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Обобщённые шахматные комплексы
и дискретная теория Морса¹

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Аннотация

Шахматные комплексы и их обобщения, как объекты, и дискретная теория Морса, как инструмент, представлены в виде объединяющей темы, связывающая различные области геометрии, топологии, алгебры и комбинаторики. Теорема Эдмондса и Фулкерсона о бутылочном горлышке (минимаксе) реализуется и интерпретируется как результат о критической точке дискретной функции Морса на сфере Бира $Bier(K)$ ассоциированного симплициального комплекса K . Мы проиллюстрируем использование «стандартных дискретных функций Морса» на обобщенных шахматных комплексах, доказав результат связности для шахматных комплексов с кратностями. Приложения включают новые результаты типа Тверберга-Ван Кампена-Флореса для разбиений симплекса без j -кратных пересечений.

Ключевые слова: шахматные комплексы, дискретная теория Морса, теорема о бутылочном горлышке, теоремы Тверберга-ван Кампена-Флореса.

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Generalized chessboard complexes and discrete Morse theory

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Abstract

Chessboard complexes and their generalizations, as objects, and Discrete Morse theory, as a tool, are presented as a unifying theme linking different areas of geometry, topology, algebra and combinatorics. Edmonds and Fulkerson bottleneck (minmax) theorem is proved and interpreted as a result about a critical point of a discrete Morse function on the Bier sphere $Bier(K)$ of an associated simplicial complex K . We illustrate the use of “standard discrete Morse functions” on generalized chessboard complexes by proving a connectivity result for chessboard complexes with multiplicities. Applications include new Tverberg-Van Kampen-Flores type results for j -wise disjoint partitions of a simplex.

Keywords: chessboard complexes, discrete Morse theory, bottleneck theorem, Tverberg-Van Kampen-Flores theorems.

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1. Introduction

Chessboard complexes and their relatives have been for decades an important theme of topological combinatorics. They have found numerous and often unexpected applications in group theory, representation theory, commutative algebra, Lie theory, discrete and computational geometry, algebraic topology, and geometric and topological combinatorics, see [1], [2], [3] [5], [15], [16], [26], [28], [33], [35], [36], [38], [39], [40].

The books [25] and [31], as well as the review papers [38] and [41], cover selected topics of the theory of chessboard complexes and contain a more complete list of related publications.

Chessboard complexes and their generalizations are some of the most studied *graph complexes* [25]. From this point of view chessboard complexes can be interpreted as relatives of L. Lovász *Hom*-complexes [29], matching complexes, clique complexes, and many other important classes of simplicial complexes.

More recently new classes of generalized chessboard complexes have emerged and new methods, based on novel *shelling techniques* and ideas from Forman's *discrete Morse theory*, were introduced. Examples include multiple and symmetric multiple chessboard complexes [22, 23], Bier complexes [18], and deleted joins of collectively unavoidable complexes, see [18] and [20, 21]. Among applications are the resolution of the balanced case of the “admissible/prescribable partitions conjecture” [23], general Van Kampen-Flores type theorem for balanced, collectively unavoidable complexes [20], and “balanced splitting necklace theorem” [21].

This paper is both a leisurely introduction and an invitation to this part of topological combinatorics, and a succinct overview of some of the ideas of discrete Morse theory, combinatorics and equivariant topology, used in our earlier papers.

New results are in Sections 5, 6 and 7. They include an alternative treatment of Edmonds and Fulkerson bottleneck (minmax) theorem (Section 5) and the construction of “standard discrete Morse functions” on generalized chessboard complexes with multiplicities (Section 6). This leads to a frequently optimal connectivity result for generalized chessboard complexes with multiplicities (Theorem 2 in Section 6), which is used in Section 7 for the proof of new Tverberg-Van Kampen-Flores type results for j -wise disjoint partitions of a simplex.

2. Chessboard complexes

Chessboard complexes naturally arise in the study of the geometry of *admissible rook configurations* on a general $(m \times n)$ -chessboard. An admissible configuration is any non-taking placement of rooks, i.e., a placement which does not allow any two of them to be in the same row or in the same column. The collection of all these placements forms a simplicial complex which is called the chessboard complex and denoted by $\Delta_{m,n}$.

More formally, the set of vertices of $\Delta_{m,n}$ is $\text{Vert}(\Delta_{m,n}) = [m] \times [n]$ and $S \subseteq [m] \times [n]$ is a simplex of $\Delta_{m,n}$ if and only if for each two distinct elements $(i_1, j_1), (i_2, j_2) \in S$ neither $i_1 = i_2$ nor $j_1 = j_2$.

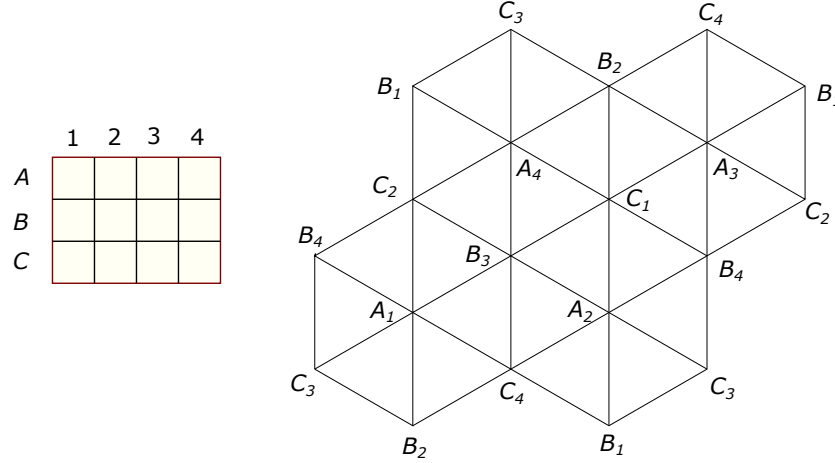
2.1. An example

Let us take a closer look at one of the simplest chessboard complexes, the complex $\Delta_{4,3}$, based on the 4×3 chessboard (see Figure 1).

The f -vector of $\Delta_{4,3}$ is $f(\Delta_{4,3}) = (12, 36, 24)$ so its Euler characteristics is $\chi(\Delta_{4,3}) = 0$. Moreover, the geometric realization of $\Delta_{4,3}$ is an orientable 2-dimensional manifold.

Indeed, the link of each vertex is isomorphic to $\Delta_{3,2}$ (= hexagonal triangulation of the circle S^1) while the link of each edge is the circle S^0 . Each 2-dimensional simplex $\sigma = \{A_i, B_j, C_k\}$ is uniquely completed to a permutation $\pi = (i, j, k, l)$ of the set $[4] = \{1, 2, 3, 4\}$ and $\text{Sign}(\sigma) := \text{Sign}(\pi)$ defines an orientation on $\Delta_{4,3}$.

From here we immediately conclude that $\Delta_{4,3}$ is a triangulation of the 2-dimensional torus T^2 . The universal covering of $\Delta_{4,3}$ is identified as the honeycomb tiling of the plane and the corresponding fundamental domain is exhibited in Figure 1. From here we can easily read off the generators of the group $H_1(\Delta_{4,3}; \mathbb{Z}) \cong \mathbb{Z}^2$ as the geodesic edge-paths connecting the three copies of vertex C_3 , shown in Figure 1.

Figure 1: Chessboard complex $\Delta_{4,3}$

2.2. Graph complexes

Let G be a finite graph with vertex set $V = V_G$ and edge set $E = E_G$. A graph complex on G is an abstract simplicial complex consisting of subsets of E . We usually interpret such a complex as a family of subgraphs of G . The study of graph complexes, with the emphasis on their homology, homotopy type, connectivity degree, Cohen-Macaulayness, etc., has been an active area of study in topological combinatorics, see [25].

The chessboard complex $\Delta_{m,n}$ can be interpreted as a graph complex of the complete bipartite graph $K_{m,n}$, where the simplices $S \subset [m] \times [n]$ are interpreted as “matchings” in $K_{m,n}$. Recall that $\Gamma \subseteq E_G$ is a matching on the graph G if each $v \in V_G$ is incident to at most one edge in Γ .

All “generalized chessboard complexes”, introduced in Section 3, can be also described as graph complexes of the graph $K_{m,n}$.

2.3. Chessboard complexes as Tits coset complexes

Perhaps the first appearance of chessboard complexes was in the thesis of Garst [16], as *Tits coset complexes*. Recall that a Tits coset complex $\Delta(G; G_1, \dots, G_n)$, associated to a group G and a family $\{G_1, \dots, G_n\}$ of its subgroups is the nerve $Nerve(\mathcal{F})$ of the associated family of cosets $\mathcal{F} = \{gG_i \mid g \in G, i \in [n]\}$. More explicitly vertices of $\Delta(G; G_1, \dots, G_n)$ are cosets gG_i and a collection $S = \{g_j G_i\}_{(i,j) \in I}$, for some $I \subseteq [n] \times G$, is a simplex of $\Delta(G; G_1, \dots, G_m)$ if and only if

$$\bigcap_{(i,j) \in I} g_j G_i \neq \emptyset.$$

If $G = S_m$ is the symmetric group and $G_i := \{\pi \in S_m \mid \pi(i) = i\}$ for $i = 1, \dots, n$, the associated Tits coset complex is the chessboard complex $\Delta_{m,n}$.

2.4. Chessboard complexes in discrete geometry

Chessboard complexes made their first appearance in discrete geometry in [40], in the context of the so called *colored Tverberg problem*.

For illustration, an instance of the *type B* colored Tverberg theorem [35, 41] claims that for each collection $C \subset \mathbb{R}^3$ of fifteen points in the 3-space, evenly colored by three colors, there exist three vertex disjoint triangles $\Delta_1, \Delta_2, \Delta_3$, formed by the points of different color, such that $\Delta_1 \cap \Delta_2 \cap \Delta_3 \neq \emptyset$.

A general form of this result was deduced in [35] from a Borsuk-Ulam type result claiming that each \mathbb{Z}_r -equivariant map

$$(\Delta_{r,2r-1})^{*(k+1)} \xrightarrow{\mathbb{Z}_r} W_r^{\oplus(d+1)} \quad (1)$$

must have a zero if $r \leq d/(d-k)$ (this is a necessary condition), r is a prime power, $\Delta_{r,2r-1}$ is a chessboard complex, and $W_r = \{x \in \mathbb{R}^r \mid x_1 + \dots + x_r = 0\}$.

The reader is referred to [41] for an overview of these and more recent results, as well as for a more complete list of references.

3. Generalized chessboard complexes

Motivated primarily by applications to problems in discrete geometry, especially the problems of Tverberg and Van Kampen-Flores type, more general chessboard complexes were introduced and studied. Closely related complexes previously emerged in algebraic combinatorics [28, 38].

These complexes are also referred to as *generalized chessboard complexes*, since the set of vertices remains the $(m \times n)$ -chessboard $[m] \times [n]$, but the criterion for $S \subseteq [m] \times [n]$ to be a simplex (“admissible rook placement”) may be quite different and vary from problem to problem.

The following definition includes most if not all of the currently studied examples and provides a natural ecological niche for all these complexes and their relatives.

DEFINITION 1. Suppose that $\mathcal{K} = \{K_i\}_{i=1}^n$ and $\mathcal{L} = \{L_j\}_{j=1}^m$ are two collections of simplicial complexes where $\text{Vert}(K_i) = [m]$ for each $i \in [n]$ and $\text{Vert}(L_j) = [n]$ for each $j \in [m]$. Define,

$$\Delta_{m,n}^{\mathcal{K},\mathcal{L}} = \Delta_{m,n}(\mathcal{K}, \mathcal{L}) \quad (2)$$

as the complex of all subsets (rook-placements) $A \subset [m] \times [n]$ such that $\{i \in [m] \mid (i, j) \in A\} \in K_j$ for each $j \in [n]$ and $\{j \in [n] \mid (i, j) \in A\} \in L_i$ for each $i \in [m]$.

Definition 1 can be specialized in many ways. Again, we focus on the special cases motivated by intended applications to the generalized Tverberg problem.

DEFINITION 2. Suppose that $\mathbf{k} = (k_i)_{i=1}^n$ and $\mathbf{l} = (l_j)_{j=1}^m$ are two sequences of non-negative integers. Then the complex,

$$\Delta_{m,n}^{\mathbf{k},\mathbf{l}} = \Delta_{m,n}^{k_1,\dots,k_n;l_1,\dots,l_m} \quad (3)$$

arises as the complex of all rook-placements $A \subset [m] \times [n]$ such that at most k_i rooks are allowed to be in the i -th row (for $i = 1, \dots, n$), and at most l_j rooks are allowed to be in the j -th column (for $j = 1, \dots, m$).

REMARK 1. The complexes $\Delta_{m,n}^{\mathbf{k},\mathbf{l}} = \Delta_{m,n}^{k_1,\dots,k_n;l_1,\dots,l_m}$ are sometimes referred to as the *chessboard complexes with multiplicities* or *multiple chessboard complexes*. Closely related are “bounded degree graph complexes”, studied in [28] and [38].

When $k_1 = \dots = k_n = p$ and $l_1 = \dots = l_m = q$, we obtain the complex $\Delta_{m,n}^{p,q}$. For the reasons which will become clear in the following section of the paper, in our earlier papers [22, 23] we focused to the case $l_1 = \dots = l_m = 1$, i.e. to the complexes,

$$\Delta_{m,n}^{k_1,\dots,k_n;\mathbf{1}} := \Delta_{m,n}^{k_1,\dots,k_n;1,\dots,1}. \quad (4)$$

In Section 6 of this paper we fill this “gap” and return to the case of general chessboard complexes with multiplicities.

3.1. n -fold j -wise deleted join

Joins and deleted joins of simplicial complexes, as well as their generalizations, have found numerous applications in topological combinatorics, see [31, Section 6.3] for motivation and an introduction.

For a simplicial complex K , the n -fold j -wise deleted join of K is

$$K_{\Delta(j)}^{*n} := \{A_1 \uplus A_2 \uplus \cdots \uplus A_n \in K^* \mid (A_1, A_2, \dots, A_n) \text{ is } j\text{-wise disjoint}\} \quad (5)$$

where an n -tuple (A_1, A_2, \dots, A_n) is j -wise disjoint if every sub-collection $\{A_{k_i}\}_{i=1}^j$, where $k_1 < k_2 < \cdots < k_j$, has an empty intersection.

It immediately follows that $K_{\Delta(j)}^{*n} \subseteq K_{\Delta(j+1)}^{*n}$ and that $K_{\Delta(n+1)}^{*n} = K^{*n}$ and $K_{\Delta(2)}^{*n} = K_{\Delta}^{*n}$ are respectively the n -fold join and the n -fold deleted join of the complex K .

A simple but very useful property of these operations is that they commute

$$(K_{\Delta(j)}^{*n})_{\Delta(k)}^{*m} \cong (K_{\Delta(k)}^{*m})_{\Delta(j)}^{*n}.$$

For example if $K = pt$ is a one-point simplicial complex we obtain the isomorphism

$$\Delta_{m,n} = ((pt)_{\Delta}^{*m})_{\Delta}^{*n} \cong ((pt)_{\Delta}^{*n})_{\Delta}^{*m} = \Delta_{n,m}.$$

A single complex K in equation (5) can be replaced by a collection $\mathcal{K} = \{K_j\}_{j=1}^n$ of complexes $K_j \subseteq 2^{[m]}$ which leads to the definition of the j -wise deleted join of \mathcal{K} ,

$$\mathcal{K}_{\Delta(j)} := \{A_1 \uplus \cdots \uplus A_n \in K_1 * \cdots * K_n \mid (A_1, \dots, A_n) \text{ is } j\text{-wise disjoint}\}.$$

All simplicial complexes described in this section are generalized chessboard complexes in the sense of Definition 1. For example if $K \subseteq 2^{[m]}$ then its n -fold j -wise deleted join is the complex

$$K_{\Delta(j)}^n \cong \Delta_{m,n}^{\mathcal{K}, \mathcal{L}}$$

where $K_1 = \cdots = K_n$ and $L_1 = \cdots = L_m = \binom{[m]}{\leq j-1}$ is the collection of all subsets of $[m]$ of cardinality strictly less than j .

3.2. Bier spheres as generalized chessboard complexes

Let $K \subsetneq 2^{[m]}$ be a simplicial complex on the ground set $[m]$ (meaning that we allow $\{j\} \notin K$ for some $j \in [m]$). The Alexander dual of K is the simplicial complex K° that consists of the complements of all nonsimplices of K

$$K^\circ := \{A^c \mid A \notin K\}.$$

By definition the “*Bier sphere*” is the deleted join $Bier(K) := K *_\Delta K^\circ$. (A face $A_1 \uplus A_2 \in Bier(K)$ is often denoted as a triple $(A_1, A_2; B)$ where $B := [m] \setminus (A_1 \cup A_2)$.)

It turns out that $Bier(K)$ is indeed a triangulation of an $(m-2)$ -dimensional sphere [4], see [31] and [30] for different, very short and elegant proofs.

The Bier sphere $Bier(K)$ is also a generalized chessboard complex where $K_1 = K, K_2 = K^\circ$ and $L_1 = \cdots = L_m = \{\emptyset, \{1\}, \{2\}\} \subset 2^{[2]}$.

Alexander r -tuples $\mathcal{K} = \{K_i\}_{i=1}^r$ of simplicial complexes were introduced in [18] as a generalization of pairs (K, K°) of Alexander dual complexes. The associated *generalized Bier complexes*, defined as the r -fold deleted joins $\mathcal{K}_{\Delta}^{*r}$ of Alexander r -tuples are also generalized chessboard complexes in the sense of Definition 1.

4. Discrete Morse theory

A discrete Morse function on a simplicial complex $K \subseteq 2^V$ is, by definition, an acyclic matching on the Hasse diagram of the partially ordered set (K, \subseteq) . Here is a brief reminder of the basic facts and definitions of discrete Morse theory.

Let K be a simplicial complex. Its p -dimensional simplices (p -simplices for short) are denoted by $\alpha^p, \alpha_i^p, \beta^p, \sigma^p$, etc. A *discrete vector field* is a set of pairs $D = \{\dots, (\alpha^p, \beta^{p+1}), \dots\}$ (called a matching) such that:

- (a) each simplex of the complex participates in at most one pair;
- (b) in each pair $(\alpha^p, \beta^{p+1}) \in D$, the simplex α^p is a facet of β^{p+1} ;
- (c) the empty set $\emptyset \in K$ is not matched, i.e. if $(\alpha^p, \beta^{p+1}) \in D$ then $p \geq 0$.

The pair (α^p, β^{p+1}) can be informally thought of as a vector in the vector field D . For this reason it is occasionally denoted by $\alpha^p \rightarrow \beta^{p+1}$ or $\alpha^p \nearrow \beta^{p+1}$ (and in this case α^p and β^{p+1} are informally referred to as the *beginning* and the *end* of the arrow $\alpha^p \rightarrow \beta^{p+1}$).

Given a discrete vector field D , a *gradient path* in D is a sequence of simplices (a zig-zag path)

$$\alpha_0^p \nearrow \beta_0^{p+1} \searrow \alpha_1^p \nearrow \beta_1^{p+1} \searrow \alpha_2^p \nearrow \beta_2^{p+1} \searrow \dots \searrow \alpha_m^p \nearrow \beta_m^{p+1} \searrow \alpha_{m+1}^p$$

satisfying the following conditions:

- 1. $(\alpha_i^p, \beta_i^{p+1})$ is a pair in D for each i ;
- 2. for each $i = 0, \dots, m$ the simplex α_{i+1}^p is a facet of β_i^{p+1} ;
- 3. for each $i = 0, \dots, m-1$, $\alpha_i \neq \alpha_{i+1}$.

A path is *closed* if $\alpha_{m+1}^p = \alpha_0^p$. A *discrete Morse function* (DMF for short) is a discrete vector field without closed paths.

Assuming that a discrete Morse function is fixed, the *critical simplices* are those simplices of the complex that are not matched. The Morse inequality [13] implies that critical simplices cannot be completely avoided.

A discrete Morse function D is *perfect* if the number of critical k -simplices equals the k -th Betty number of the complex. It follows that D is a perfect Morse function if and only if the number of all critical simplices equals the sum of all Betty numbers of K .

A central idea of discrete Morse theory, as summarized in the following theorem of R. Forman, is to contract all matched pairs of simplices and to reduce the simplicial complex K to a cell complex (where critical simplices correspond to the cells).

THEOREM 1. [13] *Assume that a discrete Morse function on a simplicial complex K has a single zero-dimensional critical simplex σ^0 and that all other critical simplices have the same dimension $N > 1$. Then K is homotopy equivalent to a wedge of N -dimensional spheres.*

More generally, if all critical simplices, aside from σ^0 , have dimension $\geq N$, then the complex K is $(N-1)$ -connected. \square

4.1. Discrete vector fields on Bier spheres

It is known that all Bier spheres are shellable, see [6] and [10]. A method of Chari [9] can be used to turn this shelling into a perfect discrete Morse function (DMF). The construction of our *perfect DMF* on a Bier sphere essentially follows this path, see [18] for more details. For the reader's convenience here we reproduce this construction since it will be needed in Section 5.

A perfect DMF on $Bier(K)$

We construct a discrete vector field D_1 on the Bier sphere $Bier(K)$ in two steps:

(1) We match the simplices

$$\alpha = (A_1, A_2; B \cup i) \text{ and } \beta = (A_1, A_2 \cup i; B)$$

iff the following holds:

- (i) $i < B$, $i < A_2$
(that is, i is smaller than all the entries of B and A_2).
- (ii) $A_2 \cup i \in K^\circ$.

Before we pass to step 2, let us observe that the non-matched simplices are labelled by $(A_1, A_2; B \cup i)$ such that $A_2 \in K^\circ$, but $A_2 \cup i \notin K^\circ$. As a consequence, for non-matched simplices $A_1 \cup B \in K$.

(2) In the second step we match together the simplices

$$\alpha = (A_1, A_2; B \cup j) \text{ and } \beta = (A_1 \cup j, A_2; B)$$

iff the following holds:

- (a) None of the simplices α and β is matched in the first step.
- (b) $j > B$, $j > A_1$.
- (c) $A_1 \cup j \in K$.

Observe that the condition (c) always holds (provided that the condition (a) is satisfied), except for the case $B = \emptyset$.

LEMMA 1. (see [18, Lemma 6.1]) *The discrete vector field D_1 is a discrete Morse function on the Bier sphere $\text{Bier}(K)$.*

Proof. Since D_1 is (by construction) a discrete vector field, it remains to check that there are no closed gradient paths. Observe that in each pair of simplices in the discrete vector field D_1 there is exactly one *migrating element*. More precisely, in the case (1) the element i migrates to A_2 , and in the case (2) the element j migrates to A_1 .

The lemma follows from the observation that (along a gradient path) the values of the migrating element that move to A_2 strictly decreases. Similarly, the values of migrating elements that move to A_1 can only increase. This is certified through the following simple case analysis: (1) After a first step pairing comes a splitting of A_2 . Then the gradient path terminates. (2) After a first step pairing (with migrating element i) comes a splitting of A_1 . The gradient path proceeds only if the splitted element is smaller than i . (2) After a second step pairing comes a splitting of A_1 . Then the gradient path terminates. (2) After a second step pairing (with migrating element i) comes a splitting of A_2 . The gradient path proceeds only if the splitted element is bigger than i .

Let us illustrate this observation by an example which captures the above case analysis. Assume we have a fragment of a gradient path that contains two matchings of type 1. We have:

$$\begin{aligned} (A_1 \cup k, A_2; B \cup i) &\rightarrow (A_1 \cup k, A_2 \cup i; B) \rightarrow \\ &(A_1, A_2 \cup i; B \cup k) \rightarrow (A_1, A_2 \cup k \cup i; B) \end{aligned}$$

The migrating elements here are i and k . The definition of the matching D_1 implies $k < i$. Otherwise $(A_1, A_2 \cup i; B \cup k)$ is matched with $(A_1, A_2; B \cup k \cup i)$, and the path would terminate after its second term. \square

It is not difficult to see that there are precisely two critical simplices in D_1 :

1. An $(n - 2)$ -dimensional simplex,

$$(A_1, A_2; i)$$

where $A_1 < i < A_2$, (this condition describes this simplex uniquely, in light of the fact that $A_1 \in K$ and $A_2 \in K^\circ$),

2. and the 0-dimensional simplex,

$$(\emptyset, \{1\}; \{2, 3, 4, \dots, n\}).$$

(Here we make a simplifying assumption that $\{1\} \in K^\circ$, which can be always achieved by a re-enumeration, except in the trivial case $K^\circ = \{\emptyset\}$.)

4.2. Discrete vector fields on generalized chessboard complexes

The construction of the discrete Morse function on the Bier sphere $Bier(K)$ illustrates the fruitful idea which can be extended and further developed to cover the case of other generalized chessboard complexes.

Examples of this construction can be found in [18] and [20], see also Section 6 for a construction of such a discrete Morse function on the multiple chessboard complex $\Delta_{m,n}^{k_1, \dots, k_n; l_1, \dots, l_m}$.

All these constructions of DMF share the same basic idea, for this reason we sometimes refer to them as *standard DMF on generalized chessboard complexes*. Note that the proofs that they indeed form an acyclic matching may vary from example to example and use some special properties of the class under investigation.

5. Edmonds-Fulkerson bottleneck extrema

In this section we connect, via discrete Morse theory, the combinatorial topology of Bier spheres with Edmonds-Fulkerson theorem on bottleneck extrema of pairs of dual clutters. We will show that there is much more than meets the eye in the standard concise treatment of this classical result of combinatorial optimization.

ABSTRACT

Let E be a finite set. Call a family of mutually noncomparable subsets of E a clutter on E . It is shown that for any clutter \mathcal{R} on E , there exists a unique clutter \mathcal{S} on E such that, for any function f from E to real numbers,

$$\min_{R \in \mathcal{R}} \max_{x \in R} f(x) = \max_{S \in \mathcal{S}} \min_{x \in S} f(x).$$

Specifically, \mathcal{S} consists of the minimal subsets of E that have non-empty intersection with every member of \mathcal{R} . The pair $(\mathcal{R}, \mathcal{S})$ is called a blocking system on E . An algorithm is described and several examples of blockings systems are discussed.

Figure 2: Edmonds-Fulkerson bottleneck theorem

Figure 2 shows the abstract of the published version of [11], which originally appeared as a RAND-corporation preprint AD 664879 in January of 1966.

This is a purely combinatorial result which is often referred to as the Edmonds-Fulkerson bottleneck lemma (theorem). Minmax theorems are ubiquitous in mathematics, notably in geometry, polyhedral combinatorics, critical point theory, game theory and other areas. One of

early examples is the minimax theorem of John von Neumann (first proven and published in 1928) which gives conditions on a function $f : C \times D \rightarrow \mathbb{R}$, defined on the product of two closed, convex sets in \mathbb{R}^n , to satisfy the minmax equality,

$$\operatorname{Res} \max_{y \in D} \min_{x \in C} f(x, y) = \max_{x \in C} \operatorname{Res} \min_{y \in D} f(x, y). \quad (6)$$

It is interesting to compare the Edmonds-Fulkerson minmax theorem with their geometric counterparts. For example in a vicinity of a non-degenerate critical point a Morse function has the form $f(x, y) = -|x|^2 + |y|^2 = -x_1^2 - \dots - x_p^2 + y_1^2 + \dots + y_q^2$. Moreover, this function satisfies the concave/convex condition of von Neumann's minmax theorem and the relation (6) is valid.

There is a formal resemblance of these results, for example the x -sections (respectively y -sections) of the convex sets $C \times D$ in (6) formally play the role of complementary clutters \mathcal{R} and \mathcal{S} from the result of Edmonds and Fulkerson. At first sight it appears to be naive and hard to expect a deeper connection between these results. Indeed, the clutter $\{C \times \{y\}\}_{y \in D}$ of y -sections is nowhere near to be the complementary clutter of the set $\{\{x\} \times D\}_{x \in C}$ of all x -sections, which is a consequence of the following lemma (see the property (3) on page 301 in [11]).

LEMMA 2. *The clutter $\mathcal{S} \subset 2^E$ is the complementary clutter of the clutter $\mathcal{R} \subset 2^E$, if and only if for each partition $E = E_0 \uplus E_1$ of E either an element of \mathcal{R} is contained in E_0 or an element of \mathcal{S} is contained in E_1 , but not both.*

In the next section we show that there does exist a geometric interpretation of the Edmonds-Fulkerson bottleneck minmax equality, provided we are willing to replace the smooth by discrete Morse theory.

5.1. Edmonds-Fulkerson minmax lemma revisited

Here we use the results from Section 4.1 to give a new proof and a new interpretation of Edmonds-Fulkerson minmax lemma. As before (Figure 2) the clutters \mathcal{R} and \mathcal{S} are both subfamilies of 2^E .

Let $\widehat{\mathcal{R}} := \{A \subseteq E \mid (\exists X \in \mathcal{R}) X \subseteq A\}$ be the upper closure of the clutter \mathcal{R} and let $K := 2^E \setminus \widehat{\mathcal{R}}$ be the complementary simplicial complex.

LEMMA 3. *Let K° be the Alexander dual of the simplicial complex $K := 2^E \setminus \widehat{\mathcal{R}}$. Then*

$$K^\circ = 2^E \setminus \widehat{\mathcal{S}}$$

is the complementary simplicial complex of the upper closure $\widehat{\mathcal{S}}$ of the clutter \mathcal{S} .

PROOF. This is an immediate consequence of Lemma 2 since the pair of complexes (K, K°) is also characterized by the property that for each partition $E = E_0 \uplus E_1$ precisely one of the following two relations $E_0 \in K$, $E_1 \in K^\circ$ is satisfied. \square

Let $f : E \rightarrow \mathbb{R}$ be a real function. We may assume that f is 1-1. Moreover, we may replace E by the set $[n]$ (where n is the cardinality of E) and assume that $f = id : [n] \rightarrow [n]$ is the identity function.

By construction and properties of the perfect DMF on the Bier sphere $Bier(K) = K *_\Delta K^\circ$, constructed in Section 4.1, there is a unique $(n - 2)$ -dimensional critical simplex $(A_1, A_2; i)$, characterized by the conditions $A_1 < i < A_2$, $A_1 \in K$, $A_2 \in K^\circ$. Let us show that

$$a := \operatorname{Res} \max_{I \in \mathcal{R}} \min_{x \in I} f(x) = f(i) = \max_{J \in \mathcal{S}} \operatorname{Res} \min_{x \in J} f(x) =: b. \quad (7)$$

Indeed, $A_1 \cup \{i\} \notin K$ implies $A_1 \cup \{i\} \in \mathcal{R}$ and from $\max_{x \in A_1 \cup \{i\}} f(x) = f(i)$ we deduce the relation $a \leq f(i)$.

For the opposite inequality observe that if $I \in \mathcal{R}$ then $I \cap (A_2 \cup \{i\}) \neq \emptyset$ (otherwise, since $A_2 \cup \{i\} \in \mathcal{S}$, Lemma 2 would be violated). Hence, $\max_{x \in I} f(x) \geq f(i)$ and $a \geq f(i)$.

The proof of the equality $b = f(i)$ is similar. \square

REMARK 2. One of the consequences is that the (algorithmic) complexity of determining the critical cell $(A_1, A_2; i)$ in the Bier sphere $Bier(K)$ is at least as big as the complexity of evaluating the maxmin (minmax) of a function on a family of sets (clutter).

6. Discrete Morse theory for chessboard complexes with multiplicities

Suppose that k_1, \dots, k_n and l_1, \dots, l_m are two sequences of non-negative integers. The generalized chessboard complex $\Delta_{m,n}^{k_1, \dots, k_n; l_1, \dots, l_m}$ contains all rooks placements on $[n] \times [m]$ table such that at most k_i rooks are in the i -th row and at most l_j rooks are in the j -th column. We use Forman's discrete Morse theory to obtain a generalization of Theorem 3.2 from [22].

THEOREM 2. *If*

$$l_1 + l_2 + \dots + l_m \geq k_1 + k_2 + \dots + k_n + n - 1 \quad (*)$$

then $\Delta_{m,n}^{k_1, \dots, k_n; l_1, \dots, l_m}$ is $(k_1 + k_2 + \dots + k_n - 2)$ -connected.

PROOF. A column (or a row) is called *full* if it contains the maximal allowed number of rooks. Otherwise, it is called *free*.

We now define a Morse matching for $\Delta = \Delta_{m,n}^{k_1, \dots, k_n; l_1, \dots, l_m}$. For a given face R we describe a face R' that is paired with R , or we recognize that R is a critical face. Let us do it stepwise.

Step 1.

Take the minimal a_1 such that either (1) there is a rook positioned at $(1, a_1)$, or (2) the a_1 column is free.

In the first case (there is a rook at $(1, a_1)$), we match R and $R' = R \setminus \{(1, a_1)\}$.

This is always possible except for the unique exception, when R contains exactly one rook at $(1, 1)$.

In the second case we match R and $R' = R \cup \{(1, a_1)\}$ provided that R' belongs to Δ . The latter condition means that the first row in R is not full.

Clearly, after Step 1 the unmatched simplices are those with full first row, empty $(1, a_1)$, and a free column a_1 .

Step 2. We match some of the simplices that are unpaired on the first step.

1. If there is a rook at $(2, a_1)$, set $a_2 := a_1$ and match R and $R' = R \setminus \{(2, a_2)\}$.
2. If

- (a) there is no rook at $(2, a_1)$, and
- (b) the number of rooks in column a_1 is smaller than $l_{a_1} - 1$,

set $a_2 := a_1$ and match R and $R' = R \cup \{(2, a_2)\}$ provided that R' belongs to Δ . The latter condition means that the second row in R is not full.

Introduce also $T(R) := 2$. Its meaning is "the column $a_1 = a_2$ has been used twice".

3. If none of the above cases holds, set $a_2 > a_1$ to be the minimal number such that either (1) there is a rook positioned at $(2, a_2)$, or (2) the a_2 column is free.

The condition (*) guarantees that a_2 is well-defined.

If there is a rook at $(2, a_2)$, we match R and $R' = R \setminus \{(2, a_2)\}$.

Otherwise, we match R and $R' = R \cup \{(2, a_2)\}$ provided that R' belongs to Δ . The latter condition means that the second row in R is free.

In this case we set $T(R) := 1$, since the column a_2 has been used once.

Clearly, after Step 2 the unmatched simplices are those with full first and second rows, empty $(2, a_2)$, and a free column a_2 .

We proceed in the same manner. During the first $k - 1$ steps, some of the simplices become matched. Unmatched simplices have first $k - 1$ rows full. They also have no rook at $(k - 1, a_{k-1})$. Each unmatched simplex R is associated a number $T(R)$.

This is how a generic step looks like:

Step k.

1. If there is a rook at (k, a_{k-1}) , then match R and $R' = R \setminus \{(k, a_k)\}$.
2. If

- (a) there is no rook at (k, a_{k-1}) , and
- (b) the number of rooks in column a_{k-1} is smaller than $l_{a_{k-1}} - T(R)$,

set $a_k := a_{k-1}$ and match R and $R' = R \cup \{(k, a_k)\}$ provided that R' belongs to Δ . The latter condition means that the k -th row in R is free.

Set $T(R) := T(R) + 1$; this means that “now the column $a_k = a_{k-1}$ has been used $T(R)$ times”.

3. Otherwise, set $a_k > a_{k-1}$ to be the minimal number such that either (1) there is a rook positioned at (k, a_k) , or (2) the a_k column is free.

Next, we match R and $R' = R \setminus \{(2, a_2)\}$ or $R' = R \cup \{(2, a_2)\}$ provided that R' belongs to Δ .

If R is not matched, set $T(R) := 1$.

REMARK. If $k < n$, then (*) guarantees that a_k is well-defined. For the last row a_n is ill-defined if and only if (*) is an equality and R has all the rows full.

Eventually we have all the rows full for non-matched simplices (except for the unique zero-dimensional simplex).

Now let us prove that the above defined matching is acyclic. Take a directed path

$$R_1 \nearrow Q_1 \searrow R_2 \nearrow Q_2 \searrow \cdots$$

Recall that $R_i \nearrow Q_i$ if and only if $Q_i = R_i \cup \{(s_i, a_{s_i})\}$, the first $s_i - 1$ rows of R_i are full, and a_{s_i} is the first free column after a_{s_i-1} .

Let us prove that (s_i, a_{s_i}) strictly decreases along the path wrt lexicographic order. This will imply the acyclicity.

For $Q_i \searrow R_{i+1}$, we have $R_{i+1} = Q_i \setminus \{(p_i, q_i)\}$ for some $(p_i, q_i) \in Q_i$ (there are no conditions when we remove a rook from Q_i). It suffices to consider the first two steps in our directed path:

$$R_1 \nearrow Q_1 = R_1 \cup \{(s_1, a_{s_1})\} \searrow R_2 = Q_1 \setminus \{(p_2, q_2)\}.$$

- If $p_2 > s_1$ or $p_2 = s_1$ and $a_{s_1} < q_2$ (the removed rook is below or right on (s_1, a_{s_1}) , the added rook at the first step) our path stop, because R_2 is paired with $R_2 \setminus \{(s_1, a_{s_1})\}$.
- If $p_2 < s_1$ or $p_2 = s_1$ and $a_{s_1} > q_2$ (the removed rook is above or left (s_1, a_{s_1})), then we have that $s_2 < s_1$ or $s_2 = s_1$ and $a_{s_2} < a_{s_1}$.

Summarizing, all critical faces (except for the unique zero-dimensional one) have all the rows full. Therefore $\Delta_{m,n}^{k_1, \dots, k_n; l_1, \dots, l_m}$ is $(k_1 + k_2 + \dots + k_n - 2)$ -connected. \square

7. Tverberg-Van Kampen-Flores type results for j -wise disjoint partitions of a simplex

Recall that a coloring of a set $S \subset \mathbb{R}^d$ is a partition $S = S_1 \uplus \dots \uplus S_k$, where S_i are the corresponding monochromatic sets. By definition a subset $C \subseteq S$ is a *rainbow set* if it contains at most 1 point from each of the color classes S_i .

THEOREM 3. *Let r be a prime power and $j \geq 1$. Suppose that $\{S_i\}_{i=1}^k$ is a collection of k finite sets of points in \mathbb{R}^d (called colors). Assume that the cardinalities $m_i = |S_i|$ satisfy the inequality $jm_i - 1 \leq r$ for each $i = 1, \dots, k$. If $(r-1)(d+1) \leq (j-1)m - 1$, where $m := m_1 + \dots + m_k$, then it is possible to partition the set $S = S_1 \uplus \dots \uplus S_k$ into r rainbow, j -wise disjoint sets $S = C_1 \uplus \dots \uplus C_r$, so that their convex hulls intersect,*

$$\text{conv}(C_1) \cap \dots \cap \text{conv}(C_r) \neq \emptyset.$$

PROOF. The rainbow sets span the multicolored simplices which are encoded as the simplices of the simplicial complex $([pt]_{\Delta(2)}^{*(m_1)}) * \dots * ([pt]_{\Delta(2)}^{*(m_k)})$. Indeed these are precisely the simplices which are allowed to have at most 1 vertex in each of k different colors. The configuration space of all r -tuples of j -wise disjoint multicolored simplices is the simplicial complex,

$$K = (([pt]_{\Delta(2)}^{*(m_1)}) * \dots * ([pt]_{\Delta(2)}^{*(m_k)}))_{\Delta(j)}^{*r}$$

Since the join and deleted join commute, this complex is isomorphic to,

$$K = ([pt]_{\Delta(2)}^{*(m_1)})_{\Delta(j)}^{*r} * \dots * ([pt]_{\Delta(2)}^{*(m_k)})_{\Delta(j)}^{*r}$$

where pt is a one-point simplicial complex.

If we suppose, contrary to the statement of the theorem, that the intersection of images of any r , j -wise disjoint multicolored simplices is empty, the associated mapping $F : K \rightarrow (\mathbb{R}^d)^{*r}$ would miss the diagonal $D \subset (\mathbb{R}^d)^{*r}$. By composing this map with the orthogonal projection to D^\perp , and after the radial projection to the unit sphere in D^\perp , we obtain a $(\mathbb{Z}/p)^\alpha$ -equivariant mapping,

$$\tilde{F} : K \rightarrow S^{(r-1)(d+1)-1}.$$

The complex $([pt]_{\Delta(2)}^{*(m_i)})_{\Delta(j)}^{*r}$ is a multiple chessboard complex $\Delta_{m_i, r}^{1, j-1}$. Since by assumption $jm_i - 1 \leq r$, this complex is $(m_i(j-1) - 2)$ -connected by the main result from [22]. Hence the complex K is $(m(j-1) - 2)$ -connected. By our assumption $m(j-1) - 2 \geq (r-1)(d+1) - 1$, so in light of Volovikov's theorem [34] such a mapping \tilde{F} does not exist. \square

The following obvious corollary of Theorem 2 is more suitable for applications in the rest of the section.

COROLLARY 1. *By interchanging the rows and the columns of the multiple chessboard complex in Theorem 2, we obtain that the complex $\Delta_{m,n}^{k_1, \dots, k_n; l_1, \dots, l_m}$ is $(l_1 + \dots + l_m - 2)$ -connected if $l_1 + \dots + l_m \leq k_1 + \dots + k_n - m + 1$.*

THEOREM 4. *Let r be a prime power. Assume that positive integers k, r, N, j and d satisfy the inequalities $(k+1)r + r - 1 \leq (N+1)(j-1)$ and $(r-1)(d+1) + 1 \leq r(k+1)$. Then for every continuous map $f : \Delta^N \rightarrow \mathbb{R}^d$ there exist r, j -wise disjoint faces of the simplex Δ^N of dimension at most k , whose images have a nonempty intersection.*

PROOF. The faces of dimension at most k form the k -skeleton $(\Delta^N)^{(k)} = [pt]_{\Delta^{(k+2)}}^{*(N+1)}$. The configuration space of all r -tuples of j -wise disjoint k -dimensional faces of this skeleton is the simplicial complex,

$$K = ([pt]_{\Delta^{(k+2)}}^{*(N+1)})_{\Delta^{(j)}}^{*r}.$$

This is a generalized chessboard complex $K = \Delta_{N+1,r}^{k+1;j-1}$. Since by our assumption $(k+1)r \leq (N+1)(j-1) - r + 1$, this complex K is by Corollary 1 $((k+1)r - 2)$ -connected.

If we suppose, contrary to the statement of the theorem, that the intersection of images of any r, j -wise disjoint k -dimensional faces is empty, the associated mapping $F : K \rightarrow (\mathbb{R}^d)^{*r}$ would miss the diagonal D .

As in the proof of the previous theorem we obtain a $(\mathbb{Z}/p)^\alpha$ -equivariant mapping,

$$\tilde{F} : K \rightarrow S^{(r-1)(d+1)-1}.$$

We have already observed that K is $((k+1)r - 2)$ -connected, and by our assumption $r(k+1) - 2 \geq (r-1)(d+1) - 1$, so in light of Volovikov's theorem [34] such a mapping \tilde{F} does not exist. \square

THEOREM 5. *Let r be a prime power. Suppose that q, r, j and d are positive integers and let $\{S_i\}_{i=1}^k \subseteq \mathbb{R}^d$ is a collection of colored points where all color classes S_i are of the same cardinality m . Then if $qr \leq m(j-1) - r + 1$ and $(r-1)(d+1) + 1 \leq qrk$, then it is always possible to partition the set $S := \cup_{i=1}^k S_i$ into r j -wise disjoint sets containing at most q points of each color, so that their convex hulls $\text{conv}(S_i)$ have a non-empty intersection.*

PROOF. The sets containing at most q points of each color span the multicolored simplices which are encoded as the simplices of the simplicial complex $([pt]_{\Delta^{(q+1)}}^{*m})^{*k}$. Indeed, these are precisely the simplices which are allowed to have at most q vertices in each of k different colors. The configuration space of all r -tuples of j -wise disjoint multicolored simplices is the simplicial complex,

$$K = (([pt]_{\Delta^{(q+1)}}^{*m})^{*k})_{\Delta^{(j)}}^{*r}.$$

Since the join and deleted join commute, this complex is isomorphic to,

$$K = (([pt]_{\Delta^{(q+1)}}^{*m})_{\Delta^{(j)}}^{*r})^{*k}.$$

If we suppose, contrary to the statement of the theorem, that the intersection of images of any r, j -wise disjoint multicolored simplices is empty, the associated mapping $F : K \rightarrow (\mathbb{R}^d)^{*r}$ would miss the diagonal D . As before, by composing this map with the orthogonal projection to D^\perp , and after the radial projection to the unit sphere in D^\perp , we obtain a $(\mathbb{Z}/p)^\alpha$ -equivariant mapping,

$$\tilde{F} : K \rightarrow S^{(r-1)(d+1)-1}.$$

The complex $([pt]_{\Delta(q+1)}^{*m})_{\Delta(j)}^{*r}$ is a multiple chessboard complex $\Delta_{m,r}^{q,j-1}$. Since we assumed $qr \leq (j-1)m - r + 1$, this complex is $(qr-2)$ -connected by Corollary 1. Hence the complex K is $(qrk-2)$ -connected. By our assumption $qrk \geq (r-1)(d+1) + 1$, so in light of Volovikov's theorem [34] such a mapping \tilde{F} does not exist. \square

THEOREM 6. *Let r be a prime power. Suppose that q, r, j and d are positive integers and let $\{S_i\}_{i=1}^k \subseteq \mathbb{R}^d$ is a collection of colored points where all color classes S_i are of the same cardinality m . If $jm - 1 \leq qr$ and $(r-1)(d+1) + 1 \leq (j-1)mk$, then it is possible to divide all points in r, j -wise disjoint sets containing at most q points of each color, so that their convex hulls $\text{conv}(S_i)$ have a non-empty intersection.*

PROOF. As before the sets containing at most q points of each color span the multicolored simplices which are encoded as the simplices of the simplicial complex $([pt]_{\Delta(q+1)}^{*m})^{*k}$. Indeed these are precisely the simplices which are allowed to have at most q vertices in each of k different colors. The configuration space of all r -tuples of j -wise disjoint multicolored simplices is the simplicial complex,

$$K = ([pt]_{\Delta(q+1)}^{*m})_{\Delta(j)}^{*r}.$$

Since the join and deleted join commute, this complex is isomorphic to,

$$K = ([pt]_{\Delta(q+1)}^{*m})_{\Delta(j)}^{*r}.$$

If we suppose, contrary to the statement of the theorem, that the intersection of images of any r, j -wise disjoint multicolored simplices is empty, the associated mapping $F : K \rightarrow (\mathbb{R}^d)^{*r}$ would miss the diagonal D . As before, from here by an equivariant deformation we obtain a $(\mathbb{Z}/p)^\alpha$ -equivariant mapping,

$$\tilde{F} : K \rightarrow S^{(r-1)(d+1)-1}.$$

The complex $([pt]_{\Delta(q+1)}^{*m})_{\Delta(j)}^{*r}$ is the multiple chessboard complex $\Delta_{m,r}^{q,j-1}$. Since we assumed $(j-1)m \leq qr - m + 1$, this complex is $((j-1)m-2)$ -connected by Corollary 1. Hence the complex K is $((j-1)mk-2)$ -connected. By our assumption $(j-1)mk \geq (r-1)(d+1) + 1$, and again this is in contradiction with Volovikov's theorem [34]. \square

For illustration let us consider a very special case of this theorem $q = 1$ and $j = 2$.

THEOREM 7. *Let r be a prime power. Given k finite sets of points in \mathbb{R}^d (called colors), of m points each, so that $2m - 1 \leq r$ and $(r-1)(d+1) + 1 \leq mk$, it is possible to divide the points in r pairwise disjoint sets containing at most 1 point of each color, so that their convex hulls intersect.*

REMARK 3. It is easy to see that the assumptions on the total number of points is the best possible, since the set of $(r-1)(d+1)$ points in the general position could not be divided in r disjoint sets whose convex hulls intersect.

7.1. A comparison with known results

It is interesting to compare results from the previous section with similar results from [7] (Section 9). Note that the proof methods are quite different. We use high connectivity of the multiple chessboard complex, established in Section 6, while the authors of [7] use the 'constraint method', relying on the 'optimal colored Tverberg theorem' from [8], as a 'black box' result.

For illustration, let us compare our Theorem 7 to Theorem 9.1 from [7].

Let us choose $k \geq 2(d+1)$ in Theorem 7 and select the smallest m satisfying the inequality $(r-1)(d+1) + 1 \leq mk$, meaning that we are allowed to assume

$$(m-1)k < (r-1)(d+1) + 1 \leq mk.$$

From here we immediately deduce the inequality $2m-1 \leq r$ and, as a consequence of Theorem 7, we have the following result.

COROLLARY 2. *Let r be a prime power. Assume $k \geq 2(d+1)$ and choose m satisfying the inequality $(r-1)(d+1) + 1 \leq mk$. Suppose that $S \subset \mathbb{R}^d$ is a set of cardinality mk , evenly colored by k colors (meaning that $S = \cup_{i=1}^k S_i$ where $|S_i| = m$ for each i). Then it is possible to select r pairwise disjoint subsets $C_i \subset S$, containing at most 1 point of each color, so that $\cap_{i=1}^r \text{conv}(C_i) \neq \emptyset$.*

This result clearly follows from Theorem 9.1 if we assume that r is a prime. Corollary 2 illustrates the phenomenon that there exist instances of the ‘optimal colored Tverberg theorem’ (Theorem 9.1 in [7]) which remain valid if the condition on r being a prime is relaxed to r is a prime power.

7.2. A remark on Tverberg A-P conjecture

In this section we briefly discuss the problem whether each admissible r -tuple is Tverberg prescribable. This problem, as formulated in [7], will be referred to as the Tverberg A-P problem or the Tverberg A-P conjecture.

DEFINITION 3. *For $d \geq 1$ and $r \geq 2$, an r -tuple $d = (d_1, \dots, d_r)$ of integers is admissible if, $[\frac{d}{2}] \leq d_i \leq d$ for all i , and $\sum_{i=1}^r (d - d_i) \leq d$. An admissible r -tuple is Tverberg prescribable if there is an N such that for every continuous map $f : \Delta^N \rightarrow \mathbb{R}^d$ there is a Tverberg partition $\{\sigma_1, \dots, \sigma_r\}$ for f with $\dim(\sigma_i) = d_i$.*

QUESTION. (Tverberg A-P problem; [7] (Question 6.9.)) Is every admissible r -tuple Tverberg prescribable?

As shown in [14], (Theorem 2.8.), the answer to the above question is negative. It was also demonstrated that a more realistic conjecture arises if the condition $[\frac{d}{2}] \leq d_i \leq d$, in the definition of admissible r -tuple, is replaced by a stronger requirement $\frac{(r-1)}{r}(d-1) \leq d_i \leq d$ for all i .

Here we remark that a positive answer to the modified question is quite straightforward in the case $r \geq d$. Indeed, in this case we have for all i

$$d_i \geq \frac{(r-1)}{r}(d-1) \geq d-1 - \frac{(d-1)}{r} > d-2.$$

So, in this case each d_i is equal to either $d-1$ or d , and the A-P conjecture reduces to the ‘balanced case’, established in [23].

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