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О ретрактах линейных конечномерных пространств, порождённых коэрцитивными отображениями [1]

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Аннотация

Рассматриваются коэрцитивные непрерывные инъективные отображения, действующие из одного линейного конечномерного пространства в другое. Доказано, что образы этих отображений являются ретрактами линейных пространств.

Ключевые слова: ретракт, коэрцитивное отображение, равномерная регулярность.

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On the retracts of finite-dimensional spaces, generated by coercive mappings

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Abstract

Coercive continuous injective mappings acting from one linear finite-dimensional space to another are considered. It is proved that the images of these mappings are retracts of linear spaces.

Keywords: retract, coercive mapping, uniform regularity

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1. Retracts of linear finite-dimensional spaces

On the seminar "Differential geometry and applications" academician A. T. Fomenko proposed the following question to the author of this paper. Under what assumption the image of a mapping $f : \mathbb{R}^n \to \mathbb{R}^k$, $k \geq n$, is a retract of the space $\mathbb{R}^k$? An answer to this question is a result below which provides a sufficient condition for $f(\mathbb{R}^n)$ to be a retract of $\mathbb{R}^k$.

Let a number $k \geq n$ and a mapping $f : \mathbb{R}^n \to \mathbb{R}^k$ be given.

THEOREM 1. *Assume that f is continuous and injective. Then for the conditions*

(a) *f is coercive (i.e. if $|x| \to +\infty$ then $|f(x)| \to +\infty$),*

(b) *there exists a continuous left inverse mapping $g : \mathbb{R}^k \to \mathbb{R}^n$ to the mapping f (i.e. $g(f(x)) = x$ for every $x \in \mathbb{R}^n$),*

(c) *$f(\mathbb{R}^n)$ is a retract of $\mathbb{R}^k$,*

the following implications take place: $(a) \Leftrightarrow (b) \Rightarrow (c)$.

PROOF. Set $Y := f(\mathbb{R}^n)$ and denote by $h : Y \to \mathbb{R}^n$ a mapping which assigns to $y \in Y$ a point $x \in \mathbb{R}^n$ such that $f(x) = y$. The existence and uniqueness of this mapping follows from the injectivity of f. Denote by h_i a function which assigns to $y \in Y$ the i-th coordinate of $h(y)$, $i = \overline{1, n}$, i.e. $h(y) = (h_1(y), ..., h_n(y))$ for every $y \in Y$.

1) Prove $(a) \Rightarrow (b)$. We first show that h is continuous. Take arbitrary $y \in Y$, $\{y_j\} \subset Y$ such that $y_j \to y$. The sequence $\{h(y_j)\}$ is bounded since otherwise there exists a subsequence $\{h(y_{j_i})\}$ such that $|h(y_{j_i})| \to \infty$ and $f(h(y_{j_i})) = y_{j_i} \to y$, and this contradicts (a).

Show that the sequence $\{h(y_j)\}$ has at most one limit point. Indeed, since f is continuous and $f(h(y_j)) = y_j \to y$, for a limit point $x \in \mathbb{R}^n$ of the sequence $\{h(y_j)\}$ equality $f(x) = y$ holds. Injectivity of f implies that such a point x is unique.

Since the sequence $\{h(y_j)\}$ is bounded and has the only limit point, this sequence converges to this limit point $x \in \mathbb{R}^n$. Continuity of f implies that $y_j = f(h(y_j)) \to f(x)$. Hence, $f(x) = y$, thus $x = h(y)$. Continuity of h is proved.

Show that Y is closed. Take a sequence $\{y_j\} \subset Y$ and a point $y \in \mathbb{R}^k$ such that $y_j \to y$. The sequence $\{h(y_j)\}$ is bounded, since otherwise it has a subsequence $\{h(y_{j_i})\}$ such that $|h(y_{j_i})| \to \infty$ and $f(h(y_{j_i})) = y_{j_i} \to y$ in contradiction to (a). Hence, the sequence $\{h(y_j)\}$ has at least one limit point $x \in \mathbb{R}^n$. The continuity of f and the relation $f(h(y_j)) = y_j \to y$ imply $f(x) = y$. Hence, Y is closed.

So, each function $h_i : Y \to \mathbb{R}$, $i = \overline{1, n}$, is a continuous function and its domain is a closed subset of $\mathbb{R}^k$. The Tietze-Urysohn extension theorem (see, for instance, [1, Theorem 2.1.8]) implies that for every $i = \overline{1, n}$ there exists a continuous function $g_i : \mathbb{R}^k \to \mathbb{R}$ such that $g_i(y) = h_i(y)$ for every $y \in Y$. Define a mapping $g : \mathbb{R}^k \to \mathbb{R}^n$ by formula $g(y) := (g_1(y), ..., g_k(y))$, $y \in \mathbb{R}^k$. Obviously g is continuous and $g(f(x)) = h(f(x)) = x$ for every $x \in \mathbb{R}^n$.

2) Prove $(b) \Rightarrow (a)$. Assume the contrary, i.e. there exist a sequence $\{x_j\} \subset \mathbb{R}^n$ and a point $y \in \mathbb{R}^k$ such that $x_j \to \infty$ and $f(x_j) \to y$ as $j \to \infty$. Put $y_j := f(x_j)$, $j = 1, 2, \ldots$. Then $g(y_j) = x_j \to \infty$ and the sequence $\{y_j\}$ converges, in contradiction to continuity of g.

3) Prove $(b) \Rightarrow (c)$. It is obvious that $y \mapsto f(g(y))$, $y \in \mathbb{R}^k$, is a retraction of $\mathbb{R}^k$ onto Y. $\square$

In the proof of the theorem it is shown that (a) implies that the image Y of f is closed. Let us show that under the assumptions of continuity and injectivity of f the closedness of Y is not sufficient for Y to be a retraction of $\mathbb{R}^k$.

EXAMPLE 1. *Let $S_1 \subset \mathbb{R}^2$ be a circle with radius one centered at the point $y_1 = (1, 0)$ and $S_2 \subset \mathbb{R}^2$ be a circle with radius one centered at the point $y_2 = (-1, 0)$. Define the mapping*

$f : \mathbb{R} \to \mathbb{R}^2$ by formula

$$f(x) = y_1 + (-\cos(4\mathrm{arctg}x), \sin(4\mathrm{arctg}x)), \quad \textit{for} \quad x \geq 0,$$

$$f(x) = y_2 + (\cos(4\mathrm{arctg}x), \sin(4\mathrm{arctg}x)), \quad \textit{for} \quad x < 0.$$

Obviously, this mapping is continuous (in particular, at the point $x = 0$, the value of f and the left-hand and the right-hand limits of f equal $(0, 0)$) and injective. Moreover, f assigns to nonnegative numbers the circle S_1 and assigns to nonpositive numbers the circle S_2. Therefore, the image $Y = S_1 \cup S_2$ of f is closed. Since Y is bounded, f is not coercive.

Show that Y is not a retract of $\mathbb{R}^2$. Assume the contrary, i.e. there exists a retraction $r : \mathbb{R}^2 \to Y$. Consider the mapping $w : Y \to S_1$, $w(y) = y$ for $y \in S_1$, $w(y) = (0, 0)$ for $y \in S_2$. Obviously the mapping $y \mapsto w(r(y))$, $y \in \mathbb{R}^2$, is a retraction of $\mathbb{R}^2$ onto S_1. Hence, a circle is a retract of a plane which is impossible (see, for instance, [2, §3.4]). Therefore, Y is not a retract of $\mathbb{R}^2$.

Remark 1. *In connection with Theorem 1 there appears the following natural question. Is the implication (c) $\Rightarrow$ (a) true? The author does not know the answer to this question yet.*

2. Images and preimages of retracts

Let us state a corollary of Theorem 1 which provides sufficient condition for an image of a retract to be a retract.

Corollary 1. *Let f be continuous, injective and coercive, $U \subset \mathbb{R}^n$ be a retract of $\mathbb{R}^n$. Then $f(U)$ is a retract of $\mathbb{R}^k$.*

Proof. Let $r : \mathbb{R}^n \to U$ be a retraction. By virtue of the proposition $(a) \Rightarrow (b)$ of Theorem 1 there exists a continuous mapping $g : \mathbb{R}^k \to \mathbb{R}^n$ such that $g(f(x)) = x$ for every $x \in \mathbb{R}^n$. Show that the mapping $y \mapsto f(r(g(y)))$, $y \in \mathbb{R}^k$, is a retraction onto $f(U)$.

Since $g(\mathbb{R}^k) = \mathbb{R}^n$ and $r(\mathbb{R}^n) = U$, then $f(r(g(\mathbb{R}^k))) = f(U)$. Further, for every $y \in f(U)$ there exists $x \in U$ such that $f(x) = y$. The definition of g implies $g(y) = g(f(x)) = x$, the definition of r and the inclusion $x \in U$ implies $r(x) = x$, thus

$$f(r(g(y))) = f(r(x)) = f(x) = y.$$

Therefore, the mapping $f(r(g(\cdot)))$ is a retraction and its image coincide with $f(U)$. $\square$

Let us now state conditions for preimage of a retract to be a retract.

Everywhere below we assume that the spaces $\mathbb{R}^n$ and $\mathbb{R}^k$ are equipped with Euclidian norms. For arbitrary linear operator $A : \mathbb{R}^k \to \mathbb{R}^n$ denote by A^* the adjoint operator, for arbitrary linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ denote by $\|A\|$ the norm of A.

Proposition 1. *Let $U \subset \mathbb{R}^n$ be a retract of $\mathbb{R}^n$, a mapping $g : \mathbb{R}^k \to \mathbb{R}^n$ be twice continuosly differentiable, the linear operator $\dfrac{\partial g}{\partial y}(y)$ be surjective for every $y \in \mathbb{R}^k$ and*

$$\exists\, c \geq 0 : \quad \left\| \frac{\partial g}{\partial y}(y)^* \left(\frac{\partial g}{\partial y}(y) \cdot \frac{\partial g}{\partial y}(y)^* \right)^{-1} \right\| \leq c \quad \forall\, y \in \mathbb{R}^k. \tag{1}$$

Then the set $g^{-1}(U) := \{y \in \mathbb{R}^k : g(y) \in U\}$ is a retract of $\mathbb{R}^n$.

Proof. Assume $g(0) = 0$, without loss of generality. Put $M := \{y \in \mathbb{R}^k : g(y) = 0\}$, and let $s : \mathbb{R}^n \to U$ be a retraction. By virtue of [3, Theorem 1] there exists a homeomorphism $F : M \times \mathbb{R}^n \to \mathbb{R}^k$ such that

$$g(F(\xi, x)) = x \quad \forall\, (\xi, x) \in M \times \mathbb{R}^n. \tag{2}$$

Denote by $a:\mathbb{R}^k\to M$ and $b:\mathbb{R}^k\to\mathbb{R}^n$ the projections of F^{-1} onto M and $\mathbb{R}^n$, respectively, i.e.

$$F^{-1}(y)=(a(y),b(y))\quad \forall\, y\in\mathbb{R}^k.$$

Show that $b=g$. Take arbitrary point $y\in\mathbb{R}^k$. We have $y=F(F^{-1}(y))=F(a(y),b(y))$. Thus,

$$g(y)=g(F(a(y),b(y)))=b(y).$$

Here, the second equality follows from (2). Thus $b=g$. This identity implies that

$$F(a(y),g(y))=F(a(y),b(y))=F(F^{-1}(y))=y\quad \forall\, y\in\mathbb{R}^k. \tag{3}$$

Define a mapping $r:\mathbb{R}^k\to g^{-1}(U)$ by formula

$$r(y):=F(a(y),s(g(y))),\quad y\in\mathbb{R}^k.$$

Show that it is well defined, i.e. $r(\mathbb{R}^k)\subset g^{-1}(U)$. For arbitrary $y\in\mathbb{R}^k$, we have $s(g(y))\in U$ by virtue of the definition of s. Thus, (2) implies

$$g(F(a(y),s(g(y))))=s(g(y))\in U.$$

So, the mapping r is well defined.

Show that r is a retraction. Take arbitrary $y\in g^{-1}(U)$. Put $x:=g(y)$. Obviously $x\in U$. We have

$$r(y)=F(a(y),s(g(y)))=F(a(y),s(x))=F(a(y),x).$$

Here, the last equality follows from the inclusion $x\in U$ since $s:\mathbb{R}^n\to U$ is a retraction. Further,

$$F(a(y),x)=F(a(y),g(y))=y.$$

Here, the last equality follows from (3). So, r is a retraction and $g^{-1}(U)$ is a retract. □

Remark 2. *The assumption of nondegeneracy of the derivatives in Proposition 1 is essential. Indeed, let $g:\mathbb{R}\to\mathbb{R}$, $g(y)=y^2$, $y\in\mathbb{R}$, $U=\{1\}$. The set U is obviously a retract of $\mathbb{R}$, however the set $g^{-1}(U)=\{-1,1\}$ is not a retract of $\mathbb{R}$ since $g^{-1}(U)$ is not connected.*

The uniform regularity assumption (1) is also essential. Indeed, let $g:\mathbb{R}\to\mathbb{R}$, $g(y)=e^y$, $y\in\mathbb{R}$, $U=\mathbb{R}$ is obviously a retract of $\mathbb{R}$, however the set $g^{-1}(U)=(0,+\infty)$ is not a retract of $\mathbb{R}$ since $g^{-1}(U)$ is not closed.

In case $n=k$, Proposition 1 is a corollary of Hadamard's global homeomorphism theorem (see, for instance, [4, Theorem 5.3.10]). Indeed, if $n=k$ the surjectivity of linear operators $\dfrac{\partial g}{\partial y}(y)$, $y\in\mathbb{R}^k$, is equivalent to there invertability and uniform regularity condition (1) takes the following form:

$$\exists\, c\geq 0:\quad \left\|\left(\frac{\partial g}{\partial y}(y)\right)^{-1}\right\|\leq c\quad \forall\, y\in\mathbb{R}^k.$$

So, g satisfies the assumptions of Hadamard's theorem. Thus, g is a homeomorphism. Hence, if U is a retract, then $g^{-1}(U)$ is a retract.

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СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Р. Энгелькинг, Общая топология, М.: Мир, 1986.
2. А. Т. Фоменко, Д. Б. Фукс, Курс гомотопической топологии, М.: Наука, 1989.
3. А. В. Арутюнов, С. Е. Жуковский, Применение методов обыкновенных дифференциальных уравнений для глобальных теорем об обратной функции, Дифф. уравнения, 2019, т. 55, N 4, с. 452–463.
4. Дж. Ортега, В. Рейнболдт, Итерационные методы решения нелинейных систем уравнений со многими неизвестными, М.: Мир, 1975.

REFERENCES

1. Engelking, R. 1986, *General topology*, Mir, Moscow (in Russian).
2. Fomenko, A. T., Fuks, D. B. 1989, *A course in homotopic topology*, Nauka, Moscow.
3. Arutyunov A. V., Zhukovskiy S. E., 2019, “Application of Methods of Ordinary Differential Equations to Global Inverse Function Theorems”, *Differential Equations*, vol. 55, no. 4, pp. 437-–448.
4. Ortega J., Reinboldt V., 1975, *Iterative methods for solving nonlinear systems of equations with many variables*, Mir Publ., Moscow.

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