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Расстояния Громова — Хаусдорфа до симплексов и некоторые приложения к дискретной оптимизации

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Аннотация

В работе изучается взаимосвязь между расстоянием Громова — Хаусдорфа и задачами дискретной оптимизации. Расстояние Громова — Хаусдорфа до метрического пространства с одинаковыми непутевыми расстояниями используется используется для решения следующих проблем: вычисление длин ребер минимального остовного дерева для конечного метрического пространства; обобщенная пробам Борсука; вычисление хроматического числа и минимального размера клинкового покрытия для простого графа.

Ключевые слова: расстояние Громова — Хаусдорфа, минимальное остовное дерево, проблема Борсука, хроматическое число, клинковое покрытие, метрическая геометрия, дискретная оптимизация

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Gromov–Hausdorff Distances to Simplexes and Some Applications to Discrete Optimisation

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Abstract

Relations between Gromov–Hausdorff distance and Discrete Optimisation problems are discussed. We use the Gromov–Hausdorff distances to single-distance metric space for solving the following problems: calculation of lengths of minimum spanning tree edges of a finite metric space; generalised Borsuk problem; chromatic number and clique cover number of a simple graph calculation problems.

Keywords: Gromov–Hausdorff distance, Minimum spanning tree, Borsuk problem, chromatic number, clique covering, metric geometry, discrete optimisation

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1. Introduction

The aim of the paper is to demonstrate close connections between the geometry of Gromov–Hausdorff distance and such popular Discrete Optimisation problems as minimum spanning tree problem, Borsuk conjecture, estimation of chromatic number and clique cover number of a simple graph. We start with a short informal review, all necessary formal definitions can be found below.

A general concept of distance is usually used to measure a difference between objects under consideration. Distances have applications in almost all spheres of human activity, from Geography to Linguistics, from Biology to Theology. A great number of beautiful examples can be found in [1]. A natural idea to compare subsets of a given metric space or, more generally, compare different metric spaces using appropriate distances, leads to appearance of so-called *hyperspaces*, i.e., metric spaces of some spaces, see, for example [2]. For subsets A and B of a fixed metric space X, a natural distance function d_H was defined by F. Hausdorff [4] as the infimum of positive numbers r such that A is contained in the r-neighbourhood of B, and vice-versa. It is well-known that this function, referred as the *Hausdorff distance*, is a metric on the family of all closed bounded subsets of the metric space X, see for example [3]. The Hausdorff distance was generalised to the case of two metric spaces X and Y by D. Edwards [5] and independently by M. Gromov [6]. They suggested to take the infimum of the values $d_H\big(\varphi(X),\psi(Y)\big)$ over all possible isometrical embeddings $\varphi\,X\to Z$

and $\psi Y \to Z$ into all possible metric spaces Z. Now this value is referred as the *Gromov–Hausdorff distance* between X and Y. It is well-known that this distance function a metric on the family of isometry classes of compact metric spaces. The corresponding hyperspace is usually denoted by $\mathcal{M}$ and is referred as the *Gromov–Hausdorff space*.

The geometry of the Gromov–Hausdorff space is rather tricky and is intensively investigated by many authors, see a review in [3]. Recently the technique of closed optimal correspondences permitted to prove that the space $\mathcal{M}$ is geodesic [7], to describe some local and all global isometries of $\mathcal{M}$, see [8] and [9]. Since finite metric spaces form an everywhere dense subset of $\mathcal{M}$, the distances to such spaces and between such spaces play an important role in the research of geometry of $\mathcal{M}$. Important classes of such spaces are formed by the ones all whose non-zero distances are the same (so-called *single-distance spaces* or *simplexes*) and by the spaces whose non-zero distances take only two different values (so-called *two-distance spaces*). The authors, together with S. Illiadis and D. Grigor'ev, see [10], [11], calculated distances from any metric space to any simplex, and, as a particular case, the distances between any simplex and any 2-distance space, see [12]. It turns out that the Gromov–Hausdorff distance from a metric space X to a simplex "feels" somehow a geometry of partitions of the space X. The latter explains some relations between the Gromov–Hausdorff distance and Discrete Optimisation problems.

Many Discrete Optimisation problems are related to Geometry, have a long history, and are either unsolved yet, or solved only in some particular cases. Due to many natural applications and total computerisation Discrete Optimisation is one of the most fast developing branch of modern Mathematics. Describe shortly the problems considered in the paper. Start with a problem of metric minimum spanning trees.

For a finite subset M of a metric space X, consider the complete graph $K(M)$ with the vertex set M, endowed with the weight function whose value on an edge $\{x, y\}$ equals to the distance $|xy|$ between the points x and y in the space X. A *minimum spanning tree on M* is a subtree of $K(M)$ with the same vertex set M and the least possible total weight. It is well-known that such a tree can be always constructed (even in a polynomial time) by a greedy algorithm such as the Kruskal algorithm [13]. Generally speaking, a minimum spanning tree on a fixed subset $M\subset X$ is not defined uniquely, but the ordered list of the weights of edges is the same for all such trees. This list is referred as an mst-*spectrum of M*. It is shown, see Section 4.1 and paper [25], that the mst-spectrum of M can be calculated in terms of the Gromov–Hausdorff distance from M to the simplexes consisting of $k = 2, \ldots, \#M$ points and such that there diameters are sufficiently large (they has to be at least twice greater than the diameter of M).

Now, let us pass to *Borsuk Problem*. In 1933, a Polish mathematician Karol Borsuk asked the following question: How many parts one needs to partition an arbitrary subset of the Euclidean space into, to obtain pieces of smaller diameters? He made the following famous conjecture: Any bounded non-single-point subset of $\mathbb{R}^n$ can be partitioned into at most $n+1$ subsets, each of which has smaller diameter than the initial subset. K. Borsuk himself proved it for $n = 2$ and for a ball in 3-dimensional space, [14] and [15]. Next, the conjecture was proved by J. Perkal (1947), and independently, by H. G. Eggleston (1955) for $n = 3$, then in 1946 by H. Hadwiger [18] and [19] for convex subsets with smooth boundaries, then for central symmetric bodies by A. S. Riesling (1971), and to this moment almost everybody believed that it is true. However, in 1993 the conjecture was suddenly disproved in general case by J. Kahn, and G. Kalai, see [20]. They constructed a counterexample in dimension $n = 1325$, and also proved that the conjecture is not valid for all $n > 2014$. This estimate was consistently improved by Raigorodskii, $n \geqslant 561$, Hinrichs and Richter, $n \geqslant 298$, Bondarenko, $n \geqslant 65$, and Jenrich, $n \geqslant 64$, see details in a review [21]. Notice that all the examples are finite subsets of the corresponding spaces, and the best known results of Bondarenko [22] and Jenrich [23] are the 2-distance subsets of the unit sphere.

On the other hand, Lusternik and Schnirelmann [16], and a bit later independently Borsuk [14] and [15], see also [17], have shown that the standard sphere and the standard ball in $\mathbb{R}^n$, $n \geqslant 2$, cannot be partitioned into $m \leqslant n$ subsets having smaller diameters. Thus, the least possible number of parts of smaller diameter, necessary to partition the sphere and the ball in $\mathbb{R}^n$ equals $n+1$.

In the present paper we consider a generalized Borsuk problem, passing to an arbitrary bounded metric space X and its partitions of an arbitrary cardinality m (not necessary finite). We give a criterion solving the Borsuk problem in terms of the Gromov–Hausdorff distance. It is shown that to verify the existence of an m-partition into subsets of smaller diameter it suffices to calculate the Gromov–Hausdorff distance from the space X to a simplex having the cardinality m and a smaller diameter than X, see Section 4.2. As a corollary, a solution to the Borsuk problem for a 2-distance space X with distances $a<b$ is obtained in terms of the clique cover number of the simple graph G with vertex set X, whose vertices x and y are connected by an edge iff $|xy|=a$.

Recall that a *clique cover* of a given simple graph is a cover of the vertex set of the graph by subsets within which every two vertices are adjacent. Each such subset is called a *clique* and is a vertex set of a complete subgraph that is also referred as a *clique*. The minimum k for which a k-element clique cover exists is called the *clique cover number* of the given graph. Further, a *graph coloring* is an assignment of labels traditionally called “colors” to vertices of a graph in such a way that no two adjacent vertices are of the same color. The smallest number of colors needed to color a graph is called its *chromatic number*. It is well-known that the clique cover can be considered as a graph coloring of the dual graph, hence the clique cover number of a graph equals to the chromatic number of the dual one. Calculation and estimation of these numbers are very hard combinatorial problems related to many other problems of Discrete Optimisation, in particular, to Borsuk conjecture, see a review in [24]. We calculate the clique cover number of a simple graph and the chromatic number of a simple graph in terms of the Gromov–Hausdorff distance from an appropriate simplex to the 2-distance spaces constructed by the graph, see Section 4.2.

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To conclude this short Introduction, the authors use the opportunity to congratulate our Teacher, Anatoly Timofeevich Fomenko, on his 75th birthday and wish him good health, beautiful results and many birthdays ahead. He stimulated us to become professional mathematicians, and teaches us to work hard, to live in the World of Mathematics, and to be optimistic both in science and in life. We are infinitely thankful for his deep influence, kind care, permanent support and attention.

2. Preliminaries

Let X be an arbitrary nonempty set. Recall that a function on $\rho\, X\times X\to\mathbb{R}$ is called a *metric* if it is non-negative, non-degenerate, symmetric, and satisfies the triangle inequality. A set with a metric is called a *metric space*. If such a function ρ is permitted to take infinite values, then we call ρ *a generalized metric*. If we omit the non-degeneracy condition, i.e., permit $\rho(x,y)=0$ for some distinct x and y, then we change the term “metric” to *pseudometric*. If ρ is only non-negative, symmetric, and $\rho(x,x)=0$ for any $x\in X$, then we call such ρ *a distance function*, instead of metric or pseudometric. As a rule, if it is not ambiguous, we write $|xy|$ for $\rho(x,y)$.

In what follows all metric spaces are endowed with the corresponding metric topology. We also use the following notations. By $\#X$ we denote the cardinality of a set X. Let X be a metric space. The closure of a subset $A\subset X$ is denoted by $\bar{A}$. For its arbitrary nonempty subset $A\subset X$ and point $x\in X$ put $|xA|=|Ax|=\inf\big\{|ax|:a\in A\big\}$. Further, for $r\geqslant 0$ put

$$B_r(x)=\big\{y\in X:|xy|\leqslant r\big\},\quad\text{and}\quad U_r(x)=\big\{y\in X:|xy|<r\big\},$$

and

$$B_r(A) = \{y \in X : |Ay| \leqslant r\}, \quad \text{and} \quad U_r(A) = \{y \in X : |Ay| < r\}.$$

2.1. Hausdorff distance

Recall the basic results concerning the Hausdorff distance. The details can be found in [3]. For a set X, by $\mathcal{P}_0(X)$ we denote the collection of all nonempty subsets of X. Let X be a metric space. For any $A, B \in \mathcal{P}_0(X)$ we put

$$d_H^1(A,B) = \max\Bigl(\sup\bigl\{|aB| : a \in A\bigr\}, \sup\bigl\{|Ab| : b \in B\bigr\}\Bigr), \tag{1}$$

$$d_H^2(A,B) = \inf\bigl\{r \in [0,\infty] : A \subset B_r(B) \ \& \ B_r(A)^B\bigr\}, \tag{2}$$

$$d_H^3(A,B) = \inf\bigl\{r \in [0,\infty] : A \subset U_r(B) \ \& \ U_r(A)^B\bigr\}. \tag{3}$$

It is well-known that these three values coincide with each other, i.e., $d_H^1(A,B) = d_H^2(AB) = d_H^3(A,B)$ for any $A, B \in \mathcal{P}_0(X)$. The value $d_H^i(A,B)$ is denoted by $d_H(A,B)$. It is easy to see that d_H is non-negative, symmetric, and $d_H(A,A) = 0$ for any nonempty $A \subset X$, thus, d_H is a generalized distance on the family $\mathcal{P}_0(X)$ of all nonempty subsets of a metric space X, moreover, it is a generalized pseudometric on $\mathcal{P}_0(X)$, i.e., it satisfies the triangle inequality. The function d_H is referred as *Hausdorff distance*.

Further, by $\mathcal{H}(X) \subset \mathcal{P}_0(X)$ we denote the set of all nonempty closed bounded subsets of a metric space X. It is well-known that the Hausdorff distance d_H is a metric on $\mathcal{H}(X)$.

In what follows, speaking about the distance in $\mathcal{H}(X)$ we will always mean the Hausdorff distance. Notice that there are different notations for this hyperspace in the literature. We use the notation $\mathcal{H}(X)$ by virtue of the fact that this is the largest natural set of subsets of a metric space which the Hausdorff distance is a metric on.

Recall a few properties of the Hausdorff distance.

Proposition 1. *Let X be an arbitrary metric space.*

1. *The mapping $f\, X \to \mathcal{P}_0(X)$ given by the formula $f\, x \mapsto \{x\}$ is an isometric embedding.*
2. *For any $A, B \in \mathcal{P}_0(X)$ we have $d_H(A,B) = d_H(A,\bar{B}) = d_H(\bar{A},B) = d_H(\bar{A},\bar{B})$.*
3. *For any $A, B \in \mathcal{P}_0(X)$ we have $d_H(A,B) = 0$ if and only if $\bar{A} = \bar{B}$.*
4. *If $Y \subset X$ is an ε-net in $A \subset X$, then $d_H(A,Y) \leqslant \varepsilon$.*

2.2. Gromov–Hausdorff distance

Let X and Y be metric spaces. A triple (X', Y', Z) consisting of a metric space Z and its two subsets X' and Y' which are isometric respectively to X and Y is be called a *realization of the pair* (X, Y). Put

$$d_{GH}(X,Y) = \inf\bigl\{r \in \mathbb{R} : \exists \text{ a realization } (X', Y', Z) \text{ of } (X,Y) \text{ such that } d_H(X', Y') \leqslant r\bigr\}.$$

Remark 1. *The value $d_{GH}(X,Y)$ is evidently non-negative, symmetric, and $d_{GH}(X,X) = 0$ for any metric space X. Thus, d_{GH} is a generalized distance function on each set of metric spaces.*

Definition 1. *The value $d_{GH}(X,Y)$ is called* the Gromov–Hausdorff distance *between the metric spaces X and Y.*

It turns out that, to define the Gromov–Hausdorff distance, it suffices to consider only metric spaces of the form $(X\sqcup Y,\rho)$, where ρ extends the original metrics of X and Y, i.e., the restrictions of ρ onto X and Y coincide with the original metrics of these metric spaces. Such ρ is called an *admissible metric for X and Y*, and the set of all admissible metrics for given X and Y is denoted by $\mathcal{D}(X,Y)$.

Proposition 2. *For any metric spaces X and Y, we have*

$$d_{GH}(X,Y)=\inf\bigl\{\rho_H(X,Y):\rho\in\mathcal{D}(X,Y)\bigr\}. \tag{4}$$

It is well-known that, on every set of metric spaces, the function d_{GH} is a generalized pseudometric. If the diameters of all spaces in the family are bounded by the same number, then d_{GH} is a pseudometric. In general, d_{GH} is not a metric, it may equal zero for distinct metric spaces. However, if we restrict ourselves to compact metric spaces considered up to an isometry, then d_{GH} is a metric.

For specific calculations of the Gromov–Hausdorff distance, other equivalent definitions of this distance are useful.

Recall that *a relation* between sets X and Y is defined as a subset of the Cartesian product $X\times Y$. Similarly to the case of mappings, for each $\sigma\in\mathcal{P}_0(X\times Y)$ and for every $x\in X$ and $y\in Y$, there are defined the *image* $\sigma(x):=\bigl\{y\in Y:(x,y)\in\sigma\bigr\}$ of any $x\in X$ and the *pre-image* $\sigma^{-1}(y)=\bigl\{x\in X:(x,y)\in\sigma\bigr\}$ of any $y\in Y$. Also, for $A\subset X$ and $B\subset Y$ their *image* and *pre-image* are defined as the union of the images and pre-images of their elements, respectively.

Let $\pi_X\ X\times Y\to X$ and $\pi_Y\ X\times Y\to Y$ be the canonical projections, i.e., $\pi_X(x,y)=x$ and $\pi_Y(x,y)=y$. The restrictions of these mappings to each relation $\sigma\subset X\times Y$ are denoted in the same way. A relation R between X and Y is called a *correspondence* if the restrictions of the canonical projections π_X and π_Y onto R are surjective. In other words, for every $x\in X$ there exists $y\in Y$, and for every $y\in Y$ there exists $x\in X$, such that $(x,y)\in R$. Thus, the correspondence can be considered as a surjective multivalued mapping. By $\mathcal{R}(X,Y)$ we denote Tthe set of all correspondences between X and Y.

If X and Y are metric spaces, then for each relation $\sigma\in\mathcal{P}_0(X\times Y)$ its *distortion* $\operatorname{dis}\sigma$ as follows

$$\operatorname{dis}\sigma=\sup\Bigl\{\bigl||xx'|-|yy'|\bigr|:(x,y),\ (x',y')\in\sigma\Bigr\}.$$

Remark 2. *For any $\sigma_1,\sigma_2\in\mathcal{P}_0(X\times Y)$ such that $\sigma_1\subset\sigma_2$, we have* $\operatorname{dis}\sigma_1\leqslant\operatorname{dis}\sigma_2$.

The next constructions establish a link between correspondences from $\mathcal{R}(X,Y)$ and admissible metrics on $X\sqcup Y$. At first, let $\rho\in\mathcal{D}(X,Y)$ be an arbitrary admissible metric for metric spaces X and Y, and suppose that $\rho_H(X,Y)<\infty$. Choose an arbitrary $r\geqslant\rho_H(X,Y)$ such that the set $R_r^\rho=\{(x,y):\rho(x,y)\leqslant r\}$ is a correspondence between X and Y (it is so for any $r>\rho_H(X,Y)$). Then $\operatorname{dis}R_r^\rho\leqslant 2r$.

Conversely, consider an arbitrary correspondence $R\in\mathcal{R}(X,Y)$. Suppose that $\operatorname{dis}R<\infty$. Extend the metrics of X and Y up to a symmetric function ρ^R defined on $X\sqcup Y$ as follows:

$$\rho^R(x,y)=\rho^R(y,x)=\inf\bigl\{|xx'|+|yy'|+\frac12\operatorname{dis}R:(x',y')\in R\bigr\}.$$

If $\operatorname{dis}R>0$, then ρ^R is an admissible metric, and $\rho_H^R(X,Y)=\frac12\operatorname{dis}R$.

The key well-known result on the relation between the correspondences and the Gromov–Hausdorff distance is the following Theorem.

Theorem 1. *For any metric spaces X and Y the equality*

$$d_{GH}(X,Y)=\frac12\inf\bigl\{\operatorname{dis}R:R\in\mathcal{R}(X,Y)\bigr\}$$

holds.

2.3. Irreducible correspondences

For arbitrary nonempty sets X and Y, a correspondence $R\in\mathcal{R}(X,Y)$ is called *irreducible* if it is a minimal element of the set $\mathcal{R}(X,Y)$ with respect to the order given by the inclusion relation. The set of all irreducible correspondences between X and Y is denoted by $\mathcal{R}^0(X,Y)$.

The following result is evident.

Proposition 3. *A correspondence $R\in\mathcal{R}(X,Y)$ is irreducible if and only if for any $(x,y)\in R$ it holds*

$$\operatorname{Res}\bigl\{\#R(x),\#R^{-1}(y)\bigr\}=1.$$

Theorem 2. *Let X, Y be arbitrary nonempty sets. Then for every $R\in\mathcal{R}(X,Y)$ there exists $R^0\in\mathcal{R}^0(X,Y)$ such that $R^0\subset R$. In particular, $\mathcal{R}^0(X,Y)\neq\emptyset$.*

Theorems 2 and 1, together with Remark 2, implies

Corollary 1. *For any metric spaces X and Y we have*

$$d_{GH}(X,Y)=\frac{1}{2}\inf\bigl\{\operatorname{dis}R\mid R\in\mathcal{R}^0(X,Y)\bigr\}.$$

Now we give another useful description of irreducible correspondences.

Proposition 4. *For any nonempty sets X, Y, and each $R\in\mathcal{R}^0(X,Y)$, there exist and unique partitions $R_X=\{X_i\}_{i\in I}$ and $R_Y=\{Y_i\}_{i\in I}$ of the sets X and Y, respectively, such that $R=\cup_{i\in I}X_i\times Y_i$. Moreover, $R_X=\cup_{y\in Y}\bigl\{R^{-1}(y)\bigr\}$, $R_Y:=\cup_{x\in X}\bigl\{R(x)\bigr\}$,*

$$\{X_i\times Y_i\}_{i\in I}=\cup_{(x,y)\in R}\{R^{-1}(y)\times R(x)\},$$

and for each i it holds $\operatorname{Res}\{\#X_i,\#Y_i\}=1$.

Conversely, each set $R=\cup_{i\in I}X_i\times Y_i$, where $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ are partitions of nonempty sets X and Y, respectively, such that for each i it holds $\operatorname{Res}\{\#X_i,\#Y_i\}=1$, *is an irreducible correspondence between X and Y.*

Let X be an arbitrary set consisting of more than one point, and m a cardinal number, $2\leqslant m\leqslant\#X$. By $\mathcal{D}_m(X)$ we denote the family of all possible partitions of the set X into m nonempty subsets.

Now let X be a metric space. Then for each $D=\{X_i\}_{i\in I}\in\mathcal{D}_m(X)$ we put

$$\operatorname{diam}D=\sup_{i\in I}\operatorname{diam}X_i.$$

Further, for any nonempty $A,B\subset X$, we put $|AB|=\inf\bigl\{|ab|:(a,b)\in A\times B\bigr\}$, and $|AB|':=\sup\bigl\{|ab|:(a,b)\in A\times B\bigr\}$. Further, for each $D=\{X_i\}_{i\in I}\in\mathcal{D}_m(X)$ we put

$$\alpha(D)=\inf\bigl\{|X_iX_j|:i\neq j\bigr\}\quad\text{and}\quad\beta(D)=\sup\bigl\{|X_iX_j|':i\neq j\bigr\}.$$

Also notice that $|X_iX_i|=0$, $|X_iX_i|'=\operatorname{diam}X_i$, and hence, $\operatorname{diam}D=\sup_{i\in I}|X_iX_i|'$.

The next result follows easily from the definition of distortion, as well as from Proposition 4.

Proposition 5. *Let X and Y be arbitrary metric spaces, $D_X=\{X_i\}_{i\in I}$, $D_Y=\{Y_i\}_{i\in I}$, $\#I\geqslant2$, be some partitions of the spaces X and Y, respectively, and $R=\cup_{i\in I}X_i\times Y_i\in\mathcal{R}(X,Y)$. Then*

$$\begin{aligned}\operatorname{dis}R&=\sup\bigl\{|X_iX_j|'-|Y_iY_j|,\,|Y_iY_j|'-|X_iX_j|:i,j\in I\bigr\}=\\&=\sup\bigl\{\operatorname{diam}D_X,\,\operatorname{diam}D_Y,\,|X_iX_j|'-|Y_iY_j|,\,|Y_iY_j|'-|X_iX_j|:i,j\in I,\,i\neq j\bigr\}\leqslant\\&\leqslant\max\bigl\{\operatorname{diam}D_X,\operatorname{diam}D_Y,\beta(D_X)-\alpha(D_Y),\beta(D_Y)-\alpha(D_X)\bigr\}.\end{aligned}$$

It will also be convenient for us to represent a relation $\sigma \in \mathcal{P}_0(X\times Y)$ as a bipartite graph. Then the degree deg of each vertex is defined: $\deg_\sigma(x) = \#\sigma(x)$ and $\deg_\sigma(y) = \#\sigma^{-1}(y)$.

Remark 3. *Notice that if $R \in \mathcal{R}^0(X,Y)$, $x \in X$, and $\deg_R(x) > 1$, then for each $x' \in X$, $x' \neq x$, it holds $R(x) \cap R(x') = \emptyset$. Therefore, if $\#X \geqslant 2$ and $\#Y \geqslant 2$, then for any $R \in \mathcal{R}^0(X,Y)$ there is no $x \in X$ such that $\{x\}\times Y\subset R$.*

2.4. Some Examples and Estimates

Here we list several simple cases of exact calculation and estimate of the Gromov–Hausdorff distance.

Example 1. *Let Y be an arbitrary ε-net of a metric space X. Then $d_{GH}(X,Y) \leqslant d_H(X,Y) \leqslant \varepsilon$. Thus, every compact metric space is approximated (according to the Gromov-Hausdorff metric) with any accuracy by finite metric spaces.*

By Δ_1 we denote a single-point metric space.

Example 2. *Then for any metric space X we have*

$$d_{GH}(\Delta_1, X) = \frac{1}{2}\operatorname{diam}X.$$

Example 3. *Let X and Y be some metric spaces, and the diameter of one of them is finite. Then*

$$d_{GH}(X,Y) \geqslant \frac{1}{2}|\operatorname{diam}X - \operatorname{diam}Y|.$$

Example 4. *Let X and Y be some metric spaces, then*

$$d_{GH}(X,Y) \leqslant \frac{1}{2}\max\{\operatorname{diam}X, \operatorname{diam}Y\},$$

in particular, if X and Y are bounded metric spaces, then $d_{GH}(X,Y) < \infty$.

For an arbitrary metric space X and a real $\lambda > 0$, by λX we denote the metric space obtained from X by multiplying all distances by λ. For $\lambda = 0$ we set $\lambda X = \Delta_1$.

Example 5. *For any bounded metric space X and any $\lambda \geqslant 0$, $\mu \geqslant 0$, we have $d_{GH}(\lambda X, \mu X) = \frac{1}{2}|\lambda - \mu|\operatorname{diam}X$, in particularly, for any $0 \leqslant a < b$ the curve $\gamma(t) := t\,X$, $t \in [a,b]$, is shortest.*

Example 6. *Let X and Y be metric spaces, then for any $\lambda > 0$ we have $d_{GH}(\lambda X, \lambda Y) = \lambda\, d_{GH}(X,Y)$. If, in addition, $d_{GH}(X,Y) < \infty$, then the equality holds for all $\lambda \geqslant 0$.*

3. Gromov–Hausdorff Distance to Simplexes

By *simplex* we call a metric space, all whose non-zero distances equal to each other. If m is an arbitrary cardinal number, then by Δ_m we denote a simplex containing m points and such that all its non-zero distances equal 1. Thus, $\lambda\Delta_m$, $\lambda > 0$, is a simplex whose non-zero distances equal λ. Also, for arbitrary metric space X and $\lambda = 0$, the space λX coincides with Δ_1 by definitoon.

3.1. The Case of Simplexes of Greater Cardinality

The next result generalizes Theorem 4.1 from [10].

THEOREM 3. *Let X be an arbitrary metric space, $m > \#X$ a cardinal number, and $\lambda \geqslant 0$, then*

$$2d_{GH}(\lambda\Delta_m, X) = \max\{\lambda, \mathrm{diam}X - \lambda\}.$$

PROOF. If X is unbounded, then $2d_{GH}(\lambda\Delta_m, X) = \infty$ by Example 3, and the required equality holds.

Now, let $\mathrm{diam}X < \infty$.

If $\#X = 1$, then $\mathrm{diam}X = 0$, and, by Example 2, we have

$$2d_{GH}(\lambda\Delta, X) = \mathrm{diam}\lambda\Delta = \lambda = \max\{\lambda, \mathrm{diam}X - \lambda\}.$$

If $\lambda = 0$, then, by Example 2, we have

$$2d_{GH}(\Delta_1, X) = \mathrm{diam}X = \max\{\lambda, \mathrm{diam}X - \lambda\}.$$

Let $\#X > 1$ and $\lambda > 0$. Choose an arbitrary $R \in \mathcal{R}(\lambda\Delta_m, X)$. Since $\#X < m$ and $\lambda > 0$, then there exists $x \in X$ such that $\#R^{-1}(x) \geqslant 2$, thus, $\mathrm{dis}R \geqslant \lambda$ and $2d_{GH}(\lambda\Delta_m, X) \geqslant \lambda$.

Consider an arbitrary sequence $(x_i, y_i) \in X\times X$ such that $|x_iy_i| \to \mathrm{diam}X$. If it contains a subsequence (x_{i_k}, y_{i_k}) such that for each i_k there exists $z_k \in \lambda\Delta$, $(z_k, x_{i_k}) \in R$, $(z_k, y_{i_k}) \in R$, then $\mathrm{dis}R \geqslant \mathrm{diam}X$ and

$$2d_{GH}(\lambda\Delta_m, X) \geqslant \max\{\lambda, \mathrm{diam}X\} \geqslant \max\{\lambda, \mathrm{diam}X - \lambda\}.$$

If such subsequence does not exist, then there exists a subsequence (x_{i_k}, y_{i_k}) such that for any i_k there exist distinct $z_k, w_k \in \lambda\Delta_m$, $(z_k, x_{i_k}) \in R$, $(w_k, y_{i_k}) \in R$, and, therefore,

$$2d_{GH}(\lambda\Delta_m, X) \geqslant \max\bigl\{\lambda, |\mathrm{diam}X - \lambda|\bigr\} \geqslant \max\{\lambda, \mathrm{diam}X - \lambda\}.$$

Thus, in the both cases we have $2d_{GH}(\lambda\Delta, X) \geqslant \max\bigl\{\lambda, \mathrm{diam}X - \lambda\bigr\}$.

Choose an arbitrary $x_0 \in X$, then, by assumption, $\#X > 1$, and, thus, the set $X\backslash\{x_0\}$ is not empty. Since $\#X < m$, then $\lambda\Delta_m$ contains a subset $\lambda\Delta'$ of the same cardinality as $X\backslash\{x_0\}$. Let $g\,\lambda\Delta' \to X\backslash\{x_0\}$ be an arbitrary bijection, and $\lambda\Delta'' = \lambda\Delta_m\backslash\lambda\Delta'$, then $\#\lambda\Delta'' > 1$. Consider the following correspondence

$$R_0 = \Bigl\{\bigl(z', g(z')\bigr) : z' \in \lambda\Delta'\Bigr\} \cup \bigl(\lambda\Delta''\times\{x_0\}\bigr)$$

and apply Proposition 5. So, we have:

$$\mathrm{dis}R_0 = \sup\{\lambda, |x_1x_1'| - \lambda, \lambda - |x_2x_2'| : x_1, x_1', x_2, x_2' \in X,\ x_1 \neq x_1',\ x_2 \neq x_2'\} = \max\{\lambda, \mathrm{diam}X - \lambda\},$$

therefore,

$$2d_{GH}(\lambda\Delta, X) = \max\{\lambda, \mathrm{diam}X - \lambda\},$$

what is required. $\square$

3.2. The Case of Simplexes with at most the Same Cardinality

Let X be an arbitrary set consisting of more than one point, $2 \leqslant m \leqslant \#X$ a cardinal number, and $\lambda > 0$. Under notations of Subsection 2.3, consider an arbitrary $D \in \mathcal{D}_m(X)$, any bijection $g\,\lambda\Delta_m \to D$, and construct the correspondence $R_D \in \mathcal{R}(\lambda\Delta_m, X)$ in the following way:

$$R_D = \bigcup_{z\in\lambda\Delta_m} \{z\}\times g(z).$$

Clearly that the correspondence R_D is irreducible. Apply Proposition 5 to calculate its distortion.

Proposition 6. *Let $X \neq \Delta_1$ be an arbitrary metric space, $2 \leqslant m \leqslant \#X$ a cardinal number, and $\lambda > 0$. Then for any $D \in \mathcal{D}_m(X)$ it holds*

$$\operatorname{dis}R_D = \max\{\operatorname{diam}D,\, \lambda - \alpha(D),\, \beta(D) - \lambda\}.$$

Proof. If X is unbounded, then $\operatorname{dis}R = \infty$ for any $R \in \mathcal{R}(\lambda\Delta_m, X)$. Since $m \geqslant 2$, for any $D = \{X_i\}_{i\in I} \in \mathcal{D}_m(X)$ we have either $\operatorname{diam}D = \infty$, or $\beta(D) = \infty$. Indeed, if $\operatorname{diam}D < \infty$ and $\beta(D) < \infty$ then for any $x, y \in X$ either $x, y \in X_i$, thus $|xy| \leqslant \operatorname{diam}D$, or $x \in X_i$, $y \in X_j$, $i \neq j$, and $|xy| \leqslant |X_iX_j|' \leqslant \beta(D)$, therefore X is bounded. Thus, for an unbounded X the both sides of the equality are infinite, thus we get what is required.

Now, let $\operatorname{diam}X < \infty$. By Proposition 5, we have

$$\operatorname{dis}R_D = \sup\bigl\{\operatorname{diam}D,\, \lambda-|X_iX_j|,\, |X_iX_j|'-\lambda\colon i,j \in I,\, i \neq j\bigr\} = \max\{\operatorname{diam}D,\, \lambda-\alpha(D),\, \beta(D)-\lambda\},$$

that completes the proof. □

Corollary 2. *Let $X \neq \Delta_1$ be an arbitrary metric space, $2 \leqslant m \leqslant \#X$ a cardinal number, and $\lambda > 0$. Then for any $D \in \mathcal{D}_m(X)$ it holds*

$$\operatorname{dis}R_D = \max\{\operatorname{diam}D,\, \lambda - \alpha(D),\, \operatorname{diam}X - \lambda\}.$$

Proof. For unbounded X the equation evidently holds.

Consider now the case of bounded X. Notice that $\operatorname{diam}D \leqslant \operatorname{diam}X$ and $\beta(D) \leqslant \operatorname{diam}X$. In addition, if $\operatorname{diam}D < \operatorname{diam}X$, and $(x_i, y_i) \in X\times X$ is a sequence such that $|x_iy_i| \to \operatorname{diam}X$, then, starting from some i, the points x_i and y_i belong to different elements of D, therefore, in this case we have $\beta(D) = \operatorname{diam}X$, and the formula is proved.

Now, let $\operatorname{diam}D = \operatorname{diam}X$, then $\beta(D) - \lambda \leqslant \operatorname{diam}X$ and $\operatorname{diam}X - \lambda \leqslant \operatorname{diam}X$, thus

$$\max\{\operatorname{diam}D,\, \lambda-\alpha(D),\, \beta(D)-\lambda\}=\max\{\operatorname{diam}X,\, \lambda-\alpha(D)\}=\max\{\operatorname{diam}D,\, \lambda-\alpha(D),\, \operatorname{diam}X-\lambda\},$$

that completes the proof. □

Proposition 7. *Let $X \neq \Delta_1$ be an arbitrary metric space, and $2 \leqslant m \leqslant \#X$ a cardinal number, and $\lambda > 0$. Then*

$$2d_{GH}(\lambda\Delta_m, X) = \inf_{D\in\mathcal{D}_m(X)} \operatorname{dis}R_D.$$

Proof. The case of unbounded X is trivial, so, let X be bounded. By Corollary 1,

$$2d_{GH}(\lambda\Delta_m, X) = \inf_{R\in\mathcal{R}^0(\lambda\Delta_m,X)} \operatorname{dis}R,$$

thus, it suffices to prove that for any irreducible correspondence $R \in \mathcal{R}^0(\lambda\Delta_m, X)$ there exists $D \in \mathcal{D}_m(X)$ such that $\operatorname{dis}R_D \leqslant \operatorname{dis}R$.

Let us choose an arbitrary $R \in \mathcal{R}^0(\lambda\Delta_m, X)$ such that it cannot be represented in the form R_D, then the partition $D^R_{\lambda\Delta_m}$ is not pointwise, i.e., there exists $x \in X$ such that $\#R^{-1}(x) \geqslant 2$, therefore, $\mathrm{dis}R \geqslant \lambda$.

Define a metric on the set $D^R_{\lambda\Delta_m}$ to be equal λ between any its distinct elements, then this metric space is isometric to a simplex $\lambda\Delta'_n$, $n \leqslant m$. The correspondence R generates naturally a correspondence $R' \in \mathcal{R}(\lambda\Delta'_n, X)$, namely, if $D^R_{\lambda\Delta_m} = \{\Delta_j\}_{j\in J}$, and $f_R\, D^R_{\lambda\Delta_m} \to D^R_X$ is the bijection generated by R, then

$$R' = \bigcup_{j\in J}\{\Delta_j\}\times f_R(\Delta_j).$$

It is easy to see that $\mathrm{dis}R = \max\{\lambda,\, \mathrm{dis}R'\}$. Moreover, R' is generated by the partition $D' = D^R_X$, i.e., $R' = R_{D'}$, thus, by Corollary 2, we have

$$\mathrm{dis}R' = \max\{\mathrm{diam}D',\, \lambda - \alpha(D'),\, \mathrm{diam}X - \lambda\},$$

and hence,

$$\mathrm{dis}R = \max\{\lambda, \mathrm{diam}D',\, \lambda - \alpha(D'),\, \mathrm{diam}X - \lambda\} = \max\{\lambda,\, \mathrm{diam}D',\, \mathrm{diam}X - \lambda\}.$$

Since $n \leqslant m$, the partition D' has a subpartition $D \in \mathcal{D}_m(X)$. Clearly, $\mathrm{diam}D \leqslant \mathrm{diam}D'$, therefore,

$$\mathrm{dis}R_D = \max\{\mathrm{diam}D,\, \lambda - \alpha(D),\, \mathrm{diam}X - \lambda\} \leqslant \max\{\mathrm{diam}D',\, \lambda,\, \mathrm{diam}X - \lambda\} = \mathrm{dis}R,$$

q.e.d. $\square$

Considering separately the trivial case of $\lambda = 0$, we get the following result.

Corollary 3. *Let $X \neq \Delta_1$ be an arbitrary metric space, $2 \leqslant m \leqslant \#X$ a cardinal number, and $\lambda \geqslant 0$. Then*

$$2d_{GH}(\lambda\Delta_m, X) = \inf_{D\in\mathcal{D}_m(X)} \max\{\mathrm{diam}D,\, \lambda - \alpha(D),\, \mathrm{diam}X - \lambda\}.$$

For any metric space X put

$$\varepsilon(X) = \inf\bigl\{|xy| : x, y \in X,\, x \neq y\bigr\}.$$

Notice that $\varepsilon(X) \leqslant \mathrm{diam}X$, and for a bounded X the equality holds if and only if X is a simplex.

Corollary 3 immediately implies the following result that is proved in [10].

Theorem 4 ([10]). *Let $X \neq \Delta_1$ be a finite metric space, $m = \#X$, and $\lambda \geqslant 0$, then*

$$2d_{GH}(\lambda\Delta_m, X) = \max\bigl\{\lambda - \varepsilon(X),\, \mathrm{diam}X - \lambda\bigr\}.$$

4. Some Applications

In this section we apply the previous results to some well-known discrete optimisation problems from Metric Geometry and Graph Theory.

4.1. Calculation mst-spectrum

The first application deals with optimal graphs, so we start from some preliminaries for the Graph Theory.

4.1.1. Elements of Graph Theory

Here we consider simple graphs only, so in what follows by a *graph* we mean a pair $G = (V, E)$ consisting of two sets V and E referred as the *vertex set* and the *edge set* of the graph G , respectively; elements of V are called *vertices*, and the ones of E are called *edges* of the graph G. The set E is a subset of the family of two-element subsets of V. If V and E are finite sets then the graph G is called *finite*.

It is convenient to use the following notations:

- If $\{v, w\} \in E$ is an edge of the graph G, then we write it just as vw or wv; further one says that an edge *vw connects the vertices v and w*, and that v and w are *the vertices of the edge vw*;

- We write $V(G)$ and $E(G)$ for the vertex set and the edge set of a graph G to underline which graph is under consideration.

Graphs $G = (V, E)$ and $H = (W, F)$ are called *isomorphic* if there exists a bijective map $f V \to W$ such that $uv \in E$ if and only if $f(u)f(v) \in F$. Such a mapping f is called an *isomorphism of the graphs G and H*. Isomorphic graphs are often identified and, therefore, are not distinguished.

Two vertices $v, w \in V(G)$ are called *adjacent* if $vw \in E(G)$. Two different edges $e_1, e_2 \in E(G)$ are called *adjacent* if they have a common vertex, i.e., if $e_1 \cap e_2 \neq \emptyset$. Each edge $vw \in E(V)$ and its vertex, i.e., v or w, are said to be *incident* to each other. The set of vertices of a graph G adjacent to a vertex $v \in V$ is called the *neighbourhood of the vertex v* and denoted by N_v. The cardinal number of edges incident to a vertex v is called the *degree of the vertex v* and is denoted by $\deg v$, so $\deg v = \#N_v$.

A *subgraph* of a graph $G = (V, E)$ is each graph $H = (W, F)$ provided that $W \subset V$ and $F \subset E$. The fact that a graph H is a subgraph of a graph G is denoted as $H \subset G$. If $W = V$ then the subgraph $H \subset G$ is called *spanning*.

On the set of all graphs, whose vertex sets lie in a given set V, the inclusion relation $\subset$ of being a subgraph defines a partial order. The smallest element in this order is the empty graph $(\emptyset, \emptyset)$; the greatest one is called the *complete graph on V* and is denoted by $K(V)$. This partial order induces the one on the set of all subgraphs of a graph $G = (V, E)$: now the smallest element is again the empty graph $(\emptyset, \emptyset)$, but the greatest one is the graph G itself.

We also need some set-theoretical operations on graphs. They are usually defined in an intuitively clear way in terms of vertex and edge sets. For example, if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs, then put $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. Also, if $G = (V, E)$ is a graph, and e is a two-element subset of V, then $G \cup e = \big(V, E \cup \{e\}\big)$; similarly for $e \in E$ put $G \backslash e = \big(V, E \backslash \{e\}\big)$.

For each $W \subset V$ the *subgraph $G(W)$ of the graph G generated by W* is defined as the graph with the vertex set W, whose edge set consists of all $e \in E$ that connects vertices from W. In other words, $G(W)$ is maximal among subgraphs of G, whose vertex sets coincides with W.

We also need a similar construction for an edges set. Namely, for $F \subset E$ the *subgraph $G(F)$ of the graph G generated by F* is defined as the graph with the edge set F, whose vertex set is the collection of all vertices of G incident to edges from F.

A finite sequence $\gamma = (v_0 = v, v_1, \ldots, v_k = w)$ of vertices of a graph G is called a *walk of length k connecting v and w* if for every $i = 1, \ldots, k$ the vertices v_{i-1} and v_i are adjacent, and the edges $e_i = v_{i-1}v_i$ are called the *edges of the walk γ*. A walk containing at least one edge is called *non-degenerate*, and the walk containing no edges, i.e., with $k = 0$, is called *degenerate*. The walk is called *closed* if $v_0 = v_n$, and it is called *open* otherwise. A *trail* is a walk with no repeated edges, a *path* is an open trail with no repeated vertices. A *circuit* is a closed trail, and a *cycle* is a circuit with no repeated vertices.

A graph G is called *connected* if each pair of its vertices are connected by a walk. Maximal (by inclusion) connected subgraphs of a graph G are called *components* of G. A graph without cycles is called *a forest*, and a connected forest is called a *tree*.

A *weighted graph* is a graph $G = (V, E)$ equipped with a *weight function* $\omega\, E \to [0, \infty)$ (sometimes it is useful to consider more general weight functions, for instance, the ones taking negative or/and negative values also). A weighted graph is denoted by (V, E, ω) or (G, ω). The *weight* $\omega(H)$ *of a subgraph* $H \subset G$ is the sum of the weights of all the edges from this subgraph: $\omega(H) = \sum_{e\in E(H)} \omega(e)$. This definition can be extended to trails, in particular, to paths, circuits and cycles, considered as the corresponding subgraphs of G. In the case of a walk $\gamma = (v_0 = v, v_1, \ldots, v_k = w)$, its weight is defined as the sum of weights of all its consecutive edges: $\omega(\gamma) = \sum_{i=1}^{n} \in (v_{i-1}v_i)$. For graphs without weight functions these notions are defined as well by assigning the weight 1 to each edge by default.

Remark 4. *As in the case of metric spaces, we sometimes won't denote the weight function explicitly. Instead of that, speaking about a weighted graph G, we denote the weight of an object x just by $|x|$. For example, for $e \in E$ by $|e|$ we mean the weight of this edge, and for a subgraph $H \subset G$ by $|H|$ we denote the weight of H, etc.*

4.1.2. Minimum Spanning Tree Problem

Let M be a metric space. We consider M as a weighted complete graph $K(M)$ whose weight function equals to the distance between the corresponding edges. By $\mathcal{T}(M)$ we denote the set of all spanning trees in $K(M)$. Put

$$\operatorname{mst}(M) = \inf_{T\in\mathcal{T}(M)} |T|$$

and call this value by the *length of minimum spanning tree on M*. Each $T \in \mathcal{T}(M)$ with $|T| = \operatorname{mst}(M)$ is call a *minimum spanning tree on M*. The set of all minimum spanning trees on M is denoted by $\operatorname{MST}(M)$.

Remark 5. *If M is finite, them $\operatorname{MST}(M) \neq \emptyset$. For infinite M the situation is rather more difficult, see [26].*

Example 7. *If all nonzero distances in M are the same, then every spanning tree in $K(M)$ is minimum, so $\operatorname{MST}(M) = \mathcal{T}(M)$.*

If $\#M = 3$, then each minimum spanning tree is obtained from the complete graph $K(M)$ by deleting the longest edge (any of them if there are several).

4.1.3. The mst-spectrum

In this Section we consider only finite metric spaces M, i.e., $\#M < \infty$.

Notice that a minimum spanning tree, generally speaking, is not uniquely defined. For $G \in \operatorname{MST}(M)$, by $\sigma(G)$ we denote the vector whose elements are the lengths of the edges of the tree G sorted in descending order. The following result is well-known, however, we present its proof for completeness.

Proposition 8. *For any $G_1, G_2 \in \operatorname{MST}(M)$ the equality $\sigma(G_1) = \sigma(G_2)$ holds.*

Proof. Recall the standard algorithm for converting one minimum spanning tree to another [13].

Let $G_1 \neq G_2$, $G_i = (M, E_i)$, then $E_1 \neq E_2$ and $\#E_1 = \#E_2$, therefore, there exists $e \in E_2 \setminus E_1$. The graph $G_1 \cup e$ has a cycle C containing the edge e, and the cycle C does not contain an edge longer than e, because $G_1 \notin \operatorname{MST}(M)$ otherwise. The forest $G_2 \setminus e$ consists of two trees whose vertex sets we denote by V' and V''. Clearly, $M = V' \sqcup V''$. The cycle C contains an edge $e' \neq e$ connecting a vertex from V' with a vertex from V''. This edge does not lie in E_2, otherwise G_2 would contain a cycle. Therefore, $e' \in E_1 \setminus E_2$.

The graph $G_2 \cup e'$ also contains some cycle C'. By the choice of e', the cycle C' also has the edge e. Similarly to the above, the length of the edge e is less than or equal to the length of the edge e', otherwise $G_2 \notin \mathrm{MST}(M)$. Therefore, $|e| = |e'|$.

Replacing the edge e' in G_1 with e, we get a tree G_1' of the same length, i.e., it is a minimum spanning tree as well, and G_1' and G_2 have one common edge more than the trees G_1 and G_2. Thus, in a finite number of steps, we rebuild the tree G_1 into the tree G_2, passing through minimum spanning trees. It remains to notice that $\sigma(G_1') = \sigma(G_1)$, therefore, $\sigma(G_1) = \sigma(G_2)$. $\square$

Proposition 8 motivates the following definition.

Definition 2. *For any finite metric space M, by $\sigma(M)$ we denote $\sigma(G)$ for an arbitrary $G \in \mathrm{MST}(M)$ and call it the* mst-spectrum of the space M.

Theorem 5. *Let M be a finite metric space and $\sigma(M) = (\sigma_1, \dots, \sigma_{n-1})$. Then*

$$\sigma_k = \max\bigl\{\alpha(D) : D \in \mathcal{D}_{k+1}(M)\bigr\}.$$

Proof. Let $G = (M, E) \in \mathrm{MST}(M)$ and the set E be ordered so that $|e_i| = \sigma_i$. Denote by $D = \{M_1, \dots, M_{k+1}\}$ the partition of the set M into the sets of vertices of the trees forming the forest $G \setminus \{e_i\}_{i=1}^k$.

Lemma 1. *We have $\alpha(D) = |e_k|$.*

Proof. Indeed, choose arbitrary M_i and M_j, $i \neq j$, take arbitrary points a_i and a_j in them, respectively, and let γ be the unique path in G, connecting a_i and a_j. Then γ contains some edge e_p, $1 \leqslant p \leqslant k$. However, due to the minimality of the tree G, we have

$$|a_i a_j| \geqslant |e_p| \geqslant \operatorname*{Res}_{1 \leqslant i \leqslant k} |e_i| = |e_k|,$$

thus $|M_i M_j| \geqslant |e_k|$, so $\alpha(D) \geqslant |e_k|$. On the other hand, the edge e_k connects some M_p and M_q, then we get $\alpha(D) \leqslant |M_p M_q| = |e_k|$. $\square$

Now consider an arbitrary partition $D' = \{M_1', \dots, M_{k+1}'\}$.

Lemma 2. *We have $\alpha(D') \leqslant \alpha(D)$.*

Proof. Due to Lemma 1, it suffices to show that $\alpha(D') \leqslant |e_k|$. Denote by E' the set consisting of all edges $e_p \in E$, each of which connects some M_i' and M_j', $i \neq j$. Since G is connected, then the set E' consists of at least k edges. On the other hand, if some M_i' and M_j', $i \neq j$, are connected by an edge $e' \in E'$, then $|M_i' M_j'| \leqslant |e'|$, hence $\alpha(D') = \mathrm{Res}\,|M_i' M_j'| \leqslant \mathrm{Res}_{e' \in E'} |e'| \leqslant |e_k|$. $\square$

Lemma 2 completes the proof. $\square$

4.1.4. Calculating mst-spectrum by Means of Gromov–Hausdorff Distances

In the present section we show that the mst-spectrum of an arbitrary n-point metric space X can be represented as a linear function on the Gromov–Hausdorff distances from this space to the $\lambda\Delta_2, \dots, \lambda\Delta_n$ for $\lambda \geqslant 2\mathrm{diam} X$.

Theorem 6. *Let X be a finite metric space, $\sigma(X) = (\sigma_1, \dots, \sigma_{n-1})$, $\lambda \geqslant 2\mathrm{diam} X$. Then*

$$\sigma_k = \lambda - 2d_{GH}(\lambda\Delta_{k+1}, X).$$

Proof. Choose any $1 \leqslant k \leqslant n-1$ and arbitrary irreducible correspondence $R \in \mathcal{R}^0(\lambda\Delta_{k+1}, X)$. By Proposition 4, there exists partitions $R_{\lambda\Delta_{k+1}} = \{Z_i\}_{i=1}^p$ and $R_X = \{X_i\}_{i=1}^p$ of $\lambda\,\Delta_{k+1}$ and X, respectively, such that $R = \cup_{i=1}^p Z_i \times X_i$, and $\operatorname{Res}\{\#Z_i, \#X_i\} = 1$ for all i. By Proposition 5, it holds $\operatorname{dis}R \geqslant \max\{\operatorname{diam}R_{\lambda\Delta_{k+1}}, \operatorname{diam}R_X\}$. Thus, if for some i we have $\#Z_i > 1$, then $\operatorname{dis}R \geqslant \lambda \geqslant 2\operatorname{diam}X$. Since $k+1 \leqslant n$, there exists R such that $\#Z_i = 1$ for all i. For such R, again by Proposition 5, we have $\operatorname{dis}R \leqslant \operatorname{diam}X$. Therefore, $\inf_{R\in\mathcal{R}^0(\lambda\Delta_{k+1},X)} \operatorname{dis}R$ is achieved on a correspondences of the latter type. By $\mathcal{R}$ we denote the set of such correspondences.

Now, if $R \in \mathcal{R}$, then it consists of $p = k+1$ elements, and $R_X \in \mathcal{D}_{k+1}(X)$. By Proposition 5, we have

$$\operatorname{dis}R = \sup\bigl\{\operatorname{diam}R_X,\, |X_iX_j|' - \lambda,\, \lambda - |X_iX_j| : 1 \leqslant i < j \leqslant k+1\bigr\} = \\ = \sup\bigl\{\lambda - |X_iX_j| : 1 \leqslant i < j \leqslant k+1\bigr\} = \lambda - \alpha(R_X),$$

where the second equality holds because for λ chosen the estimate

$$\max\bigl\{|X_iX_j|' - \lambda, \operatorname{diam}R_X\bigr\} \leqslant \operatorname{diam}X \leqslant \lambda - \operatorname{diam}X \leqslant \lambda - |X_iX_j|$$

holds for any $1 \leqslant i < j \leqslant k+1$. Corollary 1, together with above considerations, gives us

$$2d_{GH}(\lambda\Delta_{k+1}, X) = \operatorname*{Res}_{R\in\mathcal{R}} \operatorname{dis}R = \operatorname*{Res}_{R\in\mathcal{R}} \bigl(\lambda - \alpha(R_X)\bigr) = \lambda - \max_{D\in\mathcal{D}_{k+1}(X)} \alpha(D),$$

where the last equality holds because each D generates some $R \in \mathcal{R}$. It remains to apply Theorem 5 saying that $\max\bigl\{\alpha(D) : D \in \mathcal{D}_{k+1}(X)\bigr\} = \sigma_k$, thus, $2d_{GH}(\lambda\Delta_{k+1}, X) = \lambda - \sigma_k$. $\square$

Corollary 4. *Let X be a finite metric space and $\lambda \geqslant 2\operatorname{diam}X$, then*

$$\operatorname{mst}X = \lambda(\#X - 1) - 2\sum_{k=1}^{\#X-1} d_{GH}(\lambda\Delta_{k+1}, X).$$

4.2. Generalized Borsuk Problem

Classical Borsuk Problem deals with partitions of subsets of Euclidean space into parts having smaller diameters. We generalize the Borsuk problem to arbitrary bounded metric spaces and partitions of arbitrary cardinality. Let X be a bounded metric space, m a cardinal number such that $2 \leqslant m \leqslant \#X$, and $D = \{X_i\}_{i\in I} \in \mathcal{D}_m(X)$. We say that D is a partition of X into subsets having *strictly smaller diameters*, if there exists $\varepsilon > 0$ such that $\operatorname{diam}X_i \leqslant \operatorname{diam}X - \varepsilon$ for all $i \in I$.

The *Generalized Borsuk Problem*: Is it possible to partition a bounded metric space X into a given, probably infinite, number of subsets, each of which has a strictly smaller diameter than X?

We give a solution to this Problem in terms of the Gromov–Hausdorff distance.

Theorem 7. *Let X be an arbitrary bounded metric space and m a cardinal number such that $2 \leqslant m \leqslant \#X$. Choose an arbitrary number $0 < \lambda < \operatorname{diam}X$, then X can be partitioned into m subsets having strictly smaller diameters if and only if $2d_{GH}(\lambda\Delta_m, X) < \operatorname{diam}X$. If not, then $2d_{GH}(\lambda\Delta_m, X) = \operatorname{diam}X$.*

Proof. Due Corollary 3, for the λ chosen the inequality $2d_{GH}(\lambda\Delta_m, X) \leqslant \operatorname{diam}X$ is valid, and the equality holds if and only if for each $D \in \mathcal{D}_m(X)$ we have $\operatorname{diam}D = \operatorname{diam}X$. The latter means that there is no partition of the space X into m parts having strictly smaller diameters. $\square$

Corollary 5. *Let $d > 0$ be a real number, and $m \leqslant n$ cardinal numbers. By $\mathcal{M}_n$ we denote the set of isometry classes of bounded metric spaces of cardinality at most n, endowed with the Gromov–Hausdorff distance. Choose an arbitrary $0 < \lambda < d$. Then the intersection*

$$S_{d/2}(\Delta_1) \cap S_{d/2}(\lambda\Delta_m)$$

of the spheres (here the the spheres are considered as spheres in $\mathcal{M}_n$) does not contain spaces, whose cardinality is less than m, and consists exactly of all metric spaces from $\mathcal{M}_n$, whose diameters are equal to d and that cannot be partitioned into m subsets of strictly smaller diameters.

Proof. Let X belong to the intersection of the spheres, then $\operatorname{diam} X = d$ in accordance with Example 2. If $m > \#X$, then, due to Theorem 3, we have

$$2d_{GH}(\lambda\Delta_m, X) = \max\{\lambda, \operatorname{diam} X - \lambda\} < d,$$

therefore $X \notin S_{d/2}(\lambda\Delta_m)$, that proves the first statement of Corollary.

Now let $m \leqslant \#X$. Since $\operatorname{diam} X = d$ and $2d_{GH}(\lambda\Delta_m, X) = d$, then, due to Theorem 7, the space X cannot be partitioned into m subsets of strictly smaller diameters.

Conversely, each X of the diameter d, such that $m \leqslant \#X$ and X cannot be partitioned into m subsets of strictly smaller diameter, lies in the intersection of the spheres by Theorem 7. □

4.3. Clique Cover Number and Chromatic Number of a Simple Graph

Recall that a subgraph of an arbitrary simple graph G is called *a clique*, if any its two vertices are connected by an edge, i.e., the clique is a subgraph which is a complete graph itself. Notice that each single-vertex subgraph is a single-vertex clique. For convenience, the vertex set of a clique is also referred as a *clique*.

On the set of all cliques, an ordering with respect to inclusion is naturally defined, and hence, due to the above remarks, a family of maximal cliques is uniquely defined; this family forms *a covering of the graph G* in the following sense: the union of all vertex sets of all maximal cliques coincides with the vertex set $V(G)$ of the graph G.

If one does not restrict himself by maximal cliques only, then, generally speaking, one can find other families of cliques covering the graph G. One of the classical problems of Graph Theory is to calculate the minimal possible number of cliques covering a given finite simple graph G. This number is referred as the *clique cover number* and is often denoted by $\theta(G)$. It is easy to see that the value $\theta(G)$ also equals the least number of cliques whose vertex sets form a partition of $V(G)$.

Another popular problem is to find the least possible number of colors that is necessary to color the vertices of a simple finite graph G in such a way that adjacent vertices have different colors. This number is denoted by $\gamma(G)$ and is referred as the *chromatic number of the graph G*.

For a simple graph G, by G' we denote its *dual graph*, i.e., the graph with the same vertex set and the complementary set of edges (two vertices of G' are adjacent if and only if they are not adjacent in G). It is not difficult to verify, that for any simple finite graph G it holds $\theta(G) = \gamma(G')$.

Let $G = (V, E)$ be an arbitrary finite graph. Fix two real numbers $a < b \leqslant 2a$ and define a metric on V as follows: the distance between adjacent vertices equals a, and the distance between nonadjacent vertices equals b. Then a subset $V' \subset V$ has diameter a if and only if $G(V') \subset G$ is a clique. This implies that the clique cover number of G equals the least possible cardinality of partitions of the metric space V into subsets of (strictly) smaller diameter. However, this number was calculated in Theorem 7. Thus, we get the following result.

Corollary 6. *Let $G = (V, E)$ be an arbitrary finite graph. Fix two real numbers $a < b \leqslant 2a$ and define a metric on V as follows: the distance between adjacent vertices equals a, and the distance between nonadjacent ones equals b. Let m be the greatest positive integer k such that $2d_{GH}(a\Delta_k, V) = b$ (in the case when there is no such k, we put $m = 0$). Then $\theta(G) = m + 1$.*

Because of the duality between clique cover and chromatic numbers, we get the following dual result.

Corollary 7. *Let $G = (V, E)$ be an arbitrary finite graph. Fix two real numbers $a < b \leqslant 2a$ and define a metric on V as follows: the distance between adjacent vertices equals b, and the distance between nonadjacent ones equals a. Let m be the greatest positive integer k such that $2d_{GH}(a\Delta_k, V) = b$ (in the case when there is no such k, we put $m = 0$). Then $\gamma(G) = m + 1$.*

4.4. Examples

In conclusion we give several examples demonstrating how the above Corollaries can be applied.

4.4.1. An Empty Graph and a Complete Graph

Let $G = (V, E)$ be an empty graph, i.e., $E = \emptyset$. Put $n = \#V$, then $\theta(G) = n$. Now, let us calculate $\theta(G)$ by means of Corollary 6.

The metric space V constructed in Corollary 6 coincides with $b\Delta_n$, then for $k < n$ we have $2d_{GH}(a\Delta_k, V) = 2d_{GH}(a\Delta_k, b\Delta_n) = b$ because for any $R \in \mathcal{R}(a\Delta_k, b\Delta_n)$ there exists $x \in a\Delta_k$ such that $\#R(x) \geqslant 2$, thus $\operatorname{dis} R = b$. For $k \geqslant n$ we have $2d_{GH}(a\Delta_k, b\Delta_n) \leqslant \max\{a, b - a\}$. Indeed, for $k = n$ we can consider a bijection R with $\operatorname{dis} R = b - a$. For $k > n$ we can define R as follows: take some $x \in b\Delta_n$, and let $R^{-1}(x)$ consists of arbitrary $k - n + 1$ points of $a\Delta_k$; for remaining points let R be a bijection. Then $\operatorname{dis} R = \max\{a, b - a\}$. Thus, according to Corollary 6, we also have $\theta(G) = n$.

Now, let $G = (V, E)$ be a complete graph, i.e., any two its vertices are adjacent. In this case $\theta(G) = 1$. Now, let us calculate $\theta(G)$ by means of Corollary 6.

In this case the metric space V from Corollary 6 coincides with $a\Delta_n$, therefore $2d_{GH}(a\Delta_k, V) = 2d_{GH}(a\Delta_k, a\Delta_n) \leqslant \max\bigl\{\operatorname{diam}(a\Delta_k), \operatorname{diam}(a\Delta_n)\bigr\} < b$, due to Example 4, therefore $\theta(G) = 1$ according to Corollary 6.

4.4.2. Bipartite Graphs

Let $G = (V, E)$ be a complete bipartite graph, i.e., its vertex set is partitioned in two non-empty non-intersecting subsets V_1 and V_2, and its edge set E consists of all pairs v_1v_2, $v_i \in V_i$, $i = 1, 2$. In this case $\gamma(G) = 2$. Now, let us calculate $\gamma(G)$ by means of Corollary 7.

The metric space V constructed in Corollary 7 is a 2-distance space such that the distances between the points belonging to the same subset V_i equals a, and the distance between the points belonging to distinct V_i equals b, where $0 < a < b \leqslant 2a$. Then $\operatorname{diam} V = b$, so $d_{GH}(a\Delta_1, V) = b$. Further, for $a\Delta_k$, $k \geqslant 2$, let us partition the vertex set of Δ_k in two non-empty sets D_1 and D_2, and put $R = (D_1 \times V_1) \cup (D_2 \times V_2)$. Then $d_{GH}(a\Delta_k, V) \leqslant \operatorname{dist} R = \max\{a, b - a\} \leqslant a < b$. Therefore $\gamma(G) = 2$ in accordance with Corollary 7.

If $G = (V, E)$ is a bipartite graph, i.e., its vertex set is partitioned in two non-empty non-intersecting subsets V_1 and V_2 again, but its edge set E is nonempty and is contained in the edge set of the corresponding complete bipartite graph, then $\gamma(G) = 2$, and similar reasoning can be used to calculate it by means of Corollary 7.

4.4.3. Distance from Simplexes to Balls and Spheres

As it is mentioned in Introduction, Lusternik and Schnirelmann [16], and a bit later independently Borsuk [14] and [15], have shown that the least possible number of parts of smaller diameter necessary to partition a sphere and a ball in $\mathbb{R}^n$ equals $n + 1$. Then Theorem 7 implies the following result.

Corollary 8. *Let X be either the standard unit sphere S^{n-1} or the standard unit ball B^n in the Euclidean space $\mathbb{R}^n$, and $0 < \lambda < 2$. Then $d_{GH}(\lambda\Delta_k, X) < 1$ for $k \geqslant n + 1$, and $d_{GH}(\lambda\Delta_k, X) = 1$ for $k \leqslant n$.*

4.4.4. Cycles and Wheel Graph

Recall that the cycle C_n with $n \geqslant 3$ vertices is a connected simple graph with n vertices and n edges, such that all the vertices have degree 2. The graphs C_{2k} are evidently bipartite, $\gamma(C_{2k}) = 2$, and $\gamma(C_{2k+1}) = 3$.

Let X be a finite 2-distance space with non-zero distances $0 < a < b$. Construct a finite graph G_X with vertex set X connecting two vertices by an edge iff the distance between them equals b. This graph is referred as the *greater distance graph of* X.

Corollary 9. *Let X be a finite n-point 2-distance metric space with non-zero distances $a < b \leqslant 2a$, such that its greater distance graph is C_{2k+1}. Then $2d_{GH}(a\Delta_m, X) = b$ for $m = 1, 2$.*

Recall that the wheel graph W_n with n vertices is obtained from the cycle C_{n-1} by adding a single vertex and $n-1$ edges connecting this vertex with all the remaining ones. It is well-known that $\gamma(W_{2k}) = 4$ and $\gamma(W_{2k+1}) = 3$.

Corollary 10. *Let X be a finite n-point 2-distance metric space with non-zero distances $a < b \leqslant 2a$, such that its greater distance graph is W_n. If $n = 2k+1$, then $2d_{GH}(a\Delta_m, X) = b$ for $m = 1, 2$, and if $n = 2k$, then $2d_{GH}(a\Delta_m, X) = b$ for $m = 1, 2, 3$.*

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