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Квантовая интегрируемость операторов Бельтрами — Лапласа проективно эквивалентных метрик произвольных сигнатур.

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Аннотация

Мы обобщаем результат [31] на все сигнатуры.

Ключевые слова: интегрируемые системы, тензоры Киллинга, квантовые интегрируемые системы, коммутирующие операторы, квантование по Кантеру, проективно эквивалентные метрики, геодезически эквивалентные метрики, разделение переменных, нормальные формы, геометрическая теория УрЧП, голоморфно-проективно эквивалентные метрики

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Quantum integrability for the Beltrami–Laplace operators of projectively equivalent metrics of arbitrary signatures

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Abstract

We generalize the result of [31] to all signatures.

Keywords: integrable systems, Killing tensors, quantum integrable systems, Carter quantisation, commutative operators, projectively equivalent metrics, geodesically equivalent metrics, separation of variables, normal forms, geometric theory of PDE, c-projectively equivalent metrics

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Dedicated to Anatoly Timofeevich Fomenko on his 75th birthday.

1. Introduction

Let M be a smooth manifold of dimension $n \geq 2$. We say that two metrics g and \bar{g} on this manifold are *projectively equivalent*, if each g -geodesic, after a proper reparameterization, is a \bar{g} -geodesic. Theory of projectively equivalent metrics is a classical topic in differential geometry, already E. Beltrami [1] and T. Levi-Civita [26] did important contributions there. In the last two decades a group of new methods coming from integrable systems, see e.g. [27, 28, 29, 32, 33], and from Cartan geometry, see e.g. [19, 39, 44], appeared to be useful in this theory, and made it possible to solve important open problems and named conjectures, see e.g. [37, 34, 12, 40, 38].

By [27, 35] the existence of \bar{g} projectively equivalent to g allows one to construct a family $K_{ij}^{(t)}$ of Killing tensors of second degree for the metric g (we will recall the formula and the definition later, in §2.1, following later publications, e.g. [3, 34, 36]. The family $K_{ij}^{(t)}$ is polynomial in t of degree $n - 1$ so it contains at most n linearly independent Killing tensors).

In this paper we answer in Theorem 1 the following natural ‘quantization’ question: *do the corresponding second order differential operators commute?*

There are of course many possible constructions of differential operators of second order by $(0,2)$ -tensors, and, more generally, many different quantization approaches, see e.g. [9, §6]. We use the quantization procedure of B. Carter [15, Equation (6.15)] and refer to [15] and also to [2, 18] for an explanation why it is natural in many aspects. The construction is as follows: to a tensor K_{ij} , we associate an operator

$$\widehat{K} : C^\infty(M) \rightarrow C^\infty(M), \quad \widehat{K}(f) = \nabla_i K^{ij} \nabla_j f. \quad (1)$$

Above and everywhere in the paper ∇ is the Levi-Civita connection of g , we sum with respect to repeating indexes and raise the indexes of K by the metric g .

THEOREM 1. *Assume g and \bar{g} are projectively equivalent, let $K^{(t)}$ be the family of Killing tensors of second degree for g constructed with the help of \bar{g} . Then, for any $t, s \in \mathbb{R}$, the operators $\widehat{K}^{(t)}, \widehat{K}^{(s)}$ commute, that is*

$$\widehat{K}^{(t)} \widehat{K}^{(s)} - \widehat{K}^{(s)} \widehat{K}^{(t)} = 0.$$

Note that the Beltrami–Laplace operator $\Delta_g := \nabla_i g^{ij} \nabla_j$ is a linear combination of the operators of the family $\widehat{K}^{(t)}$, so all the operators $\widehat{K}^{(t)}$ commute also with Δ_g . In fact, in the proof we go in the opposite direction: we show first (combining [15, 18] and [22]) that the operators $\widehat{K}^{(t)}$ commute with Δ_g and then use this to show that the operators $\widehat{K}^{(t)}, \widehat{K}^{(s)}$ also commute mutually.

For Riemannian manifolds, Theorem 1 is known, it was announced in [30] and the proof appeared in [31]. The proof in the Riemannian case is based on direct calculations in the coordinates in which the metrics admit the so-called Levi-Civita normal form. These coordinates exist (locally, in a neighborhood of almost every point), if the $(1,1)$ -tensor $G_j^i := g^{is} \bar{g}_{sj}$ is semi-simple (at almost every point). This is always the case, for example, if one of the metrics is Riemannian. The proof from [31] can be directly generalized to the pseudo-Riemannian metrics under the additional assumption that G is semi-simple.

There are (many) examples of projectively equivalent metrics such that G has nontrivial Jordan blocks; in this situation the proof and ideas of [31] are not sufficient. Indeed, though also in this

case there exists a local description of projectively equivalent metrics [6], direct calculation of the commutators of the operators $\widehat{K}^{(t)}$ and $\widehat{K}^{(s)}$ is a complicated task because of different combinatoric possibilities for the number and the sizes of Jordan blocks and also because the description of [6] uses a description of symmetric parallel (0,2)-tensors from [11] which is quite nontrivial. For small dimensions it is possible though to prove Theorem 1 by direct calculations, in particular in dimension 2 it was done in [4, §2.2.3].

Our proof is based on another circle of ideas, it still uses the local description of [6] but replaces local calculations by a trick which is based on quite nontrivial results of different papers. We recall the necessary results in §2.

All objects in our paper are assumed to be sufficiently smooth.

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2. Basic facts about projectively equivalent metrics and Killing tensors used in the proof

2.1. Killing tensors for projectively equivalent metrics and corresponding integrals.

Let g and \bar{g} be two projectively equivalent metrics on the manifold M . Let us recall the construction of Killing tensors $K_{ij}^{(t)}$ of second degree for the metric g by using the metric \bar{g} . We consider the (1,1)-tensor L given by the formula

$$L_j^i := \left| \frac{\det(\bar{g})}{\det(g)} \right|^{\frac{1}{n+1}} \bar{g}^{i\ell} g_{\ell j}. \quad (2)$$

Here \bar{g}^{ij} is the contravariant metric dual (= inverse, i.e., $\bar{g}^{is} \bar{g}_{sj} = \delta_j^i$) to \bar{g} .

Next, consider the family $S(t)$, $t \in \mathbb{R}$, of the (1,1)-tensors, where Id is the (1,1)-tensor corresponding to the identity endomorphism, its components in the standard tensor notation are δ_j^i .

$$S(t) := \text{Comatrix}(t \text{ Id} - L). \quad (3)$$

Recall that the comatrix (or the adjugate matrix) of a (1,1)-tensor is also a (1,1)-tensor. Indeed, at points where $t \notin \text{Spectrum}(L)$, it is given by

$$\text{Comatrix}(t \text{ Id} - L) = \det(t \text{ Id} - L) (t \text{ Id} - L)^{-1}$$

and evidently corresponds to a (1,1)-tensor, and for each point the set of t not lying in the spectrum of L is everywhere dense on the real line. From the formula for the comatrix we see that the family (3) is polynomial in t of degree $n - 1$.

THEOREM 2 (Essentially, [27]). *Let g and \bar{g} be projectively equivalent. Then, for every $t \in \mathbb{R}$ the tensor*

$$K_{ij}^{(t)} := g_{ir} S(t)_j^r \quad (4)$$

is a Killing tensor for g .

In the coordinate-free notation the Killing tensor $K^{(t)}$ is given by $K^{(t)}(\xi, \nu) = g(\xi, S(t)\nu)$. Since L is g -selfadjoint, $S(t)$ is also self-adjoint so $K^{(t)}$ is symmetric with respect to the lower indexes. Recall that a (symmetric with respect to the lower indexes) tensor K_{ij} is Killing, if

$$\nabla_{(i} K_{jk)} = 0, \quad (5)$$

where the round brackets denote the symmetrization. In our paper we do not use this equation, but use the geometric definition which we recall now: a $(0,2)$ symmetric tensor $K = K_{ij}$ is Killing, if and only if the function $\tau \mapsto K(\gamma'(\tau), \gamma'(\tau))$ is constant along every naturally parameterized g -geodesic $\gamma(\tau)$. In other words, if the function $K(\gamma'(\tau), \gamma'(\tau))$ is an integral of the geodesic flow of g . It is known, that the integrals corresponding to the Killing tensors $K^{(t)}$ constructed above commute, let us recall this statement:

THEOREM 3. *Let g and \bar{g} be projectively equivalent and $K^{(t)}$ be the Killing tensors for g constructed by (4). Consider, for each $t \in \mathbb{R}$, the function $I_t : T^*M \rightarrow \mathbb{R}$ given by formula*

$$I_t(x, p) = K_{rq}^{(t)} g^{iq} g^{jr} p_i p_j. \quad (6)$$

Here $(x, p) = (x^1, \dots, x^n, p_1, \dots, p_n)$ are local coordinates on T^*M : x^i are local coordinates on M and p_i are, for each x , the coordinates on T_x^*M corresponding to the basis $\frac{\partial}{\partial x^i}$ on $T_x M$.

Then, for any $t, s \in \mathbb{R}$ the functions I_t, I_s Poisson-commute with respect to the standard Poisson bracket on T^*M , that is:

$$\sum_{i=1}^n \frac{\partial I_t}{\partial p_i} \frac{\partial I_s}{\partial x^i} - \frac{\partial I_t}{\partial x^i} \frac{\partial I_s}{\partial p_i} = 0.$$

In the Riemannian signature, Theorem 3 is due to [27]. In all signatures, it was independently proved in [3, 49].

2.2. Difference between connections of projectively equivalent metrics

We consider the $(1,1)$ -tensor L constructed by projectively equivalent metrics g and \bar{g} by (2). As it was observed in [46], see also [3, Theorem 2], it satisfies, for a certain 1-form λ_i , the following equation:

$$\nabla_k L_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}. \quad (7)$$

Here and later we use g for the covariant differentiations and for the tensor manipulations with indexes. By contracting (7) with g^{ij} , we see that the 1-form λ_i is the differential of the function $\lambda := \frac{1}{2} \text{trace}(L) = \frac{1}{2} L^s_s$.

REMARK 1. *The projectively-invariant form of this equation is due to [19], see also the survey [44] (and [12] for its two-dimensional version). It played essential role in many recent developments in the theory of projectively equivalent metrics including the solutions of two problems explicitly stated by Sophus Lie [12, 40], the proof of the discrete version of the projective Lichnerowicz conjecture [43, 51] and the proof of the Lichnerowicz conjecture for metrics of Lorentzian signature [8].*

The 1-form λ_i is closely related to the difference between the Levi-Civita connections of $\nabla = (\Gamma_{jk}^i)$ and $\bar{\nabla} = (\bar{\Gamma}_{jk}^i)$ (see e.g. [46] or [22, §2.2]): for the 1-form

$$\phi_i := -L_i^s \lambda_s \quad (8)$$

we have

$$\bar{\Gamma}_{jk}^i - \Gamma_{jk}^i = \delta_k^i \phi_j + \delta_j^i \phi_k. \quad (9)$$

From formulas (8,9) we see that if $\lambda = \text{trace}(L)$ has zero of order k at a point $p \in M$, then at this point the connections coincide up to the order $k-1$. In particular, for any tensor field T the $(k-1)$ st, and also lower order, covariant derivatives of T in ∇ and $\bar{\nabla}$ coincide in p :

$$\nabla_{i_1} \nabla_{i_2} \dots \nabla_{i_{k-1}} T \stackrel{\text{at } p}{=} \bar{\nabla}_{i_1} \bar{\nabla}_{i_2} \dots \bar{\nabla}_{i_{k-1}} T.$$

Let us recall one more important property of projectively equivalent metrics:

THEOREM 4 (Folklore, e.g. Lemma 1 in [22] or (12) in [23]). *Let g and \bar{g} be projectively equivalent metrics and L is as in (2). Then, the Ricci curvature tensor R_{ij} of g commutes with L , in the sense*

$$R_s^i L_j^s - L_s^i R_j^s = 0. \quad (10)$$

(For each $x \in M$ the formula (10) is just the formula of the commutators of two endomorphisms of $T_x M$: the first is given by the Ricci tensor with one index raised, and the other it given by L).

2.3. Perturbing the metrics in the class of projectively equivalent metrics.

Let us now show that (for any k) one can perturb the metrics g and \bar{g} in the class of projectively equivalent metrics such that at a point they remain the same up to order k and at another point the function λ is constant up to order k .

We say that two tensors or affine connections *coincide at a point p up to order k* , if their difference is zero at p and in a local coordinate system all partial derivatives up to the order k of the components of their difference are zero at the point p . This property does not depend on the choice of a coordinate system.

In particular, a function is *constant at p up to order k* if all its partial derivatives up to order k are zero at p .

THEOREM 5. *Let g and \bar{g} be projectively equivalent metrics and L is as in (2). Then, for each $k \in \mathbb{N}$ and for almost any point $p \in M$ there exists an arbitrary small neighborhood U containing p , a point $q \in U$ and a pair of projectively equivalent metrics g' and \bar{g}' on U (whose tensor (2) will be denoted by L' and the function $\frac{1}{2} \text{trace}(L')$ will be denoted by λ') such that the following holds:*

- (A) *At the point p , g coincides with g' and \bar{g} coincides with \bar{g}' up to order k .*
- (B) *At the point q , λ' is constant up to order k .*

“Almost every point” means that the set of such points contains an open everywhere dense subset.

Theorem 5 essentially follows from [5, 6], let us explain this. We consider the points p which are algebraically generic in the sense of [10, Def. 2.7]: that is, there exists a neighborhood $U \ni p$ such that at every point of the neighborhood the number of different eigenvalues of L and the number and the sizes of the Jordan blocks are the same (of course the eigenvalues are not necessary constant and usually depend on the point; by the implicit function theorem they are smooth functions near p).

Take such a point. Note that λ is the half of the sum of eigenvalues of L , counted with algebraic multiplicities. We need to find projectively equivalent metrics g' and \bar{g}' such that they coincide to order k at p with g and \bar{g} and such that all eigenvalues of L' are constant up to order k in some point q .

By the Splitting-Gluing construction [5, §§1.1, 1.2], it is sufficient to do this under the assumption that L has one eigenvalue, or one pair of complex-conjugated eigenvalues. If the geometric multiplicity of an eigenvalue is greater than one, by [6, Proposition 1], the eigenvalue is already a constant, so we are done since $g' = g$ and $\bar{g} = \bar{g}'$ are already as we want.

Let us now consider the case when L has one real eigenvalue of geometric multiplicity 1, or a pair of nonreal complex-conjugate eigenvalues of geometric multiplicity 1. In this case, the local structure of g and L near the point p are described in some coordinate system. There are 4 possible cases, the description was done in different papers, let us give the precise references where it can be found.

If eigenvalue is real and its geometric multiplicity is one (so the “splitted out” manifold is one-dimensional), then the description is trivial and was discussed e.g. in [5, Example in §2.1] or [39, Example 3 in §3.2.1].

If L has a pair of nonreal complex-conjugate eigenvalues of geometric multiplicity 1, then the description was done in [4, Theorem 2], see also [40, Theorem A].

If L , at each point of U , is conjugate to a Jordan block with real eigenvalue, the description is in [6, Theorem 4].

If L , at each point of U , is conjugate to a pair of Jordan blocks with complex-conjugated eigenvalues, the description is done in [6, Theorem 5].

In each of the above references, one sees that description is given by a formula and the only object we can choose is the eigenvalue(s) of L : in the ‘real’ case, it is a function of one variable; this function can be chosen arbitrary (with exception that one may not make it zero; though also this is allowed if we discuss not projectively equivalent metrics but ‘compatible’ in the terminology of [6], pairs (g, L)).

In the ‘nonreal’ case, the eigenvalue is a holomorphic function of one variable, again it can be chosen arbitrary (again with exception that it is never zero) in the class of holomorphic functions.

In order to prove Theorem 5, one modifies the eigenvalue such that at p it coincides with the initial eigenvalue up to order k , and is constant up to order k in some other point q . One can clearly do it for any function of one variable and for any holomorphic function of one complex variable.

2.4. Carter’s condition.

We will need the following result:

THEOREM 6. *Assume K_{ij} is a Killing tensor for g and R_{ij} is the Ricci curvature tensor. Suppose, at the point $p \in M$, we have that up to order k*

$$\nabla_i (R_s^i K_j^s - K_s^i R_j^s) = 0. \quad (11)$$

Then, the Beltrami-Laplace operator Δ_g and the operator \widehat{K} commute at the point p up to order k , that is, for every function f we have

$$(\Delta_g \widehat{K} - \widehat{K} \Delta_g) f = 0 \quad \text{at } p \text{ up to order } k$$

Theorem above is essentially due to B. Carter. Indeed, from [15, Equation (6.16)] it follows that if $\nabla_i (R_s^i K_j^s - K_s^i R_j^s)$ is zero at all points, then Δ_g and \widehat{K} commute at all points. Careful analysis of the arguments shows that the proof of Carter is valid also pointwise. Note that only a sketch of the proof is given in [15], and we recommend [18, §III(A)] of C. Duval and G. Valent, from which a more detailed proof can be extracted. More precisely, combining [18, Equations (3.11) and (3.16)] we obtain the above mentioned result of Carter.

2.5. If a Killing tensor vanishes up to a sufficiently high order at one point, then it is identically zero

THEOREM 7. *Let M be a connected manifold and g be a metric of any signature on it. Assume K is a Killing tensor of order k (i.e., K is a symmetric $(0, k)$ tensor satisfying the equation $\nabla_{(i} K_{i_1 \dots i_k)} = 0$). If K vanishes up to order k at one point, then it vanishes identically on the whole manifold.*

This theorem follows from [48] (see also [25, §3]). We will need this theorem for first and second degree Killing tensors. Note that for the first degree Killing tensors (= Killing vectors, after raising the index), Theorem 7 can be obtained by the following geometric argument: if a Killing vector field vanishes at a point q up to order 1, then the flow of this vector field acts trivially on the tangent space to q . Since it commutes with the exponential mapping, the Killing vector field must

be identically zero. For second degree Killing tensors, the proof is based on the prolongation of the Killing equation which was essentially done in [50]. For all degree Killing tensors, the prolongation of Killing equation was essentially done in [48], though formally this paper discusses special case of constant curvature metrics. Indeed, for our goal the higher order terms of the prolongation are sufficient, and they do not depend on the curvature of the metric, see e.g. the discussion in [25, §3].

3. Proof of Theorem 1.

We assume that g and \bar{g} are projectively equivalent metrics of any signature on M^n , $n \geq 2$. We consider L given by (2), the family $K^{(t)}$ of Killing tensors given by (4) and the corresponding differential operators $\hat{K}^{(t)}$. Combining Theorems 4 and 6, we see that the operators commute with Δ_g .

Let us take any $t, s \in \mathbb{R}$ and consider the commutator

$$\hat{Q} := \hat{K}^{(t)} \hat{K}^{(s)} - \hat{K}^{(s)} \hat{K}^{(t)}.$$

Our goal is to show that it vanishes; we will first show that it is (linear) differential operator of order at most 2, i.e., that when we apply \hat{Q} to a function f the higher derivatives of f vanish. This step is well-known, see e.g. [15] or [18], let us shortly recall the arguments.

Clearly, \hat{Q} is a differential operator of order at most 4, since both $\hat{K}^{(t)}$ and $\hat{K}^{(s)}$ have order 2. One immediately sees though, that the operators $\hat{K}^{(t)} \hat{K}^{(s)}$ and $\hat{K}^{(s)} \hat{K}^{(t)}$ have the same symbols, so the 4th order terms cancel when we subtract one from the other. Thus, the order of \hat{Q} is at most 3. The third order terms vanish because the integrals corresponding to $K^{(s)}$ and $K^{(t)}$ commute by Theorem 3. Indeed, direct calculations show that the symbol of the commutator of two differential operators is the Poisson bracket of their symbols.

The proof that the first and the second order terms vanish is based on another (new) argument which will use all the results recalled in §2.

First observe that there exist a symmetric (2,0) tensor Q^{ij} and the vector field V^ℓ such that

$$\hat{Q} = \nabla_i Q^{ij} \nabla_j + V^\ell \nabla_\ell.$$

Indeed, the operator \hat{Q} does not have terms of zero order, since neither $\hat{K}^{(t)}$ nor $\hat{K}^{(s)}$ have such. One can collect all second order terms in $\nabla_i Q^{ij} \nabla_j$ and declare the rest as $V^\ell \nabla_\ell$.

Since Δ_g commutes with $\hat{K}^{(t)}$ and $\hat{K}^{(s)}$, it commutes with \hat{Q} . Then, Q_{ij} is a Killing (0,2) tensor for g .

It is sufficient to show, that Q_{ij} vanishes at almost every point. It is sufficient to show this for almost every t and s . We take s and t such that the tensors $K^{(t)}, K^{(s)}$ are nondegenerate at some point. We will work in a small neighborhood of this point, in each point of which the tensors $K^{(t)}, K^{(s)}$ are nondegenerate. Now we use Theorem 5: we first take a sufficiently big k and then, for almost every point of p of this neighborhood consider the projectively equivalent metrics g' and \bar{g}' satisfying conditions (A,B) from Theorem 5.

At the point p , the metrics g and \bar{g} coincide with the metrics g' and \bar{g}' , which implies that the Killing tensor Q'_{ij} (i.e., the analog of the Killing tensor Q_{ij} constructed by g' and \bar{g}') coincides with Q_{ij} in p . Let us show that, if k is high enough, at the point q the Killing tensor Q'_{ij} vanishes up to order 2.

At the point q , the 1-form λ_i and therefore the 1-form ϕ_i (recalled in §2.2) vanishes up to (sufficiently high) order k . Then, at the point q , the difference between Levi-Civita connections ∇' of g' and of \bar{g}' vanishes up to order $k-1$, see (9). Since the Killing tensors $K'^{(s)}, K'^{(t)}$ are constructed by g', \bar{g}' using algebraic formulas, the covariant derivative in ∇' of $K'^{(s)}, K'^{(t)}$ vanishes at the point

q up to order $k - 1$. Then, up to the order $k - 1$, at the point q , the Levi-Civita connection of the (contravariant) metrics¹ $(K'^{(s)})^{ij}$, $(K'^{(t)})^{ij}$ coincide with ∇' .

Then, at the point q , the Betrami-Laplace operators of the metrics $K'^{(s)}$, $K'^{(t)}$ coincide with $\hat{K}'^{(s)}$, $\hat{K}'^{(t)}$ up to order $k - 2$. From the other side the Ricci tensor corresponding to the metric $K'^{(s)}$ commutes (in the sense of (10)) with $K'^{(t)}$, up to the terms of order $k - 3$, since it coincides up to the terms of order $k - 3$ with the Ricci tensor of g' and it commutes with L' and therefore with $S'(t)$. Then, the Carter condition (11) is fulfilled up to order $k - 4$. Then, the operators $\hat{K}^{(t)}$ and $\hat{K}^{(s)}$ commute at q up to order $k - 4$, which means that at q we have $Q'_{ij} = 0$ up to order $k - 5$. If $k > 7$, then this implies by Theorem 5 that Q'_{ij} is identically zero, which means it vanishes at p , where it coincides with Q_{ij} . Finally, $Q_{ij} = 0$ at p and since p was almost every point $Q_{ij} \equiv 0$ on the whole manifold.

REMARK 2. *In fact the reader does not need to follow the precise calculations of the necessary order above: it is clear that if k is high enough then at the point q the Levi-Civita connection of the contravariant metric corresponding to $K'^{(s)}$ (with upper indexes) coincides with that of g up to a sufficiently high order and $K'^{(t)}$ is parallel with respect to any of this connections up to a high order which means that the operators $\hat{K}^{(t)}$ and $\hat{K}^{(s)}$ commute at p up to some high order and Q' is zero up to a high order and is therefore identically zero.*

But then $\hat{Q} = V^\ell \nabla_\ell$, since it commutes with Δ_g , V^ℓ is a Killing vector field. Using the same arguments, one shows that (for a perturbed metrics g' , \bar{g}'), $V^\ell \equiv 0$, which implies that $V^\ell = 0$ at p . Since this is fulfilled for almost all points p , we obtain $V^\ell \equiv 0$. Theorem 1 is proved.

4. Open problems

4.1. Introducing potential

We assume that g and \bar{g} are projectively equivalent metrics of any signature on M^n . We consider the Killing tensors $K^{(t)}$ and the corresponding integrals I_t from Theorem 3 and ask the following questions:

Can one add functions $U^{(t)} : M \rightarrow \mathbb{R}$ to the integrals I_t such that the results still Poisson-commute? Do the corresponding differential operators, i.e., $\hat{K}^{(t)} + U^{(t)}$, still commute?

Of course it is interesting to get not one example of such functions (the trivial example $U^{(t)} = \text{const}$ always exists) but construct all such examples, at least locally.

If L is semi-simple at almost every point (which is always the case if g is Riemannian), the answer is positive, which follows from the combination of results of [24, 17], see also [16].

4.2. Generalize the result for c-projectively equivalent metrics.

Theory of projectively equivalent metrics has a natural analogue on Kähler manifolds: theory of c-projectively equivalent metrics. Let us recall the basic definition:

Let (M, g, J) be a Kähler manifold of arbitrary signature of real dimension $2n \geq 4$. A regular curve $\gamma : \mathbb{R} \supseteq I \rightarrow M$ is called *J-planar* if there exist functions $\alpha, \beta : I \rightarrow \mathbb{R}$ such that

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = \alpha \dot{\gamma}(t) + \beta J(\dot{\gamma}(t)) \text{ for all } t \in I, \quad (12)$$

where $\dot{\gamma} = \frac{d}{dt} \gamma$.

¹As explicitly indicated, we view now the Killing tensors as metrics: we first raise the indexes in (4) by g' . The result is a nondegenerate symmetric $(2, 0)$ tensor, we view it as a contravariant metric. In order to obtain an usual metric, with lower indexes, one needs to invert the matrix of $(K'^{(t)})^{ij}$.

From the definition we see immediately that the property of J -planarity is independent of the parameterization of the curve, and that geodesics are J -planar curves. We also see that J -planar curves form a much bigger family than the family of geodesics; at every point and in every direction there exist infinitely many geometrically different J -planar curves.

Two metrics g and \hat{g} of arbitrary signature that are Kähler w.r.t the same complex structure J are *c-projectively equivalent* if any J -planar curve of g is a J -planar curve of \hat{g} . Actually, the condition that the metrics are Kähler with respect to the same complex structure is not essential; it is an easy exercise to show that if any J -planar curve of a Kähler structure (g, J) is a \hat{J} -planar curve of another Kähler structure (\hat{g}, \hat{J}) , then $\hat{J} = \pm J$.

C-projective equivalence was introduced (under the name “h-projective equivalence” or “holomorphically projective correspondence”) by T. Otsuki and Y. Tashiro in [45, 47]. Their motivation was to generalize the notion of projective equivalence to the Kähler situation. Otsuki and Tashiro, see also [21, §6.2], have shown that projective equivalence is not interesting in the Kähler situation, since only simple examples are possible, and suggested c-projective equivalence as an interesting object of study instead. This suggestion appeared to be very fruitful and between the 1960s and the 1970s, the theory of c-projectively equivalent metrics and c-projective transformations was one of the main research topics in Japanese and Soviet (mostly Odessa and Kazan) differential geometry schools. Geometric structures that are equivalent to the existence of a c-projective equivalent metric were suggested independently in different branches of mathematics, see e.g. the introductions of [42] for a list and [14] for more detailed explanation on the relation to Hamiltonian 2-forms.

It appears that many ideas and many results in the theory of projectively equivalent metrics have their counterparts in the c-projective setting. For example, the use of integrable systems in the proof of the Yano-Obata conjecture [41] about c-projective transformations is very similar to that of in the Lichnerowicz conjecture [37] for projective transformations. Compare also [7, 40]. See e.g. [8, §1.2.] for one of the explanations. In particular, Theorems 2 and 3 have clear analogs: by a c-projectively equivalent metric \bar{g} one can construct second degree Killing tensors for g , and the corresponding integrals commute: see e.g. [13, Proposition 5.14], the result was initially obtained in [49, Theorem 2]. We ask the following question: can one generalize the result of the present paper to c-projectively equivalent metrics?

Do the differential operators corresponding to the Killing tensors from [13, Proposition 5.14], [49, Theorem 2] commute?

Also in the c-projective case, the Ricci tensor commutes with the analog of the tensor L . One can do it by the following tensor calculations which are similar to that of the proof of Theorem 4: take [20, Equation (7)] (which is the c-projective analog of [23, Equation (11)]), perturb the indexes by the trivial permutation and by the permutations $ikl \mapsto kli$ and $ikl \mapsto lik$ and sum the results. We obtain [23, Equation (13)] (where a_{ij} corresponds to L_{ij} in our notation). Contracting the obtained equation with g^{jk} , we obtain an analog of (10), which implies by Theorem 6 that the operators commute with the Beltrami-Laplace operator. Unfortunately, the rest of the proof can not be directly generalized to the c-projective case, since the analog of the function λ can not be a constant up to high order by [20, Corollary 3]. One can try to employ [18, Equation (3.11)] for it, but we did not manage to overcome the technical difficulties.

We do not have clear expectation how the answer would look: we tip that the operators do commute, but will not be suprised if their commutators are first order differential operators corresponding to Killing vector fields. We would like to recall here that a c-projectively equivalent metric allows one to construct Killing vector fields, see e.g. [8, §2] and [13, §5.2].

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