action on $\Lambda^2\mathbb{R}^{n*}$. Hence $\Lambda^2\mathbb{R}^{n*}$ decomposes into $[n/2] + 1$* $\mathrm{GL}(n, \mathbb{R})$*-orbits.*

(2) The $\mathrm{GL}(n, \mathbb{R})$*-orbit of a 2-form $\varphi \in \Lambda^2\mathbb{R}^{n*}$ has two connected components if and only if $n = 2k$ and φ has maximal rank.*

(3) If φ is of maximal rank, then the $\mathrm{GL}(n, \mathbb{R})$*-orbit of φ is open and its closure contains the* $\mathrm{GL}(n, \mathbb{R})$*-orbit of any degenerate 2-form on $\mathbb{R}^n$.*

The classification of $(n-2)$-forms on $\mathbb{R}^n$ has been done by Martinet [41]. Martinet used the inverse Poincaré isomorphism $P_\Omega^{-1} : \Lambda^{n-2}\mathbb{R}^{n*} \to \Lambda^2\mathbb{R}^n$ to define *the length of* $\varphi \in \Lambda^{n-2}\mathbb{R}^n$, denoted by $l(\varphi)$, to be the half of the rank of the bi-vector $P_\Omega^{-1}(\varphi)$ [4]. By Proposition 2 and Theorem 2 the map P_Ω^{-1} induces an isomorphism between the $\mathrm{GL}(n, \mathbb{R})$-orbits of degenerate $(n-2)$-forms φ on $\mathbb{R}^n$ and degenerate bivectors $P_\Omega^{-1}(\varphi)$ on $\mathbb{R}^n$.

• If $2l(\varphi) < n$ then φ has the following canonical form

$$\varphi = \sum_{i=1}^{l(\varphi)} \alpha_1 \wedge \cdots \alpha_{2i-2} \wedge \alpha_{2i+1} \wedge \cdots \wedge \alpha_n. \tag{1}$$

By Theorem 2 (2) the orbit $\mathrm{GL}(n, \mathbb{R}) \cdot P_\Omega^{-1}(\varphi)$ is connected, and hence by Proposition 2 the orbit $\mathrm{GL}(n, \mathbb{R}) \cdot \varphi$ is connected.

• If $2l(\varphi) = n$, and $l(\varphi)$ is odd, then using Lemma 1 and Theorem 2(2) we conclude that the set of $(n-2)$-forms of length l consists of two open connected $\mathrm{GL}(n, \mathbb{R})$-orbits that correspond to the sign of $\lambda = \lambda_\Omega(\varphi)$ where

$$P_\Omega^{-1}(\varphi) = e_1 \wedge e_2 + \cdots + e_{2k-1} \wedge e_{2k},$$

$$\Omega = \lambda\, \alpha_1 \wedge \cdots \wedge \alpha_n,$$

$$\varphi = \lambda \sum_{i=1}^{l(\varphi)} \alpha_1 \wedge \cdots \alpha_{2i-2} \wedge \alpha_{2i+1} \wedge \cdots \wedge \alpha_n \text{ and } \lambda = \pm 1. \tag{2}$$

• If $2l(\varphi) = n$ and $l(\varphi)$ is even, using the same argument as in the previous case, we conclude that the set of (n-2)-forms of length l consists of one open $\mathrm{GL}(n, \mathbb{R})$-orbit, which has two connected components.

To summarize Martinet's result, we assign the sign $s_\Omega(\varphi)$ of a (n-2)-form $\varphi \in \Lambda^{n-2}\mathbb{R}^n$ to be the number $\lambda_\Omega(\varphi)^{l(\varphi)}$ if $2l(\varphi) = n$, and to be 1, if $2l(\varphi) < n$.

THEOREM 3. *(cf. [41, §5]) (1) The length $l(\varphi)$ and the sign $s_\Omega(\varphi)$ of a (n-2)-form $\varphi \in \Lambda^{n-2}\mathbb{R}^{n*}$ form a complete system of invariants of the standard* $\mathrm{GL}(n, \mathbb{R})$*-action on $\Lambda^{n-2}\mathbb{R}^{n*}$.*

(2) The $\mathrm{GL}(n, \mathbb{R})$*-orbit of a (n-2)-form $\varphi \in \Lambda^2\mathbb{R}^{n*}$ has two connected components if and only if $n = 2k$, $l(\varphi) = n/2$ and l is even.*

[4] the rank of a k-vector is defined similarly as the rank of a k-form.

2.3. Classification of 3-forms and 6-forms on $\mathbb{R}^9$

We observe that the vector space $\Lambda^k\mathbb{R}^{n*}$ is a real form of the complex vector space $\Lambda^k\mathbb{C}^{n*}$. Hence, for any $\varphi \in \Lambda^k\mathbb{R}^{n*}$ the orbit $\mathrm{GL}(n,\mathbb{R})\cdot\varphi$ lies in the orbit $\mathrm{GL}(n,\mathbb{C})\cdot\varphi$. We shall say that $\mathrm{GL}(n,\mathbb{R})\cdot\varphi$ is a *real form of the complex orbit* $\mathrm{GL}(n,\mathbb{C})\cdot\varphi$. It is known that every complex orbit has only finitely many real forms [3, Proposition 2.3]. Thus, the problem of classifying of the $\mathrm{GL}(n,\mathbb{R})$-orbits in $\Lambda^k\mathbb{R}^n$ can be reduced to the problem of classifying the real forms of the $\mathrm{GL}(n,\mathbb{C})$-orbits on $\Lambda^k\mathbb{C}^n$. The classification of $\mathrm{GL}(n,\mathbb{C})$-orbits on $\Lambda^3\mathbb{C}^n$ is trivial, if $n \leqslant 5$, cf. Proposition 2. For $n = 6$ it was solved by W. Reichel [50]; for $n = 7$ it was solved by J. A. Schouten [57]; for $n = 8$ it was solved by Gurevich in 1935, see also [23]; and for $n = 9$ it was solved by Vinberg-Elashvili [65]. In fact Vinberg-Elashvili classified $\mathrm{SL}(9,\mathbb{C})$-orbits on $\Lambda^3\mathbb{C}^9$, which are in 1-1 correspondence with $\mathrm{SL}(9,\mathbb{C})$-orbits in $\Lambda^3\mathbb{C}^{9*}$ and $\mathrm{SL}(9,\mathbb{C})$-orbits on $\Lambda^6\mathbb{C}^{9*}$. Since the center of $\mathrm{GL}(9,\mathbb{C})$ acts on $\Lambda^3\mathbb{C}^9\setminus\{0\}$ with the kernel $\mathbb{Z}_3$, it is not hard to obtain a classification of $\mathrm{GL}(9,\mathbb{C})$-orbits on $\Lambda^3\mathbb{C}^9$, and hence on $\Lambda^3\mathbb{C}^{9*}$ and on $\Lambda^6\mathbb{C}^{9*}$ from the classification of the $\mathrm{SL}(9,\mathbb{C})$-orbits on $\Lambda^3\mathbb{C}^9$.

As we have remarked before, there are infinitely many $\mathrm{GL}(n,\mathbb{C})$-orbits on $\Lambda^3\mathbb{C}^9$, and to solve this complicated classification problem Vinberg-Elashvili made an important observation that the standard $\mathrm{SL}(9,\mathbb{C})$-action on $\Lambda^3\mathbb{C}^9$ is equivalent to the action of the adjoint group $G_0^{\mathbb{C}}$ (also called the θ-group) of the $\mathbb{Z}_3$-graded complex simple Lie algebra

$$\mathfrak{e}_8 = \mathfrak{g}_{-1}^{\mathbb{C}} \oplus \mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{g}_1^{\mathbb{C}} \tag{3}$$

where $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{sl}(9,\mathbb{C})$, $\mathfrak{g}_1^{\mathbb{C}} = \Lambda^3\mathbb{C}^3$, $\mathfrak{g}_{-1}^{\mathbb{C}} = \Lambda^3\mathbb{C}^{9*}$ and $G_0^{\mathbb{C}} = \mathrm{SL}(9,\mathbb{C})/\mathbb{Z}_3$ is the connected subgroup, corresponding to the Lie subalgebra $\mathfrak{g}_0^{\mathbb{C}}$, of the simply connected Lie group $E_8^{\mathbb{C}}$ whose Lie algebra is $\mathfrak{e}_8$.

Remark 1. *Let $\mathfrak{g}^{\mathbb{C}}$ be a complex Lie algebra. Any $\mathbb{Z}_m$-grading $\mathfrak{g}^{\mathbb{C}} := \oplus_{i\in\mathbb{Z}_m}\mathfrak{g}_i^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$ defines an automorphism $\sigma \in \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$ of order m by setting $\sigma(x) := \epsilon^i x$ where $\epsilon = \exp(2\sqrt{-1}\pi/m)$ and $x \in \mathfrak{g}_i^{\mathbb{C}}$. Conversely, any $\sigma \in \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$ of order m defines a $\mathbb{Z}_m$-grading $\mathfrak{g}^{\mathbb{C}} := \oplus_{i\in\mathbb{Z}_m}\mathfrak{g}_i^{\mathbb{C}}$ by setting $\mathfrak{g}_i^{\mathbb{C}} := \{x \in \mathfrak{g}^{\mathbb{C}} |\, \sigma(x) = \epsilon^i x\}$.*

In [65, §2.2] Vinberg and Elashvili considered the automorphism $\theta^{\mathbb{C}}$ of order 3 on $\mathfrak{e}_8$ associated to the $\mathbb{Z}_3$-gradation in (6) [5]. To describe $\theta^{\mathbb{C}}$ we recall the root system Σ of $\mathfrak{e}_8$:

$$\Sigma = \{\varepsilon_i - \varepsilon_j, \pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)\},\ (i,j,k \text{ distinct}), \sum_{i=1}^{9}\varepsilon_i = 0\}.$$

Remark 2. *Given a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ let us choose a Cartan subalgebra $\mathfrak{h}_0^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$. Let Σ be the root system of $\mathfrak{g}^{\mathbb{C}}$. Denote by $\{H_\alpha, E_\alpha |\, \alpha \in \Sigma\}$ the Chevalley system in $\mathfrak{g}^{\mathbb{C}}$ i.e., $H_\alpha \in \mathfrak{h}_0^{\mathbb{C}}$ and E_α is the root vector corresponding to α such that for any $H \in \mathfrak{h}_0^{\mathbb{C}}$ we have $[H, E_\alpha] = \alpha(H)E_\alpha$, $[H_\alpha, E_\alpha] = 2E_\alpha$ and $[E_\alpha, E_{-\alpha}] = H_\alpha$ [28, §32.2]. Then*

$$\mathfrak{g}^{\mathbb{C}} = \oplus_{\alpha\in\Sigma_s^+}\langle H_\alpha\rangle_{\mathbb{C}} \oplus_{\alpha\in\Sigma^+}\langle E_\alpha\rangle_{\mathbb{C}} \oplus_{\alpha\in\Sigma^+}\langle E_{-\alpha}\rangle_{\mathbb{C}} \tag{4}$$

where $\Sigma^+ \subset \Sigma$ denote the system of positive roots, and Σ_s^+ - the subset of simple roots.

The automorphism $\theta^{\mathbb{C}}$ of order 3 on $\mathfrak{e}_8$ is defined as follows

$$\theta^{\mathbb{C}}_{|\langle H_\alpha, E_\alpha,\, \alpha=\varepsilon_i-\varepsilon_j\rangle_{\mathbb{C}}} = Id,$$

$$\theta^{\mathbb{C}}_{|\langle E_\alpha, \alpha=(\varepsilon_i+\varepsilon_j+\varepsilon_k)\rangle_{\mathbb{C}}} = \exp(i2\pi/3)\cdot Id,$$

$$\theta^{\mathbb{C}}_{|\langle E_\alpha, \alpha=-(\varepsilon_i+\varepsilon_j+\varepsilon_k)\rangle_{\mathbb{C}}} = \exp(-i2\pi/3)\cdot Id.$$

[5] Automorphisms of finite order of semisimple Lie algebras have been classified earlier independently by Wolf-Gray [66] and Kac [31].

Remark 3. *Let* $\{H_\alpha, E_\alpha | \alpha \in \Sigma\}$ *be the Chevalley system of a complex semisimple Lie algebra* $\mathfrak{g}^{\mathbb{C}}$. *Then* $\{H_\beta, E_\alpha | \alpha \in \Sigma, \beta \in \Sigma_s^+\}$ *is a basis of the normal form* $\mathfrak{g}$, *also called split real form, of* $\mathfrak{g}^{\mathbb{C}}$. *The normal form of the complex simple Lie algebra* $\mathfrak{e}_8$ *is denoted by* $\mathfrak{e}_{8(8)}$, *and the normal form of* $\mathfrak{sl}(n,\mathbb{C})$ *is the real simple Lie algebra* $\mathfrak{sl}(n,\mathbb{R})$. *Clearly the Lie subalgebra* $\mathfrak{e}_{8(8)}$ *has the induced* $\mathbb{Z}_3$*-grading from the one on* $\mathfrak{e}_8$ *defined in (3) (note that* $\mathfrak{e}_{8(8)}$ *is not invariant under* $\theta^{\mathbb{C}}$*), i.e., we have*

$$\mathfrak{e}_{8(8)} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{5}$$

where $\mathfrak{g}_i = \mathfrak{e}_{8(8)} \cap \mathfrak{g}_i^{\mathbb{C}}$ *is a real form of* $\mathfrak{g}_i^{\mathbb{C}}$ *for* $i \in \{-1,0,1\}$. *Hence there is a 1-1 correspondence between* $\mathrm{SL}(9,\mathbb{R})$*-orbits on* $\Lambda^3\mathbb{R}^{9*}$ *and the adjoint action of the subgroup* G_0, *corresponding to the Lie subalgebra* $\mathfrak{g}_0$, *of the Lie group* $G_0^{\mathbb{C}}$.

Now let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$. Based on (5), (3), Remark 3, and following [65, §1], [34, Lemma 2.5], we shall call a nonzero element $x \in \Lambda^3\mathbb{F}^9$ *semisimple*, if its orbit $\mathrm{SL}(9,\mathbb{F})\cdot x$ is closed in $\Lambda^3\mathbb{F}^9$, and *nilpotent*, if the closure of its orbit $\mathrm{SL}(9,\mathbb{F})\cdot x$ contains the zero 3-vector. Our notion of semisimple and nilpotent elements agrees with the notion of semisimple and nilpotent elements in semisimple Lie algebras [65], [34], see also [11] for an equivalent definition of semisimple and nilpotent elements in homogeneous components of graded semisimple Lie algebras.

Example 12. *([65, §4.4]) Let* $x \in \Lambda^3\mathbb{F}^9$ *be a degenerate vector of rank* $r \leqslant 8$, *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$. *(The definition of the rank of a k-vector can be defined in the same way as the definition of the rank of a k-form). Then for any* $\lambda \in \mathbb{R}$ *there exists an element* $g \in \mathrm{SL}(9,\mathbb{F})$ *such that* $g\cdot x = \lambda \cdot x$. *Hence the closure of the orbit* $\mathrm{SL}(9,\mathbb{F})\cdot x$ *contains* $0 \in \Lambda^3\mathbb{F}^9$ *and therefore x is a nilpotent element.*

Proposition 3. *Every nonzero* 3*-vector* x *in* $\Lambda^3\mathbb{F}^9$ *can be uniquely written as* $x = p + e$, *where* p *is a semisimple* 3*-vector,* e *- a nilpotent 3-vector, and* $p \wedge e = 0$.

Proposition 3 has been obtained by Vinberg-Elashvili in [65] for the case $\mathbb{F} = \mathbb{C}$. To prove Proposition 3 for $\mathbb{F} = \mathbb{R}$, we use the Jordan decomposition of a homogeneous element in a real $\mathbb{Z}_m$-graded Lie semisimple algebra and a version of the Jacobson-Morozov-Vinberg theorem for real graded semisimple Lie algebras [34, Theorem 2.1].

Using Proposition 3, Vinberg-Elashvili proposed the following scheme for their classification of 3-vectors on $\mathbb{C}^9$. First they classified semisimple 3-vectors p. The $\mathrm{SL}(9,\mathbb{C})$-equivalence class of semisimple 3-vectors p has dimension 4 - the dimension of a maximal subspace consisting of commuting semisimple elements in $\mathfrak{g}_1$. Then the equivalence classes of semisimple elements p are divided into seven types according to the type of the stabilizer subgroup $\mathrm{St}(p)$ and the subspace $E(p) := \{x \in \Lambda^3\mathbb{C}^9 | p\wedge x = 0\}$. We assign a 3-vector on $\mathbb{F}^9$ to the same family as its semisimple part. Then Vinberg-Elashvili described all possible nilpotent parts for each family of 3-vectors. When the semisimple part is p, the latter are all the nilpotent 3-vectors e of the space $E(p)$. The classification is made modulo the action of $\mathrm{St}_{\mathrm{SL}(9,\mathbb{C})}(p)$. Note that there is only finite number of nilpotent orbits in $E(p)$ for any semisimple 3-vector p. Therefore the dimension of the orbit space $\Lambda^3\mathbb{C}^9/\mathrm{SL}(9,\mathbb{C})$ is 4, which is the dimension of the space of all semisimple 3-vectors.

To classify semisimple elements $p \in \Lambda^3\mathbb{C}^9$ and nilpotent elements in $E(p)$ Vinberg-Elashvili developed further the general method invented by Vinberg [61, 62, 63, 64] for the study of the orbits of the adjoint action of the θ-group on $\mathbb{Z}_m$-graded semisimple complex Lie algebras.

Vinberg's method has been developed by Antonyan for classification of 4-forms in $\mathbb{C}^8$, which we shall describe in more detail in Subsection 2.5, by Lê [34] and Dietrich-Faccin-de Graaf [12] for real graded semisimple Lie algebras. As a result, we have partial results concerning the orbit space of the standard $\mathrm{SL}(9,\mathbb{R})$-action on $\Lambda^3\mathbb{R}^{9*}$ (as well as partial results concerning the orbit space of the standard action of $\mathrm{SL}(8,\mathbb{R})$ on $\Lambda^4\mathbb{R}^{8*}$ we mentioned above). By Proposition 3, and following Vinberg-Elashvili scheme, the classification of the orbits of $\mathrm{SL}(9,\mathbb{R})$-action on $\Lambda^3\mathbb{R}^9$ can be reduced to the classification of semisimple elements p in $\Lambda^3\mathbb{R}^9$, which is the same as the classification of real forms of $\mathrm{SL}(9,\mathbb{C})$-orbits of semisimple elements p in $\Lambda^3\mathbb{C}^9$ (the classification of the $\mathrm{SL}(9,\mathbb{C})$-orbits has been given in [65]) and the classification of nilpotent elements $e \in \Lambda^3\mathbb{R}^9$ such that $e \wedge p = 0$. Note that e is a nilpotent element in the semisimple component $Z(p)'$ of the zentralizer $Z(p)$ of the semisimple element p. Thus the latter problem is reduced to the classification of real forms of complex nilpotent orbits in $\mathbb{Z}(p)'_{\otimes\mathbb{C}}$, and the classification of the latter orbits has been done in [65]. Lê's method [34] and Dietrich-Faccin-de Graaf's method of classification of nilpotent orbits of real graded Lie algebras [12] give partial information on the real forms of these nilpotent orbits. We shall discuss a similar scheme of classification of 4-forms on $\mathbb{R}^8$ in Subsection 2.5. Currently we consider the Galois cohomology method for classification of 3-forms on $\mathbb{R}^9$ promising [4], and therefore we include an appendix outlining the Galois cohomology method in this paper.

2.4. Classification of 3-forms and 5-forms on $\mathbb{R}^8$

The classification of 3-vectors (and hence 3-forms) on $\mathbb{R}^8$ has been given by Djokovic in [11]. Similar to [65], see (3), Djokovic made an important observation that for $\mathbb{F} = \mathbb{R}$ (resp. for $\mathbb{F} = \mathbb{C}$) the standard $\mathrm{GL}(8,\mathbb{F})$-action on $\Lambda^3\mathbb{F}^8$ is equivalent to the action of the adjoint group $\mathrm{Ad}\, G_0$ of the $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \mathfrak{e}_{8(8)}$ (resp. $\mathfrak{g} = \mathfrak{e}_8$) on the homogeneous component $\mathfrak{g}_1$ of degree 1, where

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3. \tag{6}$$

Here $\mathrm{Ad}\, G_0 = \mathrm{GL}(8,\mathbb{F})/\mathbb{Z}_3$ [11, Proposition 3.2], $\mathfrak{g}_{-3} = \mathbb{F}^{8*}$, $\mathfrak{g}_{-2} = \Lambda^2\mathbb{F}^8$, $\mathfrak{g}_{-1} = \Lambda^3\mathbb{F}^{8*}$, $\mathfrak{g}_0 = \mathfrak{gl}(8,\mathbb{F})$, $\mathfrak{g}_1 = \Lambda^3\mathbb{F}^8$, $\mathfrak{g}_2 = \Lambda^2\mathbb{F}^{8*}$, $\mathfrak{g}_3 = \mathbb{F}^8$.

Since there is only finite number of $\mathrm{GL}(n,\mathbb{F})$-orbits in $\mathfrak{g}_1$, any element in $\mathfrak{g}_1$ is nilpotent. To study nilpotent elements in $\mathfrak{g}_1 = \Lambda^3\mathbb{R}^8$, as Vinberg-Elashvili did for complex nilpotent 3-vectors on $\Lambda^3\mathbb{C}^9$, Djokovic used a real version of Jacobson-Morozov-Vinberg's theorem that associates with each nilpotent element $e \in \mathfrak{g}_1$ a semisimple element $h(e) \in \mathfrak{g}_0$ and a nilpotent element $f \in \mathfrak{g}_{-1}$ that satisfy the following condition [11, Lemma 6.1]

$$[h, e] = 2e,\ [h, f] = -2f,\ [e, f] = h. \tag{7}$$

Element h is defined by e uniquely up to conjugation and $h = h(e)$ is called *a characteristic* of e [11, Lemma 6.2], see also [34, Theorem 2.1] for a general statement. Given e and h, element f is defined uniquely. A triple (h, e, f) in (7) is called *an $\mathfrak{sl}_2$-triple*, which we shall denote by $\mathfrak{sl}_2(e)$. With help of $\mathfrak{sl}_2(e)$-triples Djokovic classified real forms of nilpotent orbits $\mathrm{GL}(8,\mathbb{C}) \cdot e$, where $e \in \mathfrak{g}_1 = \Lambda^3\mathbb{C}^8$, as follows. Denote by $Z_{\mathrm{GL}(8,\mathbb{C})}(\mathfrak{sl}_2(e))$ the centralizer of $\mathfrak{sl}_2(e)$ in $\mathrm{GL}(8,\mathbb{C})$. Let $\Phi = \mathbb{Z}_2$ be the Galois group of the field extension of $\mathbb{C}$ over $\mathbb{R}$. Then Djokovic proved that there is a bijection from the Galois cohomology $(\Phi, Z_{\mathrm{GL}(8,\mathbb{C})}(\mathfrak{sl}_2(e)))$ to the set of $\mathrm{GL}(8,\mathbb{R})$-orbits contained in $\mathrm{GL}(8,\mathbb{C}) \cdot e$ [11, Theorem 8.2]. A similar argument has been first used by Revoy [51] and later by Midoune and Noui for classification of alternating forms in dimension 8 over a finite field [43]. Recall that classification of $\mathrm{GL}(8,\mathbb{C})$-orbits has been obtained by Gurevich and later this classification is also re-obtained by Vinberg-Elashvili in their classification of 3-vectors on $\mathbb{C}^9$. There are altogether 23 $\mathrm{GL}(8,\mathbb{C})$-orbits on $\Lambda^3\mathbb{C}^8$. In [11] Djokovic gave another proof of this classification using the $\mathbb{Z}$-graded Lie algebra $\mathfrak{e}_8$ in (6). Finally Djokovic computed the related Galois cohomology to obtain the number of real forms of each complex orbit and he also found a canonical representation of each $\mathrm{GL}(8,\mathbb{R})$-orbit on $\Lambda^3\mathbb{R}^8$. The space $\Lambda^3\mathbb{R}^8$ decomposes into 35 $\mathrm{GL}(8,\mathbb{R})$-orbits.

Remark 4. *Since there is only finite number of* $\mathrm{GL}(8,\mathbb{R})$*-orbits on* $\Lambda^3\mathbb{R}^{8*}$*, there exists* $\varphi \in \Lambda^3\mathbb{R}^{8*}$ *such that the orbit* $\mathrm{GL}(8,\mathbb{R}) \cdot \varphi$ *is open in* $\Lambda^3\mathbb{R}^{8*}$*. Such a* 3*-form* φ *is called stable. Clearly any stable* 3*-form* φ *is nondegenerate, i.e.,* $\mathrm{rk}\varphi = 8$*. In general, a* k*-form* φ *on* $\mathbb{R}^n$ *is called stable, if the orbit* $\mathrm{GL}(n,\mathbb{R}) \cdot \varphi$ *is open in* $\Lambda^k\mathbb{R}^n$*. Clearly any symplectic form is stable. It is not hard to see that if* $\varphi \in \Lambda^k\mathbb{R}^n$ *is open, and* $k \geqslant 2$*, then either* $k = 3$ *and* $n = 5, 6, 7, 8$*, or* $k = 4$ *and*

$n = 6, 7$, or $k = 5$ and $n = 8$. Stable forms on $\mathbb{R}^8$ have been studied in deep by Hitchin [26], Witt [68] and later by Lê-Panak-Vanžura in [38], where they classified all stable forms on $\mathbb{R}^n$ (they proved that stable k-forms exist on $\mathbb{R}^n$ only in dimensions $n = 6, 7, 8$ if $3 \leqslant k \leqslant n - k$), and determined their stabilizer groups [38, Theorem 4.1].

Remark 5. *Djokovic's classification of 3-vectors on $\mathbb{R}^8$ contains the classification of 3-vectors on $\mathbb{R}^6$ and the classification of 3-vectors on $\mathbb{R}^7$ by Theorem* 1. The classification of 3-forms on $\mathbb{R}^7$ has been first obtained by Westwick [67] by adhoc method. There are 8 equivalence classes of multisymplectic 3-forms on $\mathbb{R}^7$, which are the real forms of 5 equivalent classes of multisymplectic 3-forms on $\mathbb{C}^7$, and there are 6 equivalence classes of 3-forms on $\mathbb{R}^6$, which are the real forms of 5 equivalence classes of 3-forms on $\mathbb{C}^6$. The stabilizer of 3-forms in $\mathbb{R}^6$ has been determined in [25] and the stabilizer of multisymplectic 3-forms in $\mathbb{R}^7$ has been defined in [6]. The stabilizer of 3-forms on $\mathbb{F}^7$ has been described by Cohen-Helminck in [8, Theorem 2.1] for any algebraically closed field $\mathbb{F}$.

Remark 6. *There are 21 equivalence classes of multisymplectic 3-forms on $\mathbb{R}^8$ which are the real forms of 13 equivalence classes of multisymplectic 3-forms on $\mathbb{C}^8$ [11, §9]. A complete list of the stabilizer groups $\mathrm{St}_{\mathrm{GL}(8,\mathbb{R})}(\varphi)$ of each multi-symplectic 3-form φ on $\mathbb{R}^8$ has not been obtained till now according to our knowledge. The stabilizer $\mathrm{St}_{\mathrm{GL}(8,\mathbb{C})}(\varphi)$ has been obtained by Midoune in his PhD Thesis [42], see also [43]. In [11] Djokovic computed the dimension of each $\mathrm{GL}(8,\mathbb{R})$-orbit in $\Lambda^3\mathbb{R}^8$ and the centralizer $Z_{\mathrm{GL}(8,\mathbb{R})}(\mathfrak{sl}_2(e))$ for each nilpotent element $e \in \mathfrak{e}_{8(8)}$. It follows that the stabilizer algebra $Z_{\mathfrak{gl}(8,\mathbb{R})}(\varphi)$ of 3-forms $\varphi \in \Lambda^3\mathbb{R}^8$ forms a complete system of invariants of the $\mathrm{GL}(8,\mathbb{R})$-action on $\Lambda^3\mathbb{R}^8$. In Proposition 4 below we show that the stabilizer of any multisymplectic 3-form φ on $\mathbb{R}^8$ is not connected.*

Proposition 4. *For any multisymplectic 3-form $\varphi \in \Lambda^3\mathbb{R}^{8*}$ we have $\mathrm{St}_{\mathrm{GL}(8,\mathbb{R})}(\varphi) \cap \mathrm{GL}^-(8,\mathbb{R}) \neq \emptyset$. Hence the $\mathrm{GL}(8,\mathbb{R})$-orbit of any 3-form on $\mathbb{R}^8$ is connected.*

Proof. For each equivalence class of a 3-form φ of rank 8 we choose a canonical element φ_0 in the Djokovic's list [11, p. 36-37]. Then we find an element $g \in \mathrm{St}_{\mathrm{GL}(8,\mathbb{R})}(\varphi_0) \cap \mathrm{GL}^-(8,\mathbb{R})$. Hence the $\mathrm{GL}(n,\mathbb{R})$-orbit of each multisymplectic 3-form on $\mathbb{R}^8$ is connected. If φ is not multisymplectic, the orbit $\mathrm{GL}(8,\mathbb{R}) \cdot \varphi$ is connected by Proposition 2. This completes the proof of Proposition 4. □

Proposition 4 and Proposition 2 imply immediately the following

Corollary 1. *(cf. [53, Proposition 4.1]) The Poincaré map P_Ω induces an isomorphism between $\mathrm{GL}(8,\mathbb{R})$-orbits on $\Lambda^3\mathbb{R}^8$ and $\mathrm{GL}(8,\mathbb{R})$-orbit on $\Lambda^5\mathbb{R}^{8*}$. Each $\mathrm{GL}(8,\mathbb{R})$-orbit on $\Lambda^5\mathbb{R}^8$ is connected.*

2.5. Classification of 4-forms on $\mathbb{R}^8$

Classification of 4-forms on $\mathbb{C}^8$, whose equivalence is defined via the standard action of $\mathrm{SL}(8,\mathbb{C})$, has been given by Antonyan [1], following the scheme proposed by Vinberg-Elashvili for the classification of 3-vectors on $\mathbb{C}^9$. In [34] Lê proposed a scheme of classification of 4-forms on $\mathbb{R}^8$ as application of her study of the adjoint orbits in $\mathbb{Z}_m$-graded real semisimple Lie algebras. In this subsection we outline Antonyan's method and Lê's method.

Let $\mathbb{F} = \mathbb{C}$ (resp. $\mathbb{R}$). Denote by $\mathfrak{g}$ the exceptional complex simple Lie algebra $\mathfrak{e}_7$ (rep. $\mathfrak{e}_{7(7)}$ - the split form of $\mathfrak{e}_7$). The starting point of Antonyan's work on the classification on 4-vectors on $\mathbb{C}^8$ (resp. the starting point of Lê's scheme of classification of 4-forms on $\mathbb{R}^8$) is the following observation, cf. (3), (5). The standard $\mathrm{GL}(8,\mathbb{F})$-action on $\Lambda^4\mathbb{F}^8$ is equivalent to the action of the θ-group of the $\mathbb{Z}_2$-graded simple Lie algebra

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{8}$$

on its homogeneous component $\mathfrak{g}_1$, which is isomorphic to $\Lambda^4\mathbb{F}^8$. Here $\mathfrak{g}_0 = \mathfrak{sl}(8, \mathbb{F})$.

Let us describe the components $\mathfrak{g}_0$ and $\mathfrak{g}_1$ in (8) for the case $\mathbb{F} = \mathbb{C}$ using the root decomposition of $\mathfrak{e}_7$. Recall that $\mathfrak{e}_7$ has the following root system:

$$\Sigma = \{\varepsilon_i - \varepsilon_j, \varepsilon_p + \varepsilon_q + \varepsilon_r + \varepsilon_s, | \, i \neq j, (p, q, r, s \text{ distinct}), \sum_{i=1}^{8} \varepsilon_i = 0\}.$$

By Remark 1, the $\mathbb{Z}_2$-grading on $\mathfrak{e}_7$ is defined uniquely by an involution $\theta^{\mathbb{C}}$ of $\mathfrak{e}_7$. In terms of the Chevalley system of $\mathfrak{e}_7$, see Remark 2, the involution $\theta^{\mathbb{C}}$ is defined as follows:

$$\theta^{\mathbb{C}}{}_{|\mathfrak{h}_0} = Id,$$

$$\theta^{\mathbb{C}}(E_\alpha) = E_\alpha, \text{ if } \alpha = \varepsilon_i - \varepsilon_j,$$
$$\theta^{\mathbb{C}}(E_\alpha) = -E_\alpha, \text{ if } \alpha = \varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon_l.$$

Note that $\theta := \theta^{\mathbb{C}}_{|\mathfrak{g}=\mathfrak{e}_{7(7)}}$ is an involution of $e_{7(7)}$ and it defines the induced $\mathbb{Z}_2$-gradation from $\mathfrak{e}_7$ on $\mathfrak{e}_{7(7)}$.

Following the Vinberg-Eliashivili scheme of the classification of 3-vectors on $\mathbb{C}^9$, Antonyan classified $\mathrm{SL}(8, \mathbb{C})$-equivalent 4-vectors on $\mathbb{C}^8$ by using the Jordan decomposition (Proposition 3). First he classified all semisimple 4-vectors on $\mathbb{C}^8$ using Vinberg's theory on finite automorphisms of semisimple algebraic groups [61], which has been employed by Vinberg-Elashvili for the classification of semisimple 3-vectors as we mentioned above. Next we include each semisimple element $x \in \mathfrak{g}_1$ of the $\mathbb{Z}_2$-graded complex Lie algebra $\mathfrak{e}_7$ into a Cartan subalgebra of $\mathfrak{g}_1$, which is defined as a maximal subspace in $\mathfrak{g}_1$ consisting of commuting semisimple elements [63] (this definition is also applied to real or complex $\mathbb{Z}_m$-graded semisimple Lie algebras $\mathfrak{g}$). If $\mathfrak{g}$ is a complex $\mathbb{Z}_m$-graded sesmisimple Lie algebra, then all the (complex) Cartan subalgebras in $\mathfrak{g}_1$ are conjugate under the action of the adjoint group $G_0^{\mathbb{C}}$. To reduce the classification of semisimple elements in $\mathfrak{g}_1$ further we introduce the notion of the Weyl group $W(\mathfrak{g}, \mathcal{C})$ of a complex $\mathbb{Z}_m$-graded semisimple Lie algebra $\mathfrak{g}$ w.r.t. to a Cartan subalgebra $\mathcal{C} \subset \mathfrak{g}_1$ as follows. Let $G^{\mathbb{C}}$ be the connected semisimple Lie algebra having the Lie algebra $\mathfrak{g}$ and $G_0^{\mathbb{C}}$ the Lie subgroup of the $G^{\mathbb{C}}$ having the Lie algebra $\mathfrak{g}_0$. We define

$$N_0(\mathcal{C}) := \{g \in G_0 | \, \forall x \in \mathcal{C} \; g(x) \in \mathcal{C}\},$$

$$Z_0(\mathcal{C}) := \{g \in G_0 | \, \forall x \in \mathcal{C} \; g(x) = x\}.$$

Then $W(\mathfrak{g}, \mathcal{C}) := N_0(\mathcal{C})/Z_0(\mathcal{C})$. The Weyl group $W(\mathfrak{g}, \mathcal{C})$ is finite, moreover $W(\mathfrak{g}, \mathcal{C})$ is generated by complex reflections, which implies that the algebra of $W(\mathfrak{g}, \mathcal{C})$-invariants on $\mathcal{C}$ is free [61]. Furthermore, two semisimple elements in $\mathcal{C}$ belong to the same $G_0^{\mathbb{C}}$-orbit if and only if they are in the same orbit of the $W(\mathfrak{g}, \mathcal{C})$-action on $\mathcal{C}$. Antonyan showed that the Weyl group $W(\mathfrak{e}_7, \mathcal{C})$ has order 2903040 and the generic semisimple element has trivial stabilizer. He also found a basis of a Cartan algebra $\mathcal{C} \subset \mathfrak{g}_1$, which is also a Cartan subalgebra of the Lie algebra $\mathfrak{e}_7$. Thus the set of $\mathrm{SL}(8, \mathbb{C})$-equivalent semisimple 4-vectors on $\mathbb{C}^8$ has dimension 7. This set is divided into 32 families depending on the type of the stabilizer of the action of the Weyl group $W(\mathfrak{e}_7, \mathcal{C})$ on the Cartan algebra $\mathcal{C}$. For the classification of nilpotent elements and mixed 4-vectors on $\mathbb{C}^8$ Antonyan used the Vinberg method of support [64].

Lê suggested the following scheme of classification of the $\mathrm{SL}(8, \mathbb{R})$-orbits on $\Lambda^4\mathbb{R}^8$ [34]. Observe that we also have the Jordan decomposition of each element in $\Lambda^4\mathbb{R}^8$ into a sum of a semisimple element and a nilpotent element [34, Theorem 2.1], as in Proposition 3. First, we classify semisimple elements, using the fact that every Cartan subspace $\mathcal{C} \subset \mathfrak{g}_1$ is conjugated to a standard Cartan subspace $\mathcal{C}_0$ that is invariant under the action of a Cartan involution $\tau_{\mathfrak{u}}$ of the $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{e}_{7(7)}$ [47]. The set of all standard Cartan subspaces $\mathcal{C}_0 \subset \mathfrak{g}_1 \subset \mathfrak{g} = \mathfrak{e}_{7(7)}$, and more generally, the set of all standard Cartan subspaces $\mathcal{C} \subset \mathfrak{g}_1$ in any $\mathbb{Z}_2$-graded real semisimple Lie algebra $\mathfrak{g}$, has been classified by Matsuki and Oshima in [47]. Lê decomposed each semisimple element into a sum of an elliptic semisimple element, i.e., a semisimple element whose adjoint action on $\mathfrak{g}_{\otimes\mathbb{C}} = \mathfrak{e}_7$ has purely imaginary eigenvalues, and a real semisimple element, i.e., a semisimple element whose adjoint action on $\mathfrak{g}_{\otimes\mathbb{C}} = \mathfrak{e}_7$ has real eigenvalues, cf. [52] for a similar decomposition of semisimple elements in a real sesimsimple Lie algebra. The classification of real semisimple elements and commuting elliptic semisimple elements in $\mathcal{C}_0 \subset \mathfrak{g}_1$ is then reduced to the classification of the orbits of the Weyl groups of associated $\mathbb{Z}_2$-graded symmetric Lie algebras on their Cartan subalgebras [34, Corollary 5.3]. As in [65] and [1], the classification of mixed 4-vectors on $\mathbb{R}^8$ is reduced to the classification of their semisimple parts and the corresponding nilpotent parts. The classification of nilpotent parts can be done using algorithms in real algebraic geometry based on Lê's theory of nilpotent orbits in graded semisimple Lie algebras [34], that develops further Vinberg's method of support also called carrier algebra. In [12] Dietrich-Faccin-de Graaf developed Vinberg's method further and applied their method to classification of the orbits of homogeneous nilpotent elements in certain graded real semisimple Lie algebras. In particular, they have a new proof for Djokovic's classification of 3-vectors on $\mathbb{R}^8$.

Remark 7. *(1) The method of θ-group has been extended by Antonyan and Elashvili for classifications of spinors in dimension 16 [2].*

(2) Many results of classifications of k-vectors over the fields $\mathbb{R}$ and $\mathbb{C}$ have their analogues over other fields $\mathbb{F}$ and their closures $\overline{\mathbb{F}}$ [43]. Over the field $\mathbb{F} = \mathbb{Z}_2$ the classification of 3-vectors in $\mathbb{F}^n$ is related to some open problems in the theory of self-dual codes [49]. Till now there is no classification of 3-vectors in $\mathbb{F}^n$ if $n \geqslant 9$ and $\mathbb{F} \neq \mathbb{C}$.

3. Geometry defined by differential forms

In this section we briefly discuss several results and open questions on the existence of differential k-forms of given type on a smooth manifold, where $k = 2, 3, 4$.

• Assume that $k = 2$ and φ is a closed 2-form with constant rank on M^n, then φ is called *a pre-symplectic form* [60]. Till now there is no general necessary and sufficient condition for the existence of a pre-symplectic form φ on a manifold M^n except the case that φ is a symplectic form. Necessary conditions for the existence of a symplectic form φ on M^{2n} are the existence of an almost complex structure on M^{2n} and if M^{2n} is closed, the existence of a cohomology class $a \in H^2(M^{2n}; \mathbb{R})$ with $a^n > 0$. If M^{2n} is open, a theorem of Gromov [18, 19] asserts that the existence of an almost complex structure is also sufficient, his argument has been generalized in [13] and used in the proof of Theorem 4(2) below. Taubes using Seiberg-Witten theory proved that there exist a closed 4-manifold M^4 admitting an almost complex structure and $a \in H^2(M, \mathbb{R})$ such that $a^2 \neq 0$ but M^4 has no symplectic structure [59]. Note that for any symplectic form ω on M^{2n} there exists uniquely up to homotopy an almost complex structure J on M^{2n} that is compatible with ω, i.e., $g(X, Y) := \omega(X, JY)$ is a Riemannian metric on M^{2n}. Connolly-Lê-Ono using the Seiberg-Witten theory showed that a half of all homotopy classes of almost complex structures on a certain class of oriented compact 4-manifolds is not compatible with any symplectic structure [9].

• Manifolds M^{2n} endowed with a nondegenerate conformally closed 2-form ω, i.e., $d\omega = \theta \wedge \omega$ for some closed 1-form θ on M^{2n}, are called *conformally symplectic manifolds*. A necessary condition for the existence of nondegenerate 2-form ω on M^{2n} is the existence of an almost complex structure on TM^{2n}, which is equivalent to the existence of a section J of the associated bundle $\mathrm{SO}(2n)/\mathrm{U}(n)$, see [56] where a necessary condition for the existence of a section J has been determined in terms of the Whitney-Stiefel characteristic classes. We don't have necessary and sufficient conditions for the existence of a general conformally symplectic form on M^{2n}, except the existence of an almost complex structure on M^{2n}. In [39] Lê-Vanžura proposed new cohomology theories of locally conformal symplectic manifolds.

• Assume that $k = 3$ and φ is a stable 3-form on M^8. In [46] Noui and Revoy proved that the Lie algebra of the stabilizer of φ is a real form of the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$. Hence stable 3-forms on $\mathbb{R}^8$ are equivalent to the Cartan 3-forms on the real forms $\mathfrak{sl}(3, \mathbb{R})$, $\mathfrak{s}u(1, 2)$ and $\mathfrak{s}u(3)$ of the complex Lie algebra $\mathfrak{sl}(3, \mathbb{C})$. Later in [38] Lê-Panak-Vanžura reproved the Noui-Revoy result by associating to each 3-form on $\mathbb{R}^8$ various bilinear forms, which are invariants of the $\mathrm{GL}(8, \mathbb{R})$-action on $\Lambda^3\mathbb{R}^{8*}$, and studied properties of these forms. They computed the stabilizer group of a stable form $\varphi \in \Lambda^3\mathbb{R}^{8*}$ and found a necessary and sufficient condition for a closed orientable manifold M^8 to admit a stable 3-form [38, Proposition 7.1]. In [36] Lê initiated the study of geometry and topology of manifolds admitting a Cartan 3-form associated with a simple compact Lie algebra.

• Necessary and sufficient conditions for a closed connected 7-manifold M^7 to admit a multisymplectic 3-form has been determined in [54], see also Appendix 4 below. There are two equivalence classes of stable 3-forms on $\mathbb{R}^7$ with the stablizer groups G_2 and $\tilde{G}_2$ respectively. Since G_2 and $\tilde{G}_2$ are exceptional Riemannian and pseudo Riemannian holonomy groups, manifolds M^7 admitting stable 3-form of G_2-type (resp. of $\tilde{G}_2$-type) are in focus of research in Riemannian geometry (respectively in pseudo Riemannian geometry) [30], [35], [32]. As we have mentioned, the study of geometries of stable forms in dimension 6,7, 8 have been initiated by Hitchin [25, 26].

• It is worth noting that the algebra of parallel forms on a quaternion Kähler manifold is generated by the quaternionic 4-form, the algebra of parallel forms on a Spin(7)-manifold is generated by the self-dual Cayley 4-form. Riemannian manifolds admitting parallel 2-forms of maximal rank are Kähler manifolds, which are the most studied subjects in geometry, in particular in the theory of minimal submanifolds, see e.g., [37].

4. Manifolds admitting a $\tilde{G}_2$-structure

In 2000 Hitchin initiated the study of geometries defined by differential forms [25], and subsequently in [26] he initiated the study of geometries defined by stable forms. The latter geometries have been investigated further in [68], [38]. A necessary and sufficient condition for a manifold M to admit a stable form φ of G_2-type, i.e., the stabilizer of φ is isomorphic to the group G_2, has been found by Gray [20]. In this Appendix we state and prove a necessary and sufficient condition for a manifold M to admit a stable form φ of $\tilde{G}_2$-type. We recall that a 3-form φ on $\mathbb{R}^7$ is called of $\tilde{G}_2$-type, if it lies on the $\mathrm{GL}(\mathbb{R}^7)$-orbit of a 3-form

$$\varphi_0 = \theta_1 \wedge \theta_2 \wedge \theta_3 + \alpha_1 \wedge \theta_1 + \alpha_2 \wedge \theta_2 + \alpha_3 \wedge \theta_3.$$

Here α_1, α_2 are 2-forms on $\mathbb{R}^7$ which can be written as

$$\alpha_1 = y_1 \wedge y_2 + y_3 \wedge y_4,\ \alpha_2 = y_1 \wedge y_3 - y_2 \wedge y_4,\ \alpha_3 = y_1 \wedge y_4 + y_2 \wedge y_3$$

and $(\theta_1, \theta_2, \theta_3, y_1, y_2, y_3, y_4)$ is an oriented basis of $\mathbb{R}^{7*}$.

Bryant showed that $\mathrm{St}_{\mathrm{GL}(7,\mathbb{R})}(\varphi_0) = \tilde{G}_2$ [7]. He also proved that $\tilde{G}_2$ coincides with the automorphism group of the split octonians [7].

Theorem 4. *(1) Suppose that M^7 is a compact 7-manifold. Then M^7 admits a 3-form of $\tilde{G}_2$-type, if and only if M^7 is orientable and spinnable. Equivalently the first and second Stiefel-Whitney classes of M^7 vanish.*

(2) Suppose that M^7 is an open manifold which admits an embedding to a compact orientable and spinnable 7-manifold. Then M^7 admits a closed 3-form φ of $\tilde{G}_2$-type.

Proof. First we recall that the maximal compact Lie subgroup of $\tilde{G}_2$ is SO(4). This follows from the Cartan theory on symmetric spaces. We refer to [27, p. 115] for an explicit embedding of SO(4) into G_2. The reader can also check that the image of this group is also a subgroup of $\tilde{G}_2 \subset \mathrm{GL}(\mathbb{R}^7)$. We shall denote this image by $SO(4)_{3,4}$.

Now assume that a smooth manifold M^7 admits a $\tilde{G}_2$-structure. Then it must be orientable and spinnable, since the maximal compact Lie subgroup $\mathrm{SO}(4)_{3,4}$ of G_2 is also a compact subgroup of the group G_2.

Lemma 2. *Assume that M^7 is compact, orientable and spinnable. Then M^7 admits a $\tilde{G}_2$-structure.*

Proof. Since M^7 is compact, orientable and spinable, M^7 admits a SU(2)-structure [16]. Since $SU(2)$ is a subgroup of $SO(4)_{3,4}$, M^7 admits a $SO(4)_{3,4}$-structure. Hence M^7 admits a $\tilde{G}_2$-structure. □

This completes the proof of the first assertion of Theorem 4.

Let us prove the last statement of Theorem 4. Assume that M^7 is a smooth open manifold which admits an embedding into a compact orientable and spinnable 7-manifold. Taking into account the first assertion of Theorem 4, there exists a 3-form φ on M^7 of $\tilde{G}_2$-type. We shall use the following theorem due to Eliashberg-Mishachev to deform the 3-form φ to a closed 3-form $\bar{\varphi}$ of $\tilde{G}_2$-type on M^7.

Let M be a smooth manifold and $a \in H^p(M, \mathbb{R})$. For a subspace $\mathcal{R} \subset \Lambda^p M$ we denote by $Clo_a\mathcal{R}$ the subspace of the space $\Gamma(M, \mathcal{R})$ of smooth sections $M \to \mathcal{R}$ that consists of closed p-forms $\omega \in \Gamma(M, \mathcal{R}) \subset \Omega^p(M)$ such that $[\omega] = a \in H^p(M, \mathbb{R})$. Denote by $\mathrm{Diff}(M)$ the diffeomorphism group of M.

Proposition 5 (Eliashberg-Mishashev Theorem). *([13, 10.2.1]) Let M be an open manifold, $a \in H^p(M, \mathbb{R})$ and $\mathcal{R} \subset \Lambda^p M$ an open* Diff(M)*-invariant subset. Then the inclusion*

$$Clo_a\mathcal{R} \hookrightarrow \Gamma(M, \mathcal{R})$$

is a homotopy equivalence. In particular,

- any p-form $\omega \in \Gamma(M, \mathcal{R})$ is homotopic in $\mathcal{R}$ to a closed form $\bar{\omega}$;

- any homotopy $\omega_t \in \Gamma(M, \mathcal{R})$ of p-forms which connects two closed forms ω_0, ω_1 such that $[\omega_0] = [\omega_1] = a \in H^p(M, \mathbb{R})$ can be deformed in $\mathcal{R}$ into a homotopy of closed forms $\bar{\omega}_t$ connecting ω_0 and ω_1 such that $[\omega_t] = a$ for all t.

Let $\mathcal{R}$ be the space of all 3-forms of $\tilde{G}_2$-type on M^7. Clearly this space is an open $\mathrm{Diff}(M^7)$-invariant subset of $\Lambda^3 M^7$. Now we apply the Eliashberg-Mishashev theorem to the 3-form φ^3 of $\tilde{G}_2$-type whose existence has been proved above. This completes the proof of Theorem 4. □

5. Classification of orbits over a nonclosed field of characteristic 0

by Mikhail Borovoi

We consider a linear algebraic group G with group of k-points $G(k)$ over an algebraically closed field k of characteristic 0. Assume that G acts on a k-variety X with set of k-points $X(k)$, and assume that we know the classification of $G(k)$-orbits in $X(k)$, e.g., $k = \mathbb{C}$, $G = \mathrm{GL}(9, \mathbb{C})$, $X = \Lambda^3\mathbb{C}^9$. Let k_0 be a subfield of k such that k is an algebraic closure of k_0. We write $\Gamma = \mathrm{Gal}(k/k_0)$ for the Galois group of the extension k over k_0. If $k_0 = \mathbb{R}$, then $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$, where γ is the complex conjugation. Assume that we have compatible k_0-forms G_0 of G and X_0 of X. We wish to classify $G_0(k_0)$-orbits in $X_0(k_0)$. We start with one G-orbit Y in X. We check whether Y is Γ-stable. If not, then Y has no k_0-points. Assume that Y is Γ-stable. Then the Γ-action on Y defines a k_0-model Y_0 of Y. Now G_0 acts on Y_0 over k_0. We say that Y_0 is (a twisted form of) a homogeneous space of G_0. We ask

(1) whether Y_0 has k_0-points;

(2) if the answer to (1) is positive, we wish to classify $G(k_0)$-orbits in $Y_0(k_0)$.

Question (1) is treated in [5]. Assume that for our Y, the answer to question (1) is Yes. Let $y_0 \in Y_0(k_0)$, and let $H_0 = \mathrm{St}_{G_0}(y_0)$. Then we may write $Y_0 = G_0/H_0$. The Galois group $\Gamma = \mathrm{Gal}(k/k_0)$ acts compatibly on $G_0(k) = G(k)$, $H_0(k) = H(k)$, and $Y_0(k) = Y(k) = G(k)/H(k)$.

Theorem 5 ([58], Section I.5.4, Corollary 1 of Proposition 36). *There is a canonical bijection between the set of orbits $Y_0(k_0)/G_0(k_0)$ and the kernel $\ker[H^1(k_0, H_0) \to H^1(k_0, G_0)]$.*

Here $H^1(k_0, H_0) := H^1(\Gamma, H_0(k))$.

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