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Распознавание и табулирование 3-многообразий до сложности 13[1]

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Аннотация

В работе кратко описывается полная таблица ориентируемых замкнутых неприводимых 3-многообразий сложности $\leqslant 13$, методы ее построения и проверки, кроме того, формулируется ряд гипотез касающихся роста числа многообразий различных типов. В приложении дается сжатое объяснение использованных понятий.

Ключевые слова: трехмерное многобразие, сложность многообразия, специальные спайны , табулирование трехмерных многообразий.

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Recognition and tabulation of 3-manifolds up to complexity 13

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Abstract

We describe in breaf the complete table of closed irreducible orientable 3-manifolds of complexity $\leqslant 13$, and method of its creation and verification. Also we formulate a conjectures concerning the growth of the number of some kinds of manifolds. The appendix contains a short explanation of used terminology.

Keywords: three-dimensional manifolds, complexity of manifold, special spines, tabulation of three-dimensional manifolds.

Bibliography: 25 titles.

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Dedicated to Anatoly Timofeevich Fomenko on his 75th birthday.

1. Statement and history of the problem

By *manifold*, we understand a compact connected 3-dimensional manifold with boundary (the boundary may be empty, in this case we speak about *closed* manifolds). Our project is related to the following goals.

(A) *Algorithmic recognition of manifolds.* Given two manifolds (given by combinatorial data; say by decomposition into simplexes), to understand whether they are homeomorphic[2].

(B) *Efficient tabulation of manifolds.* To create a table (=list equipped with additional information such as Thurston geometry type, some algebraic-topological information and the values of Turaev-Viro invariants [23]) of all manifolds up to a certain complexity[3] c together with an effective comparing method of any two manifolds of complexity $\leqslant c$.

[2]Recall that in dimension 3, the topological, smooth and PL categories are equivalent
[3]we will recall the definition of complexity in the appendix below; complexity of a manifold is a non-negative integer and for each $c \geq 0$ the number of closed irreducible manifolds of complexity c is finite

Of course, both goals are fundamental, they stay in the center of modern three-dimensional topology and were studied from different perspectives. Clearly, these two goals are closely related. Indeed, having an effective and fast algorithm for the goal (A), one can list all manifolds up to certain complexity allowing having homeomorphic manifolds in the list (there are naive and more sophisticated ways to do it), compare the manifolds from the list pairwisely with the help of the algorithm and kill the duplicates. A natural modification of this naive procedure will also label each manifold with its complexity.

In the other direction, the goal (B) explicitly includes the goal (A) restricted to the manifolds of complexity $\leqslant c$.

In theory, the goal (A) can be considered to be solved: there exists an algorithm that, given two manifolds, decides whether they are homeomorphic. It is explained in the book [17, §6]. The goal (A) was formulated at least in 1962 by W. Haken [5]. Nontrivial steps in the solution of this problem are due in particular to G. Hemion [6] and as a consequence the goal (A) was announced to be solved [8, 26]. Later, a crucial gap in the proof of [6] was found, see the discussion in [17, §6.1]. The problem was finally solved in [17, Theorem 6.6.1].

Unfortunately, using this algorithm to compare two even relatively simple manifolds would exceed the abilities of the modern computers. In simple words, in theory the algorithm exists, in practice it does not help, i.e., the situation in general recognition problem is similar to the situation in its following important special case. Given a closed manifold, how to understand whether it is homeomorphic to the sphere. There is an algorithm of doing it, it is based on the ideas/works of A. Thompson [20], and is is explained in details in [14]. The algorithm, at least in its initial version, is so slow though that there is no sense to realize it on the computer. From the other hand now, because of Perelmann's proof of the Poincare conjecture, there exists a much faster algorithm that answers whether a manifold is the sphere: one needs to check whether the fundamental group of the manifold is trivial, and the operation is a relatively "cheap" from the calculation point of view.

The results which we report in this note are related to the goal (B). In the last 15 years [16, §2] we (together with other mathematicians, see the list of coauthors in the references below) actively worked on the table of manifolds. The new result of this note is the table of all closed orientable irreducible manifolds of complexity 13. The result related to complexity $\leqslant$ 12 is published, for example, in [25].

2. On the table of 3-manifolds of complexity $\leqslant 13$

2.1. The exact numbers of pairwise nonhomeomorphic closed orientable irreducible 3-manifolds of complexity $\leqslant 13$

Here $n(c)$ denotes the number of closed orientable irreducible 3-manifolds of complexity c.

c	0	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
$N(c)$	3	2	4	7	14	31	74	175	436	1154	3078	8421	23448	66197	103041

The definition of complexity of a manifold is given in appendix below. Additionally we list there the manifolds of complexity 0.

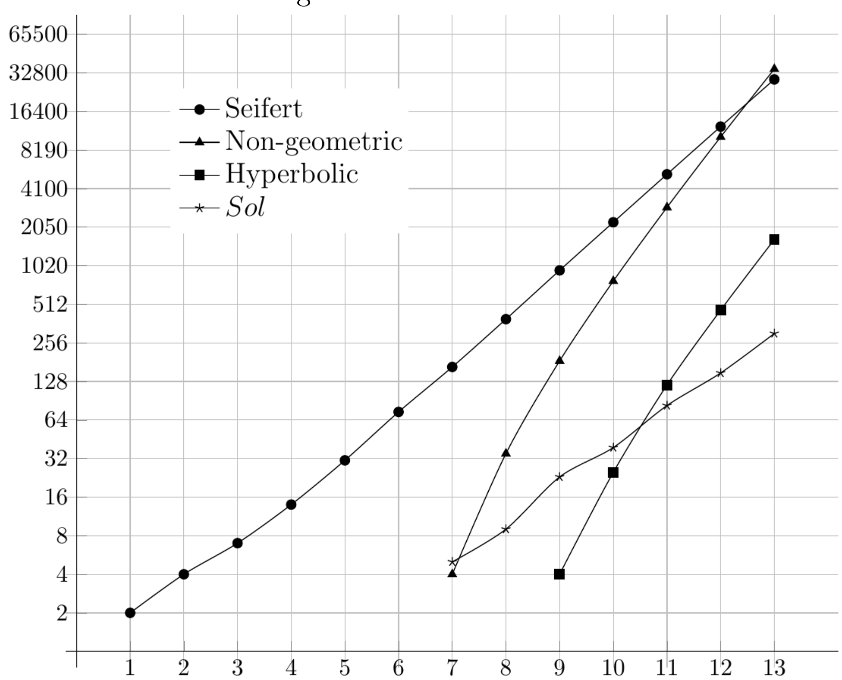
2.2. An information about manifolds of complexity $\leqslant 13$ from the point of view of Thurston's classification

Recall that Thurston proved that there are 8 geometries: $E^3, S^3, S^2\times R, H^2\times R, \widetilde{SL_2R}, Nil, Sol$ and H^3, [22]. A 3-manifold allows not more than 1 of them. Following table gives an information about manifolds of complexity $\leqslant 13$ according to the classification of Thurston.

c	$S^2 \times R$	E^3	$H^2 \times R$	S^3	Nil	$\widetilde{SL_2R}$	Sol	H^3	Non-geometric
0	0	0	0	3	0	0	0	0	0
1	0	0	0	2	0	0	0	0	0
2	0	0	0	4	0	0	0	0	0
3	0	0	0	7	0	0	0	0	0
4	0	0	0	14	0	0	0	0	0
5	0	0	0	31	0	0	0	0	0
6	0	6	0	61	7	0	0	0	0
7	0	0	0	117	10	39	5	0	4
8	0	0	2	214	14	162	9	0	35
9	0	0	0	414	15	513	23	4	185
10	0	0	8	798	15	1416	39	25	777
11	0	0	4	1582	15	3696	83	120	2921
12	0	0	24	3118	15	9324	149	461	10357
13	0	0	9	6222	15	22916	303	1641	35091

The first column of the table above contains zeros only because, as it well known, there exist exactly 2 closed orientable 3-manifolds having $S^2 \times R$ geometry and both these manifolds are reducible. Therefore, they do not involve in our table.

Below we visualize the same information as in the table above. The diagram contains not 9 but 4 graphs. Here we use the fact that a manifold having a geometry of the first 6 types is a seifert manifold. All values are shown on a logarithmic scale.



2.3. An information about manifolds of complexity $\leqslant 13$ from the point of view of another classification

Now we consider manifolds in the table from the other point of view, we partition the manifolds into 4 types depending on the type (seifert or hyperbolic) of blocks which are involved in the JSJ-decomposition of a manifold under consideration:

1. S — seifert manifolds,

2. h — hyperbolic manifolds,

3. C_S — manifolds which are not seifert but can be glued from seifert blocks only,

4. C_h — non-hyperbolic manifolds of which JSJ-decomposition involves a hyperbolic block.

The classification above do not coincide with classical Thurston's classification, at the same time it is clear that our classification is closely related to Thurston's one. The fact that a manifold is of the type S is equivalent to the fact that the manifold has a geometry of the first 6 types. The h-manifolds are exactly the hyperbolic manifolds. The main difference concerns the other manifolds. We separate out C_h-manifolds from all other non-geometric manifolds. C_S-manifolds are manifolds having geometry Sol together with non-geometric manifolds which are not C_h. The partition of composite manifolds into two different types C_S and C_h, in particular, is motivated by a consideration below (see 3.4). Note also following relationship with knot theory. a satellite knot is a knot which can be placed in a regular neighborhood of some other knot. One can expand the classification to the manifolds which are the complement of prime knots. Then a satellite knots (and probably no other) would give C_h-manifolds.

The table below shows how the numbers of manifolds of 4 types defined above increase depending on the complexity.

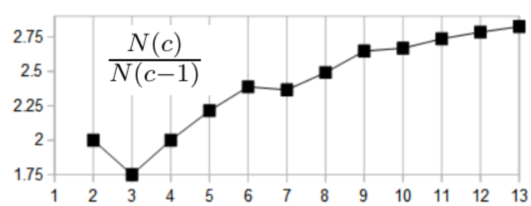
c	0	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
S	3	2	4	7	14	31	74	166	392	942	2237	5297	12481	29162	50812
h	0	0	0	0	0	0	0	0	0	4	25	120	461	1641	2251
C_S	0	0	0	0	0	0	0	9	44	208	816	3001	10482	35177	49737
C_h	0	0	0	0	0	0	0	0	0	0	0	3	24	217	244

3. On the growth of the number of some types of manifolds

3.1. On the growth of the total number of manifolds

Recall that if a manifold is glued from n tetrahedra then its complexity is at most n [25, Theorem 2.2.5]. In fact for most manifolds their complexity is precisely the minimal number of tetrahedra the manifold can be glued from, see [25, Theorems 2.2.6 and 2.2.7] and the discussion around and also [11]. The number of distinct gluings of n tetrahedra increases (depending on n) very fast. A theoretical consideration and computing experiments show the the growth is faster than exponential. We list exact numbers of distinct gluings of 1, 2, 3 and 4 tetrahedra: $11, 169, 5959, 405607$ ([19]). However, our results shows that if we consider not all gluings but gluings which gives close orientable irreducible manifolds only then the growth is exponential.

Here we visualize the value $\frac{N(c)}{N(c-1)}$, i.e., the ratio of next member of the sequence by previous one. The value is informative in the case we concern a sequence like geometric progression.



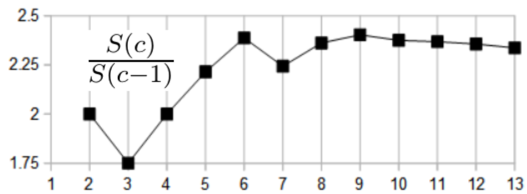
The graph above allows us to formulate the following conjecture concerning the growth of the number of closed orientable irreducible manifolds:

$$\lim_{c\to\infty} \frac{N(c)}{2.5^c} > 0, \quad \lim_{c\to\infty} \frac{N(c)}{3^c} < \infty.$$

3.2. On the growth of the number of seifert manifolds

We see that most manifolds of low complexity are Seifert (which was expected and even proved for very low complexity [1, 12, 15]). Our table allows to conjecture that the growth of the number of seifert manifolds is not so fast as the growth of the total number of manifolds, but it is exponential also.

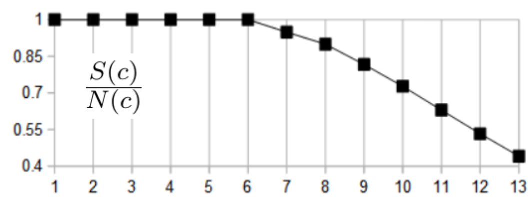
The graph on the right illustrates the supposition. Here we visualize the value $\frac{S(c)}{S(c-1)}$, where $S(c)$ denotes the number of seifert manifolds of complexity c. For sufficiently large c the ratio is closed to 2.3 and decreases slowly.



We congecture that

$$\lim_{c\to\infty} \frac{S(c)}{2^c} > 0, \quad \lim_{c\to\infty} \frac{S(c)}{2.5^c} < \infty.$$

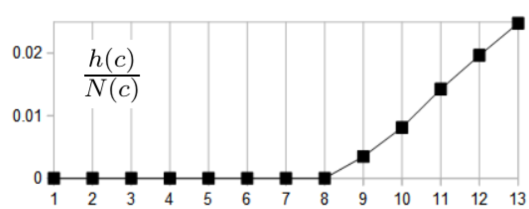
At the same time the proportion between the number of seifert manifolds of a complexity and the total number of manifolds of the complexity (i.e., the value $\frac{S(c)}{N(c)}$) monotonously decreases starting at $c = 6$. Probably this is a consequence of the facts that $\frac{S(c)}{2.5^c}$ approaches to 0 while $\frac{N(c)}{2.5^c}$ approaches to infinity.



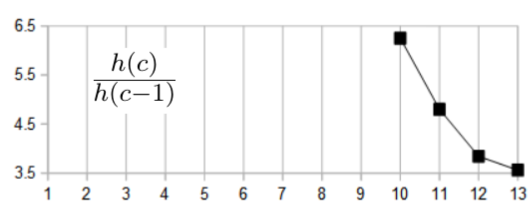
3.3. On the growth of the number of hyperbolic manifolds

We also see that relatively few manifolds in our table are hyperbolic, which was actually not expected: recall that there it is generally believed that “most” manifolds are hyperbolic. In folklore, this statement is attributed to M. Gromov (rather as a conjecture or a general direction of research that as a claim) and is possibly nowhere published. Partial results include [2, 7], see also discussion in [9, §2].

In our table hyperbolic manifolds appear at $c = 9$. The proportion between hyperbolic and all manifolds (i.e., the value $\frac{h(c)}{N(c)}$ where $h(c)$ denotes the number of hyperbolic manifolds of complexity c) is not large but increases monotonously.



At the same time the speed of the growth (we mean the value $\frac{h(c)}{h(c-1)}$) decreases monotonously and very fast.



Now we know not much about the complexity of hyperbolic manifolds, but we conjecture that the proportion between hyperbolic manifolds and all manifolds (contrary to the opinion mentioned above) approaches not to 1 but to 0:

$$\lim_{c\to\infty} \frac{h(c)}{N(c)} = 0.$$

3.4. On the growth of the number of C_S and C_h manifolds

It is understood, that we do not have enough information to formulate a well-grounded conjectures, however, we think that for sufficiently large values of complexity the dominating kind of manifolds is not hyperbolic manifolds but the union $C_S \cup C_h$, and probably C_h will dominate C_S, i.e.,

$$\lim_{c\to\infty} \frac{C_S(c)+C_h(c)}{N(c)} = 1, \quad \lim_{c\to\infty} \frac{C_S(c)}{C_h(c)} = 0.$$

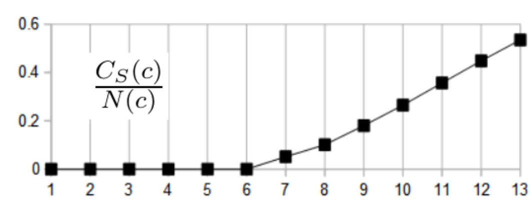
Here $C_S(c)$ and $C_h(c)$ denote, respectively, the numbers of C_S and C_h manifolds having complexity c.

These suppositions are based on following consideration.

A manifold of these two types (C_S and C_h), by definition, can be glued from seifert and hyperbolic manifolds with toric (maybe disconnected) boundary. The number of such manifolds of a complexity is much more and increases faster than the number of closed manifolds of the same complexity. The complexity of glued manifold usually is greater than the sum of complexities of parts of which it is glued from (it is necessary to take into account a "complexity" of the gluings). However, seemingly the growth of the number of blocks which one can use for gluing (more precisely, the growth of the number of combinations of blocks) determinates the growth of the number of closed manifolds composed from these blocks. Our supposition that C_h dominates C_S for $c \to \infty$ is based on the observation that the number of hyperbolic blocks increases faster than the number of seifert blocks. A closed seifert manifold also can be glued from seifert blocks. It takes place if very specific gluings are used. The proportion between such gluing and all possible gluings decreases for $c \to \infty$. The fact explains the decreasing of the proportion of seifert manifolds which was mentioned above.

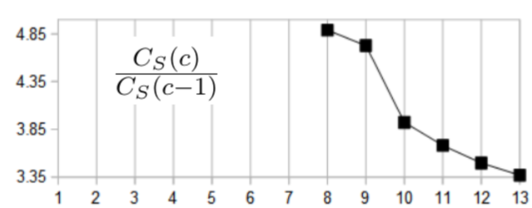
Additionally we note the following information from our table which, as we think, corroborates the congecture above.

C_S-manifolds appear at complexity 7. Then the proportion $\frac{C_S(c)}{N(c)}$ increases monotonously. At $c =$ $= 13$ the value is already more than a half. The total number (the sum over all values of complexity) of C_S manifolds is almost equal to the total number of seifert manifolds (see the table in § 2.3).



Taking into account trends which were mentioned above we can think that at complexity 14 C_S-manifolds will leave seifert manifolds behind.

The speed of the growth of $C_S(c)$ (we mean the value $\frac{C_S(c)}{C_S(c-1)}$) decreases but within our table it remains greater than the corresponding value for the total number of manifolds.



The proportion of hyperbolic manifolds also increases monotonously and fast but it is essentially less and the speed decreases faster than it takes place for C_S-manifolds. Additionally at $c = 11$ the C_h-manifolds appear. The proportion of C_h-manifolds is very little but the number of the manifolds increases very fast (3 at $c = 11$, 24 at $c = 12$, 217 at $c = 13$). Also note that the speed of the growth increases: $\frac{217}{24} > \frac{24}{3}$, while if we denote by $n(c)$ the number of manifolds of any other type (S, C_S or h) then the ratio decreases, i.e., $\frac{n(c+1)}{n(c)} < \frac{n(c)}{n(c-1)}$ for any c for C_S and h manifolds and for $c \geq 9$ for S-manifolds.

4. On the way of obtaining of our table

Our table was created in three steps.

Firstly we have enumerated all manifolds of complexity $\leqslant$ 13. More precisely, we have enumerated their special spines with $\leqslant 13$ real vertices (we recall the definitions of special spine in appendix below). The obtained list contained many duplicates. The step was much more longer than two other steps. It was necessary to use supercomputer.

Then each manifold was recognized using the method described in [17, §7]. As a result each manifold was labeled with a "name" which contains an information about the structure of the manifold. The name of a seifert manifold is its base surface and parameters of its exceptional fibers. The name of a manifold having non-trivial JSJ-decomposition is composed from the names of blocks involved in the decomposition and a description of their gluings. The name of a hyperbolic manifold is a representation of the manifold as a Dehn filling of a hyperbolic manifold with toric (not necessary connected) boundary.

Finally we have removed all duplicates. The ways we do it are different for manifolds of different types. Seifert manifolds and composite manifolds gluing from seifert blocks only (S and C_S manifolds) can be labeled with canonical names which are uniquely defined and can be obtained starting with any of admissible name of the manifold. Hence for manifolds of these types it is possible to find duplicates by comparing of names only. For hyperbolic manifolds and composite manifolds gluing from seifert and hyperbolic blocks (h and C_h manifolds) our methods do not give canonical names. So to prove that two manifolds of these types are nonhomeomorphic we compare their first homology groups and the values of Turaev-Viro invariants. It is interesting to note that the invariants of relatively low order were enough. Almost all pairs were distinguished by invariants of order $\leqslant 8$. 42 pairs were distinguished by invariants of order 9. And only one pair was distinguished by invariants of order 10.

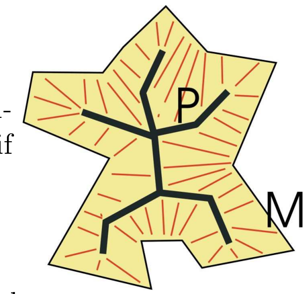
Of course, if the numbers of the order 100000 appear, it is necessary to double-check the result. There are an "internal" methods to check the table, but the most convincing was the comparison of our list with the lists independently obtained by other groups. There are at least two more scientific groups successfully working in the tabulation/recognition problem of 3-dimensional manifolds. The group lead by B. Burton (initially University of Melbourne, now University of Queensland) created a program called Regine. It is available at `https://regina-normal.github.io/`. "Regine" is very powerful and very useful tool for a research in the area of 3-dimensional topology. In 2012 using the program the Burton's group has created a table of manifolds of complexity $\leqslant$ 12 (the result seemingly is not properly published). The other group is at ENS Pisa lead by C. Petronio. B. Martelli and C. Petronio went [10] up to complexity 9. Tables obtained by these two groups coincide with corresponding subsets of our table. Till now we do not know about some other list of manifolds of complexity > 12. So we have nothing to compare our list with.

Our table and the program which we used are available online `http://www.matlas.math.csu.ru/`.

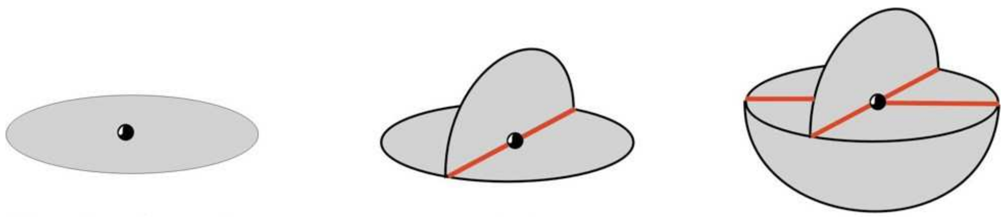
The "inner" language of our algorithm and of our computer program is based on the theory of special spines, but of course our program understands also many other popular combinatorial ways of describing the manifolds (for example, surgery presentation and singular triangulation) automatically translating them to the language of special spines.

Appendix: informal explanation of terminology of 3-dimensional topology used above

Let M be a connected compact 3-dimensional manifold with boundary. An imbedded 2-dimensional CV-complex P is a spine of M, if $M \setminus P = \partial M \times (0, 1]$. A two-dimensional analog is on the picture.



A spine is *simple*, if it has a nice local structure as on the picture below.

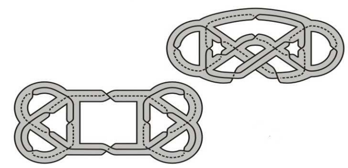


Nonsingular point *Triple line* *True vertex*

A *True Vertex* of a spine is a point having a neighborhood like in the figure above on the right. A simple spine allows a natural stratification. Strata are of dimension 0, 1 and 2. A spine is *special* if each its 1-stratum is a 1-cell and each its 2-stratum is a 2-cell.

Each closed connected 3-manifold with non-empty boundary has a special spine [17, Theorem 1.1.13]. If the boundary is empty, we remove a 3-dimensional ball from the manifold and obtain a manifold with the boundary. A special spine allows one to reconstruct the manifold [17, Theorem 1.1.17].

A graphical presentation of two special spines are on the picture. It is clearly a combinatorial object, the information necessary to reconstruct the special spine is the vertices, the 1-dimensional edges between the vertices (i.e., the 1-skeleton), and also the information how the 2-cells are glued near the vertices. One can give it as a word in a certain alphabet consisting of the number of vertices plus 3 symbols.



Note that not every special polyhedron corresponds to a manifold — there exist so-called *unthickenable* special polyhedra that can not be special spines of 3-manifolds. It is quite easy to understand whether a special polyhedron is unthickenable, see the discussion in [25, §2.2] and [17, discussion starting from page 9].

The *complexity* of a manifold is the minimal numbers of true vertices in its simple spine. It is a finite number. In this notes we concern with closed irreducible manifolds only. Within the class of manifolds for each complexity c there exists finitely many pairwise nonhomeomorphic manifolds of complexity c. Note that the class contains exactly 3 manifolds of complexity 0 (i.e., having a simple spines with no vertex). They are $S^3, \mathbb{R}p^3$ and lens space $L(3, 1)$. Of course, the manifolds have special spines (with non-zero number of true vertices) also, but in the case of these three manifolds the minimum of the number of true vertices in a spine reaches on a simple spines with no vertices.

One of the first spectacular applications of the theory of complexity is due to A.T. Fomenko. and the first author ([12], see also [13] and [17, 2.5.1]). They have found all closed hyperbolic manifolds of the lowest complexity 9 (there are precisely four such) and calculated their volumes. The result of the calculation were the numbers ≈ 0.94272, ≈ 0.98139, ≈ 1.01494 and ≈ 1.26370. Recall that by the Mostow Rigidity Theorem [18] the Riemannian metric of constant negative sectional curvature is unique on every closed 3-manifold on which it exists, which makes the notion "volume of closed hyperbolic manifold" well-defined. The manifold with the volume ≈ 0.98139 was previously studied by W. Thurston [21] and he suggested it as a candidate for the hyperbolic manifold of the smallest volume. Since $0.94272 < 0.98139$ (even taking into account the numerical error of calculations), this our result proved that the conjecture of Thurston is wrong. We conjectured [12, Conjecture 1] that the hyperbolic manifold with volume ≈ 0.94139 is the one with the smallest volume. The conjecture was proved in [3, Corollary 1.3], see also the discussion in [4].

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