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О нелокальной модифицированной гравитации[1]

И. Димитриевич, Б. Драгович, З. Ракич, Е. Станкович

Димитриевич Иван — доктор наук, профессор, математический факультет Белградского университета (г. Белград, Сербия).
e-mail: ivand@matf.bg.ac.rs

Драгович Бранко — доктор наук, профессор, Институт физики Белградского университета; Математический институт САНУ (г. Белград, Сербия).
e-mail: dragovich@ipb.ac.rs

Ракич Зоран — доктор наук, профессор, математический факультет Белградского университета (г. Белград, Сербия).
e-mail: zrakic@matf.bg.ac.rs

Станкович Елена — доктор наук, профессор, педагогический факультет Белградского университета (г. Белград, Сербия).
e-mail: jelenagg@gmail.com

Аннотация

За последние сто лет многие существенные гравитационные явления были предсказаны и обнаружены Общей теорией относительности (GR), которая до сих пор остается лучшей теорией гравитации. Тем не менее, из-за великих наблюдательных открытий 20-го века некоторые (квантовые) теоретические и (астрофизические и космологические) феноменологические трудности современной гравитации были мотивацией для поиска более общей теории гравитации, чем ОТО. В результате были рассмотрены многие модификации ТО. Одним из многообещающих недавних исследований является нелокальная модифицированная гравитация. В этой статье мы представляем обзор некоторых нелокальных гравитационных моделей с их точными космологическими решениями, в которых нелокальность выражается аналитической функцией от оператора Даламбера — Бельтрами $\Box$. Некоторые из полученных решений содержат эффекты, которые обычно присваиваются темной материи и темной энергии.

Ключевые слова: Нелокальная измененная гравитация, точные космологические решения, темная материя, темная энергия, общая теория относительности и конечная производная гравитация.

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On Nonlocal Modified Gravity

I. Dimitrijević, B. Dragovich, Z. Rakić, J. Stanković

Dimitrijević Ivan — Doctor of Sciences, Professor, Faculty of Mathematics, University of Belgrade (Belgrade, Serbia).
e-mail: ivand@matf.bg.ac.rs

Dragovich Branko — Doctor of Sciences, Professor, Institute of Physics, University of Belgrade; Mathematical Institute SANU (Belgrade, Serbia).
e-mail: dragovich@ipb.ac.rs

Rakić Zoran — Doctor of Sciences, Professor, Faculty of Mathematics, University of Belgrade (Belgrade, Serbia).
e-mail: zrakic@matf.bg.ac.rs

Stanković Jelena — Doctor of Sciences, Professor, Teacher Education Faculty, University of Belgrade (Belgrade, Serbia).
e-mail: jelenagg@gmail.com

Abstract

In the last hundred years many significant gravitational phenomena have been predicted and discovered by General Relativity (GR), which is still the best theory of gravity. Nevertheless, due to the great observational discoveries of 20th century some (quantum) theoretical and (astrophysical and cosmological) phenomenological difficulties of modern gravity have been motivation to search more general theory of gravity than GR. As a result, many modifications of GR have been considered. One of promising recent investigations is Nonlocal Modified Gravity. In this article we present a review of some nonlocal gravity models with their exact cosmological solutions, in which nonlocality is expressed by an analytic function of the d'Alembert–Beltrami operator $\Box$. Some of obtained solutions contain effects which are usually assigned to the dark matter and dark energy.

Keywords: Nonlocal modi
ed gravity, exact cosmological solutions, dark matter, dark energy, general relativity, in
nite derivative gravity.

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1. Introduction

In 2015, General relativity (GR), known also as Einstein theory of gravity, celebrated its first hundred years is considered as one of the most profound and beautiful physical theories with great phenomenological achievements and nice theoretical properties. GR has important astrophysical implications predicting existence of black holes, gravitational redshift, gravitational lensing and gravitational waves[2], it has been tested and quite well confirmed in the Solar system, and it has been also used as a theoretical laboratory for gravitational investigations at other spacetime scales.

Despite of just mentioned phenomenological successes and many nice theoretical properties, GR is not complete theory of gravity. For example, attempts to quantize GR lead to the problem of nonrenormalizability. GR also contains singularities like the Big Bang and black holes. In cosmology, it predicts existence of about 95% of additional new kind of matter, which makes dark side of the universe. Namely, if GR is the gravity theory for the universe as a whole and if the universe is homogeneous and isotropic with the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric at the cosmic scale, then it contains about 68% of *dark energy*, 27% of *dark matter*, and only about 5% of *visible matter* [63], which are not verified in laboratory conditions and have not so far seen in particle physics. If a physical theory contains singularities then it has to be modified in that domain. Because of that, there are many attempts to modify General relativity. Motivations for its modification usually come from quantum gravity, string theory, astrophysics and cosmology (for a review, see [55, 58, 15, 16, 57]). We are mainly interested in cosmological reasons to modify GR, i.e. to find such extension of Einstein theory of gravity which will not contain the Big Bang singularity and offer another possible description of the universe acceleration and large velocities in galaxies instead of mysterious dark energy and dark matter. In the case that dark energy and dark matter really exist it is still interesting to know is there a modified gravity which can imitate the same or similar effects.

Any well founded modification of the Einstein theory of gravity should be a generalization of the general theory of relativity, and consequently it should be verified at least on the dynamics of the Solar system. On the mathematical level it should be formulated within the pseudo-Riemannian geometry in terms of covariant quantities and take into account equivalence of the inertial and gravitational mass. It means that the Ricci scalar R in gravity Lagrangian $\mathcal{L}_g$ of the Einstein-Hilbert action should be replaced by an appropriate function which may contain not only R but also some scalar covariant constructions (as norms of Ricci and curvature tensor, etc.) which are possible in the pseudo-Riemannian geometry. Since there are infinitely many possibilities for such functions (i.e. its modifications), and since so far there is no guiding theoretical principle which could make appropriate choice between all possibilities, our task is very complicated. In this context the Einstein-Hilbert action is the simplest one, i.e. it can be viewed as realization of the principle of simplicity in construction of $\mathcal{L}_g$.

Modifications of GR were started a few years after its birth adding cosmological constant in the Lagrangian of Hilbert-Einstein action, later by adding Gauss-Bonnet invariant, replacing R with $f(R)$ ($f(R)$ modified gravity, see [45]) and recently nonlocal modifications, which are promising modern approaches towards more complete theory of gravity. Motivation for nonlocal modification of general relativity can be found in string theory which is nonlocal theory and contains gravity. We present here a review of our results on nonlocal gravity. In particular, we pay special attention to models in which nonlocality is expressed by an analytic function of the d'Alembert operator $\Box = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$ like nonlocality in string theory.

In Section 2 we give some preliminaries on cosmology and mention a few different approaches to nonlocal modified gravity. Section 3 contains a general modified action with an analytic nonlocality and we derive the corresponding equations of motion for nonlocality of the form $\mathcal{H}(R)\,\mathcal{F}(\Box)\,\mathcal{G}(R)$. In Sect. 4. we presented several different models of nonlocality which are special cases of general model from previous Section. In all models we found some cosmological solutions for appropriate scaling factor. We emphasize here the case when the nonlocality is given by $\mathcal{H}(R) = \mathcal{G}(R) = \sqrt{R - 2\,\Lambda}$ with scaling factor of the form $a(t) = At^{\frac{2}{3}}\,e^{\frac{\Lambda}{14}\,t^2}$ in the flat universe. This model gives some effects usually attributed to the dark matter and dark energy. Finally, Sect. 5. is devoted to the perturbations over de Sitter background. We obtained some cosmological solutions and investigated stability of them.

[2] which were experimentally discovered in 2016. [1].

At the end we showed that equations of motion of gravitational waves in our nonlocal modified gravity coincide with the corresponding equations in GR.

2. Preliminaries

2.1. Metric

General theory of relativity, i.e. Einstein theory of gravity (EG) assumes that the universe is four dimensional homogeneous and isotropic pseudo-Riemannian manifold M with metric $(g_{\mu\nu})$ of signature $(1,3)$. There exist three types of homogeneous and isotropic simple connected spaces of dimension 3:

- flat space $\mathbb{R}^3$ (of curvature equal 0),
- sphere $\mathbb{S}^3$ (of constant positive sectional curvature),
- hyperbolic space $\mathbb{H}^3$ (of constant negative sectional curvature).

The universe is homogeneous and isotropic (observation data) manifold, the generic metric in these spaces is of the form (Friedmann – Lemaître – Robertson – Walker metric, (FLRW)):

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \qquad k \in \{-1, 0, 1\}, \tag{1}$$

where $a(t)$ is a *scaling factor* which describes the evolution (in time) of the universe and parameter k describes the curvature of the space. In the FLRW metric the Ricci scalar (scalar curvature) is

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) \tag{2}$$

and

$$\Box = -\partial_t^2 - 3H\partial_t. \tag{3}$$

where $H = \dot{a}/a$ is the Hubble parameter, which describes the expansion of the universe. We use natural system of units in which speed of light is $c = 1$.

2.2. Einstein-Hilbert action.

GR is based on Einstein-Hilbert action:

$$S = \int_M \Big(\frac{R - 2\Lambda}{16\,\pi\, G} + \mathcal{L}_m\Big)\sqrt{-g}\; d^4x, \tag{4}$$

where R is scalar curvature of M, $g = \det(g_{\mu\nu})$ is determinant of metric tensor, Λ is cosmological constant and $\mathcal{L}_m$ is Lagrangian of matter.

By variation of the action S with respect to $g_{\mu\nu}$ we obtain Einstein equations of motion:

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = 8\,\pi\, G\, T_{\mu\nu}, \tag{5}$$

where $T_{\mu\nu}$ is the energy momentum tensor, $g_{\mu\nu}$ is metric tensor, $R_{\mu\nu}$ is Ricci tensor and R is scalar curvature.

2.3. Friedman equations.

The energy momentum tensor for an ideal fluid (matter in cosmology) is given by

$$T = \mathsf{diag}(-\rho\, g_{00}, g_{11}p, g_{22}p, g_{33}p), \tag{6}$$

where ρ is energy density and p is pressure. Using the conservation law we get

$$0 = \nabla_\mu T^\mu_0 = -\dot{\rho} - 3\frac{\dot{a}}{a}(\rho + p). \tag{7}$$

Since in the cosmology holds $p = w\rho$, where w is usually a constant, we have that equation (7) has general solution $\rho = Ca^{-3(1+w)}$.

The basic types of matter in the universe are: cosmic dust - $w = 0\,(\rho_m = C\,a^{-3})$, and radiation - $w = 1/3$ ($\rho_r = C\,a^{-4}$). Nowadays, the ratio $\frac{\rho_m}{\rho_r} \approx 10^6$, in the early universe radiation was dominant. Introducing the cosmological constant $\Lambda\,(\neq 0)$ is equivalent to the (dark) energy of vacuum. From Einstein equation one can find energy-momentum tensor for vacuum,

$$T_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}, \tag{8}$$

and see that $\rho = -p = \frac{\Lambda}{8\pi G}$. It is clear that energy density is a constant which does not depend on scaling factor a. Let us note that, if the universe is expanding for large values of cosmic time energy of vacuum will become dominant to the energy densities of matter and radiation.

Now, Einstein equation (5) implies *Friedmann equations*

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{9}$$

There are several of cosmological parameters which describe the state of the universe, the most important are: Hubble parameter H; the deceleration parameter

$$q = -\frac{\ddot{a}\,a}{\dot{a}^2}\ , \tag{10}$$

which measures the cosmic acceleration of the universe; the parameters of density

$$\Omega = \frac{8\,\pi\,G}{3\,H^2}\,\rho = \rho/\rho_c, \quad \text{where} \quad \rho_c = \frac{3\,H^2}{8\,\pi\,G} \quad \text{is a critical energy density,} \tag{11}$$

$\Omega_i = \rho_i/\rho_c$, and ρ_i are mass densities of matter, radiation and dark energy, and others.

From Friedmann equations (9) follows,

$$\Omega - 1 = \frac{k}{H^2\,a^2}, \tag{12}$$

and the sign of constant k is determined by Ω, and it shows which of three FLRW metrics describes the universe. More precisely we have,

- $\Omega < 1, \quad \rho < \rho_c, \quad k = -1.$
- $\Omega = 1, \quad \rho = \rho_c, \quad k = 0.$
- $\Omega > 1, \quad \rho > \rho_c, \quad k = +1.$

Let us mention some cosmological parameters obtained from Planck 2018 [63] which describe the current state of the universe. The current Planck results for the ΛCDM universe are:

- $H_0 = (67.40 \pm 0.50)$ km/s/Mpc – Hubble parameter;
- $\Omega_m = 0.315 \pm 0.007$ – matter density parameter;
- $\Omega_\Lambda = 0.685$ – Λ density parameter;
- $t_0 = (13.801 \pm 0.024) \cdot 10^9$ yr – age of the universe;
- $w_0 = -1.03 \pm 0.03$ – ratio of pressure to energy density.

2.4. Nonlocal modified gravity.

As we mentioned in Section 1, GR has certain deficiencies, and should be modified. One of the most prominent approaches is nonlocal modification.

A nonlocal modified gravity model contains an infinite number of spacetime derivatives of the d'Alembert operator $\Box$, in the form of a power series expansion of it. In this article, we are mainly interested in nonlocality expressed in the form of an analytic function $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$, where coefficients f_n should be determined from various theoretical and phenomenological conditions. Some conditions are related to the absence of tachyons and ghosts.

Here, it is worth to mention some other interesting approaches to the nonlocal gravity, as approaches containing $\Box^{-1}$ (see e.g., [19, 18, 61, 56, 41, 62, 38, 39, 42, 43, 5, 53] and references therein), nonlocal models with power of the inverse d'Alembert operator, i. e. with $\Box^{-n}$, which are proposed to explain the late time cosmic acceleration without dark energy. Such models have the form

$$S = \frac{1}{16\pi G} \int \sqrt{-g}\, (R + L_{NL})\, d^4x, \tag{13}$$

where two typical examples are: $L_{NL} = R\, f(\Box^{-1}R)$ (see a review [55, 19] and references therein), and $L_{NL} = -\frac{1}{6}m^2 R\Box^{-2}R$ (see a review [31] and references therein).

Nonlocal models with $\mathcal{F}(\Box) = \sum\limits_{n=0}^{\infty} f_n \Box^n$ are mainly considered to improve general relativity in its ultraviolet region, unlike models with $\Box^{-1}$ and $\Box^{-2}$ which intend to modify gravity in its infrared sector. It may happen that there will be more than one modification of general relativity, which are valid at the different scales. Namely, any physical theory has a domain of validity, which depends on some conditions, including spatial scale and complexity of the system. It is natural that validity of general relativity is also restricted. At very short and very large cosmic distances may act different gravity theories.

Some other aspects of nonlocal gravity models have been considered, see e.g. [14, 11, 12, 54, 13, 31, 46] and references therein.

Our motivation to modify gravity in an analytic nonlocal way comes mainly from string theory, in particular from string field theory (see the very original effort in this direction in [2]) and p-adic string theory [3, 4, 10, 34, 35, 36, 60]. Since strings are one-dimensional extended objects, their field theory description contains spacetime nonlocality expressed by some exponential functions of d'Alembert operator $\Box$.

At classical level analytic non-local gravity has proven to alleviate the singularity of the Black-hole type because the Newtonian potential appears regular (tending to a constant) on a universal basis at the origin [37, 8, 6]. Also there was significant success in constructing classically stable solution for the cosmological bounce [8, 9, 44, 47, 50].

Analysis of perturbations revealed a natural ability of analytic non-local gravities to accommodate inflationary models. In particular, the Starobinsky inflation was studied in details and new predictions for the observable parameters were made [17, 49]. Moreover, in the quantum sector infinite derivative gravity theories improve renormalization, see e.g. while the unitarity is still preserved [51, 52, 49] (note that just a local quadratic curvature gravity was proven to be renormalizable while being non-unitary [59]). Later, we shall also investigate some conditions on the analytic function $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$, in order to escape unphysical degrees of freedom like ghosts and tachyons, and to have good behavior in quantum sector (see [5, 6, 7, 37]).

3. The equations of motion

Models of nonlocal gravity which we mainly investigate are given by the following action

$$S = \frac{1}{16\pi G} \int_M \left(R - 2\Lambda + \mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R) + \mathcal{L}_m\right) \sqrt{-g}\, d^4x, \tag{14}$$

where M is a pseudo-Riemannian manifold of signature $(1,3)$ with metric $(g_{\mu\nu})$, $\mathcal{F}(\Box) = \sum\limits_{n=0}^{\infty} f_n \Box^n$, $\mathcal{H}$ and $\mathcal{G}$ are differentiable functions of the scalar curvature R and Λ is cosmological constant, and $\mathcal{L}_m$ is Lagrangian of matter.

Now, we will give an overview of deriving of the equations of motion (EOM) for the action without $\mathcal{L}_m$ (since this part in cosmology is known) and without proofs which could be find in [26, 29]. Firstly, we are starting with some technical lemmas which are proved by using standard variational calculus and Stokes theorem. Note that variations of the metric tensor elements and their first derivatives are zero on the boundary of manifold M, i.e. $\delta g_{\mu\nu}|_{\partial M} = 0$, $\delta^n \partial_{x_\lambda} g_{\mu\nu}|_{\partial M} = 0, n \in \mathbb{N}$.

Lemma 1. *Let M be a pseudo-Riemannian manifold. Then the following basic relations hold:*

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu}, \tag{15}$$

$$\delta\sqrt{-g} = -\frac{1}{2} g_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}, \tag{16}$$

$$\delta\Gamma^{\lambda}_{\mu\nu} = -\frac{1}{2}\left(g_{\nu\alpha}\nabla_\mu \delta g^{\lambda\alpha} + g_{\mu\alpha}\nabla_\nu \delta g^{\lambda\alpha} - g_{\mu\alpha} g_{\nu\beta} \nabla^\lambda \delta g^{\alpha\beta}\right), \tag{17}$$

where g is the determinant of the metric tensor.

Lemma 2. *The variation of Riemman tensor, Ricci tensor and scalar curvature satisfy the following relations*

$$\delta R^{\alpha}_{\mu\beta\nu} = \nabla_\beta \delta\Gamma^{\alpha}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\alpha}_{\mu\beta}, \tag{18}$$

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\lambda}_{\mu\lambda}, \tag{19}$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} - K_{\mu\nu}\delta g^{\mu\nu}, \tag{20}$$

$$\delta\nabla_\mu\nabla_\nu\psi = \nabla_\mu\nabla_\nu\delta\psi - \nabla_\lambda\psi\,\delta\Gamma^{\lambda}_{\mu\nu}, \tag{21}$$

where $K_{\mu\nu} = \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box$, and where ψ is a scalar function.

Lemma 3. *For every scalar function $\mathcal{H}(R)$ holds*

$$\int_M \mathcal{H} g_{\mu\nu}(\Box\delta g^{\mu\nu})\sqrt{-g}\, d^4x = \int_M g_{\mu\nu}(\Box\mathcal{H})\delta g^{\mu\nu}\sqrt{-g}\, d^4x, \tag{22}$$

$$\int_M \mathcal{H}\nabla_\mu\nabla_\nu\delta g^{\mu\nu}\sqrt{-g}\, d^4x = \int_M \nabla_\mu\nabla_\nu\mathcal{H}\, \delta g^{\mu\nu}\sqrt{-g}\, d^4x, \tag{23}$$

$$\int_M \mathcal{H} K_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\, d^4x = \int_M K_{\mu\nu}\mathcal{H}\, \delta g^{\mu\nu}\sqrt{-g}\, d^4x. \tag{24}$$

Lemma 4. *Let $\mathcal{H}(R)$ and $\mathcal{G}(R)$ be scalar functions such that $\delta\mathcal{G}|_{\partial M}=0$. Then for all $n\in\mathbb{N}$ one has*

$$\int_M \mathcal{H}\delta\Box^n\mathcal{G}\sqrt{-g}\,d^4x=\frac12\sum_{l=0}^{n-1}\int_M S_{\mu\nu}(\Box^l\mathcal{H},\Box^{n-1-l}\mathcal{G})\delta g^{\mu\nu}\sqrt{-g}\,d^4x + \int_M \Box^n\mathcal{H}\,\delta\mathcal{G}\sqrt{-g}\,d^4x, \tag{25}$$

where $S_{\mu\nu}(A,B)=g_{\mu\nu}\nabla^\alpha A\nabla_\alpha B+g_{\mu\nu}A\Box B-2\nabla_\mu A\nabla_\nu B$.

Theorem 1. *Let $\mathcal{H}$ and $\mathcal{G}$ be scalar functions of scalar curvature, then*

$$\int_M \mathcal{H}\delta(\sqrt{-g})\,d^4x=-\frac12\int_M g_{\mu\nu}\mathcal{H}\delta g^{\mu\nu}\sqrt{-g}\,d^4x, \tag{26}$$

$$\int_M \mathcal{H}\delta R\sqrt{-g}\,d^4x=\int_M (R_{\mu\nu}\mathcal{H}-K_{\mu\nu}\mathcal{H})\,\delta g^{\mu\nu}\sqrt{-g}\,d^4x, \tag{27}$$

$$\int_M \mathcal{H}\delta(\mathcal{F}(\Box)\mathcal{G})\sqrt{-g}\,d^4x=\int_M (R_{\mu\nu}-K_{\mu\nu})\big(\mathcal{G}'\mathcal{F}(\Box)\mathcal{H}\big)\,\delta g^{\mu\nu}\sqrt{-g}\,d^4x + \frac12\sum_{n=1}^{\infty}f_n\sum_{l=0}^{n-1}\int_M S_{\mu\nu}(\Box^l\mathcal{H},\Box^{n-1-l}\mathcal{G})\delta g^{\mu\nu}\sqrt{-g}\,d^4x, \tag{28}$$

where $S_{\mu\nu}(A,B)=g_{\mu\nu}\nabla^\alpha A\nabla_\alpha B+g_{\mu\nu}A\Box B-2\nabla_\mu A\nabla_\nu B$.

Now, we can find the variation of the considered action (14). In order to calculate δS we introduce the following auxiliary actions

$$S_0=\int_M (R-2\Lambda)\sqrt{-g}\,d^4x, \tag{29}$$

$$S_1=\int_M \mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\sqrt{-g}\,d^4x. \tag{30}$$

Action S_0 is Einstein-Hilbert action and its variation is

$$\delta S_0=\int_M (G_{\mu\nu}+\Lambda g_{\mu\nu})\,\delta g^{\mu\nu}\sqrt{-g}\,d^4x. \tag{31}$$

Lemma 5. *Variation of the action S_1 is*

$$\begin{aligned}\delta S_1=&-\frac12\int_M g_{\mu\nu}\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\delta g^{\mu\nu}\sqrt{-g}\,d^4x\\ &+\int_M (R_{\mu\nu}W-K_{\mu\nu}W)\,\delta g^{\mu\nu}\sqrt{-g}\,d^4x\\ &+\frac12\sum_{n=1}^{\infty}f_n\sum_{l=0}^{n-1}\int_M S_{\mu\nu}(\Box^l\mathcal{H}(R),\Box^{n-1-l}\mathcal{G}(R))\delta g^{\mu\nu}\sqrt{-g}\,d^4x,\end{aligned} \tag{32}$$

where $W=\mathcal{H}'(R)\mathcal{F}(\Box)\mathcal{G}(R)+\mathcal{G}'(R)\mathcal{F}(\Box)\mathcal{H}(R)$.

Proof. Variation δS_1 is equal to

$$\begin{aligned}\delta S_1=\int_M \Big(&\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\delta(\sqrt{-g})+\delta\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\sqrt{-g}\\ &+\mathcal{H}(R)\delta(\mathcal{F}(\Box)\mathcal{G}(R))\sqrt{-g}\Big)d^4x.\end{aligned} \tag{33}$$

All the terms in the previous formula are obtained by Theorem 1. In particular (26) yields

$$\int_M \mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\delta(\sqrt{-g})\ d^4x = -\frac{1}{2}\int_M g_{\mu\nu}\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R)\ \delta g^{\mu\nu}\sqrt{-g}\ d^4x. \tag{34}$$

Also, from equation (27) we get

$$\begin{aligned}\int_M \delta(\mathcal{H}(R))\mathcal{F}(\Box)\mathcal{G}(R)\ \sqrt{-g}\ d^4x &= \int_M \mathcal{H}'(R)\delta R\ \mathcal{F}(\Box)\mathcal{G}(R)\ \sqrt{-g}\ d^4x \\ = \int_M \Big(R_{\mu\nu}\mathcal{H}'(R)\mathcal{F}(\Box)\mathcal{G}(R) &- K_{\mu\nu}\left(\mathcal{H}'(R)\mathcal{F}(\Box)\mathcal{G}(R)\right)\Big)\ \delta g^{\mu\nu}\sqrt{-g}\ d^4x.\end{aligned} \tag{35}$$

The last term is calculated by (28).

$$\begin{aligned}&\int_M \mathcal{H}(R)\delta(\mathcal{F}(\Box)\mathcal{G}(R))\ \sqrt{-g}d^4x \\ &= \int_M \Big(R_{\mu\nu}\mathcal{G}'(R)\mathcal{F}(\Box)\mathcal{H}(R) - K_{\mu\nu}\left(\mathcal{G}'(R)\mathcal{F}(\Box)\mathcal{H}(R)\right)\Big)\ \delta g^{\mu\nu}\sqrt{-g}\ d^4x \\ &+ \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\int_M S_{\mu\nu}\left(\Box^l\mathcal{H}(R), \Box^{n-1-l}\mathcal{G}(R)\right)\ \delta g^{\mu\nu}\sqrt{-g}\ d^4x.\end{aligned} \tag{36}$$

Adding equations (34), (35) and (36) together proves the Lemma. □

Theorem 2. *Variation of the action* (14) *is equal to zero iff*

$$0 = \hat{G}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R) + (R_{\mu\nu}W - K_{\mu\nu}W) + \frac{1}{2}\Omega_{\mu\nu}, \tag{37}$$

where

$$W = \mathcal{H}'(R)\mathcal{F}(\Box)\mathcal{G}(R) + \mathcal{G}'(R)\mathcal{F}(\Box)\mathcal{H}(R), \tag{38}$$

$$\Omega_{\mu\nu} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} S_{\mu\nu}\left(\Box^l\mathcal{H}(R), \Box^{n-1-l}\mathcal{G}(R)\right). \tag{39}$$

Proof. The proof of Theorem 2 is evident from the Lemma 5 and Theorem 1. □

Corollary 1. *Under the assumptions of previous theorem holds:*

(1) $\nabla^\mu \hat{G}_{\mu\nu} = 0$,

(2) Equations of motion (37) are invariant on the replacement of functions $\mathcal{G}$ *and* $\mathcal{H}$.

Since, in the case of FLRW metric only two of four (diagonal) non-trivial equations are linearly independent trace and 00 component, the equation (37) is equivalent to the following system:

$$\begin{aligned}&4\Lambda - R - 2\mathcal{H}(R)\,\mathcal{F}(\Box)\,\mathcal{G}(R) + R\,W + 3\,\Box\,W \\ &+ \sum_{n=1}^{\infty} f_n \sum_{\ell=0}^{n-1}\left(\partial_\mu\Box^\ell\mathcal{H}(R)\,\partial^\mu\Box^{n-1-\ell}\mathcal{G}(R) + 2\Box^\ell\mathcal{H}(R)\Box^{n-\ell}\mathcal{G}(R)\right) = 0,\end{aligned} \tag{40}$$

$$\begin{aligned}&G_{00} + \Lambda g_{00} - \frac{1}{2}g_{00}\mathcal{H}(R)\,\mathcal{F}(\Box)\,\mathcal{G}(R) + R_{00}\,W - K_{00}\,W \\ &+ \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{\ell=0}^{n-1}\big(g_{00}\,g^{\alpha\beta}\,\partial_\alpha\Box^\ell\mathcal{H}(R)\,\partial_\beta\Box^{n-1-\ell}\mathcal{G}(R) \\ &- 2\partial_0\Box^\ell\mathcal{H}(R)\partial_0\Box^{n-1-\ell}\mathcal{G}(R) + g_{00}\,\Box^\ell\mathcal{H}(R)\Box^{n-\ell}\mathcal{G}(R)\big) = 0.\end{aligned} \tag{41}$$

4. Cosmological solutions of EOM

The search for a general solution of the scale factor $a(t)$ of equations (40) and (41) is a very ambitious, and because of that we use the following ansatzs, see [22, 23, 24, 32, 27, 28, 45]:

- $\Box R = rR + s$, where r and s are constants.
- $\Box R = qR^2$, where q is a constant.
- $\Box R = qR^3$, where q is a constant.
- $\Box^n R = c_n R^{n+1},\ n \geq 1$, where c_n are constants.
- $\Box (R + R_0)^m = p\,(R + R_0)^m$, where $m \in \mathbb{Q}$ and R_0, p are constants.

In fact these ansatzs make some constraints on possible solutions, but on the other hand they simplify formalism to find a particular solution.

We consider several models of nonlocal gravity without matter which are described by the action (14),

$$S = \frac{1}{16\pi G}\int_M \Big(R - 2\Lambda + \mathcal{H}(R)\,\mathcal{F}(\Box)\,\mathcal{G}(R)\Big)\sqrt{-g}\,d^4x, \tag{42}$$

for the following choice of functions $\mathcal{H}$ and $\mathcal{G}$:

1. $\mathcal{H}(R) = R$, $\mathcal{G}(R) = R$.
2. $\mathcal{H}(R) = R^{-1}$, $\mathcal{G}(R) = R$.
3. $\mathcal{H}(R) = R^p$, $\mathcal{G}(R) = R^q$.
4. $R = \text{const}$.
5. $\mathcal{H}(R) = (R + R_0)^m$, $\mathcal{G}(R) = (R + R_0)^m$.
6. $\mathcal{H}(R) = \mathcal{G}(R) = \sqrt{R - 2\,\Lambda}$.

Let us mention that for all cases we consider different scale factors, and cases 1., 2., 5. are not special case of 3. Now we will give short overview on the all six models.

4.1. Case 1: $\mathcal{H}(R) = R$, $\mathcal{G}(R) = R$.

In this model we use linear ansatz (fore more details see [23, 32, 28], with scaling factor $a(t) = a_0(\sigma e^{\lambda t} + \tau e^{-\lambda t}),\ a_0 > 0,\ \lambda, \sigma, \tau \in \mathbb{R}$ and firstly, we have:

Lemma 6. *(i1) For* $n \in \mathbb{N},\ r, s \in \mathbb{R}$ *holds*

$$\Box^n R = r^n(R + \frac{s}{r}),\ n \geq 1, \qquad \mathcal{F}(\Box)R = \mathcal{F}(r)R + \frac{s}{r}(\mathcal{F}(r) - f_0).$$

(i2) For given scaling factor hold

$$H(t) = \frac{\lambda(\sigma e^{\lambda t} - \tau e^{-\lambda t})}{\sigma e^{\lambda t} + \tau e^{-\lambda t}}, \quad R(t) = \frac{6\left(2\,a_0^2\ \lambda^2\left(\sigma^2 e^{4t\lambda} + \tau^2\right) + k\,e^{2t\lambda}\right)}{a_0^2\left(\sigma e^{2t\lambda} + \tau\right)^2},$$

$$\Box\,R = -\frac{12\,\lambda^2 e^{2t\lambda}\left(4\,a_0^2\,\lambda^2\sigma\tau - k\right)}{a_0^2\left(\sigma e^{2t\lambda} + \tau\right)^2}.$$

(i3) $\quad \Box R = 2\lambda^2 R - 24\lambda^4, \qquad r = 2\lambda^2,\ s = -24\lambda^4.$

Using previous Lemma, and EOM we have the following theorem.

Theorem 3. *The scaling factor of the form $a(t) = a_0\,(\sigma\, e^{\lambda t} + \tau\, e^{-\lambda t})$ is a solution of EOM in the following three cases:*

$$\textit{Case 1.}\quad \mathcal{F}\left(2\lambda^2\right) = 0, \quad \mathcal{F}'\left(2\lambda^2\right) = 0, \quad f_0 = -\frac{1}{8\Lambda}. \tag{43}$$

$$\textit{Case 2.}\quad 3k = 4\,a_0^2\,\Lambda\,\sigma\,\tau. \tag{44}$$

$$\textit{Case 3.}\quad \mathcal{F}\left(2\lambda^2\right) = \frac{1}{12\,\Lambda} + \frac{2}{3}\,f_0, \quad \mathcal{F}'\left(2\lambda^2\right) = 0, \quad k = -4\,a_0^2\,\Lambda\,\sigma\,\tau. \tag{45}$$

In all three cases holds $3\,\lambda^2 = \Lambda$.

From previous theorem easily follows:

In the *Case 1.* there exist a solution for arbitrary σ, τ and a_0

$$a(t) = a_0(\sigma e^{\lambda t} + \tau e^{-\lambda t}),$$

where $\mathcal{F}$ satisfies (43) and for arbitrary $k = 0, \pm 1$. This solution generalize the case $a(t) = a_0 \cosh\left(\sqrt{\frac{\Lambda}{3}}t\right)$, which is obtained in [8].

In the *Case 2.* we obtained a family of solutions for arbitrary $\sigma \neq 0$, a_0 and arbitrary analytic function $\mathcal{F}$

$$a(t) = a_0\left(\sigma e^{\lambda t} + \frac{3\,k}{4\,a_0^2\,\Lambda\,\sigma} e^{-\lambda\,t}\right).$$

The *Case 3.* also gives a family of solutions

$$a(t) = a_0\left(\sigma e^{\lambda t} - \frac{k}{4\,a_0^2\,\Lambda\,\sigma} e^{-\lambda\,t}\right),$$

where function $\mathcal{F}$ satisfies conditions (45).

Let us mention that in the *Case 2.* for $k = 0$ equation (44) and conditions (45) in *Case 3.* coincide, and consequently σ or τ must vanish, and we have two nonsingular de Sitter solutions, see ([9]),

$$a_1(t) = a_0 e^{\lambda t}, \qquad a_2(t) = a_0 e^{-\lambda t},$$

and that all solutions satisfies

$$\ddot{a}(t) = \lambda^2 a(t) > 0.$$

4.2. Case 2: $\mathcal{H}(R) = R^{-1}$, $\mathcal{G}(R) = R$.

In this model it is evident from action given by (42) that nonlocal term $R^{-1}\mathcal{F}(\Box)R$ is invariant under the transformation $R \longrightarrow c\,R$, $c \in \mathbb{R}^*$. Let us remark that in this case f_0 plays role of cosmological constant, and because of that we put $\Lambda = 0$. We are searching for the solutions of EOM with the scaling factor of the form $a(t) = a_0|t - t_0|^\alpha$. For more details see [24, 32, 28]

Firstly we have a lemma.

Lemma 7. *Let us consider the nonlocality of the form $\mathcal{H}(R) = R^{-1}$, $\mathcal{G}(R) = R$, with the scaling factor $a(t) = a_0|t - t_0|^\alpha$, and for $\Lambda = 0$. Then*

(i1) $R(t) = 6\big(\alpha(2\alpha - 1)(t - t_0)^{-2} + \frac{k}{a_0^2}(t - t_0)^{-2\alpha}\big)$.

(i2) $\Box\, R = q\, R^2$, *where q depend on α.*

Proof. (i1) It is easy to obtained this formula from the expression of R given in (2). (i2) Imposing condition $\Box R = q R^2$, we find

$$\alpha(2\alpha-1)(q\alpha(2\alpha-1)-(\alpha-1))(t-t_0)^{-4}+\frac{qk^2}{a_0^4}(t-t_0)^{-4\alpha} \\ +\frac{\alpha k}{3a_0^2}(1-\alpha+6q(2\alpha-1))(t-t_0)^{-2\alpha-2}=0. \tag{46}$$

The equation (46) is satisfied for all values of time t in six cases:

1. $k=0,\ \alpha=0,\ q\in\mathbb{R}$,
2. $k=0,\ \alpha=\frac{1}{2},\ q\in\mathbb{R}$,
3. $k=0,\ \alpha\neq 0$ and $\alpha\neq\frac{1}{2}$, $q=\frac{\alpha-1}{\alpha(2\alpha-1)}$,
4. $k=-1,\ \alpha=1,\ q\neq 0,\ a_0=1$,
5. $k\neq 0,\ \alpha=0,\ q=0$,
6. $k\neq 0,\ \alpha=1,\ q=0$.

In the cases (1), (2) and (4) we have $R=0$ and therefore R^{-1} is not defined. The case (5) yields a solution which does not satisfy equations of motion. □

The cases (3) and (6) from above lemma need additional investigations (for details, see [24]) which we will omit here, giving the following solutions.

Theorem 4. *The scale factor $a(t)=a_0|t-t_0|^\alpha$ for $\Lambda=0$ is a solution of EOM in the following cases:*

(3) For $k=0$, $\alpha\neq 0$, $\alpha\neq\frac{1}{2}$ and $\frac{3\alpha-1}{2}\in\mathbb{N}$, the coefficients of $\mathcal{F}$ are

$$f_0=0,\qquad f_1=-\frac{3\alpha(2\alpha-1)}{2(3\alpha-2)},$$
$$f_n=0\qquad \textit{for}\quad 2\leq n\leq\frac{3\alpha-1}{2},$$
$$f_n\in\mathbb{R}\qquad \textit{for}\quad n>\frac{3\alpha-1}{2}.$$

(6) For $k\neq 0$, the coefficients of $\mathcal{F}$ are

$$f_0=0,\qquad f_1=-\frac{s}{4},\qquad f_n\in\mathbb{R},\quad n\geq 2,$$

where $s=6\big(1+\frac{k}{a_0^2}\big)$.

Let us remark here that in the both cases $k=0,\alpha\neq 0,1/2$, the obtained solutions have not as its background Minkowski space, and as a special case of (6) is the solution $a(t)=|t-t_0|$ for $k=-1$ which corresponds to the Milne model of the universe.

4.3. 3. Case: $\mathcal{H}(R)=R^p$, $\mathcal{G}(R)=R^q$.

We consider the model given by $\mathcal{H}(R)=R^p$, $\mathcal{G}(R)=R^q$ with the scaling factor $a(t)=a_0\,e^{-\frac{\gamma}{12}t^2}$, $\gamma\in\mathbb{R}$, for details see [25, 20] Let us mention if $\gamma=0$ then M is a Minkowski space which is a solution of EOM (37) for $\Lambda=0$. In the following analysis is independent of the sign of γ, and one can obtain models in which the universe is expanding ($\gamma<0$) and collapsing ($\gamma>0$). Using appropriate formulas, firstly we have technical lemma.

Lemma 8. *In the model of nonlocality given by* $\mathcal{H}(R) = R^p$, $\mathcal{G}(R) = R^q$ *with metric* $a(t) = a_0\, e^{-\frac{\gamma}{12}t^2}$, $\gamma \in \mathbb{R}$, *hold*

$$\text{(1)} \quad H(t) = -\frac{1}{6}\,\gamma\, t, \quad R(t) = \frac{1}{3}\,\gamma\,(\gamma\, t^2 - 3), \quad R_{00} = \frac{1}{4}\,(\gamma - R). \tag{47}$$

$$\text{(2)} \quad \Box\, R^p = p\,\gamma R^p - \frac{p}{3}\,(4p-5)\,\gamma^2\, R^{p-1} - \frac{4}{3}\,p\,(p-1)\,\gamma^3 R^{p-2}. \tag{48}$$

Equation (48) implies that linear space $V_p = span\{1, R, R^2, \dots, R^p\}$ is invariant under the action of $\Box$. Using identities from previous lemma one can show that trace and 00 equation are polynomial in R of degree $p+q$. If we consider only leading coefficients of both equations, for $p \neq q$ we obtain an linearly dependent system

$$p\mathcal{F}(q\gamma)(q-p+2) + q\mathcal{F}(p\gamma)(q-p-2) = 0, \tag{49}$$

$$(-q - \frac{1}{2}p(q-p))\mathcal{F}(q\gamma) + (-\frac{1}{2}q(q-p) + q)\mathcal{F}(p\gamma) = 0. \tag{50}$$

Similarly, in the case when $p = q$ we will obtain linearly dependent system. So, we have the following theorem.

Theorem 5. *Let us consider nonlocality given by* $\mathcal{H}(R) = R^p$, $\mathcal{G}(R) = R^q$ *with scaling factor* $a(t) = a_0\, e^{-\frac{\gamma}{12}t^2}$, $\gamma \in \mathbb{R}$. *Then:*

(i1) for any $p, q \in \mathbb{N}$ *trace and 00 equation are equivalent.*

(i2) The trace equation is of polynomial type of degree $p+q$ *in* R, *with coefficients depending on* $f_0 = \mathcal{F}(0)$, $\mathcal{F}(\gamma)$, $\dots, \mathcal{F}(p\gamma)$, $\mathcal{F}'(\gamma)$, $\dots, \mathcal{F}'(q\gamma)$.

(i3) for $p = q = 1$, *trace equation is satisfied iff* $\gamma = -12\Lambda$, $\mathcal{F}'(\gamma) = 0$ *and* $f_0 = \frac{3}{2\gamma} - 8\mathcal{F}(\gamma)$. *In this case system has infinitely many solutions.*

Remark 1. *1. In this model, for the most simple case* $p = q = 1$, *the scaling factor of the form* $a(t) = a_0\, e^{\Lambda\, t^2}$, *firstly was considered by Koshelev and Vernov in the paper [44].*

2. The exact solutions, for $1 \le q \le p \le 4$, *are found by I. Dimitrijević in his PhD thesis. It is not possible to obtain the solution for arbitrary* p *and* q *in closed form.*

4.4. Case 4: $R = R_0 = \text{const}$.

In this case we do not have any condition on functions $\mathcal{H}$ and $\mathcal{G}$, for details see [26, 27, 28, 21]. The condition $R = R_0$ plays role of an ansatz, and immediately one get

$$6\Big(\frac{\ddot a}{a} + (\frac{\dot a}{a})^2 + \frac{k}{a^2}\Big) = R_0. \tag{51}$$

After the change of variable $b(t) = a^2(t)$ we obtain a second order linear differential equation with constant coefficients

$$3\,\ddot b - R_0\, b = -6\,k. \tag{52}$$

Depending on the sign of R_0 there exist the following solutions for $b(t)$,

$$\begin{aligned} R_0 > 0, &\qquad b(t) = \frac{6\,k}{R_0} + \sigma\, e^{\sqrt{\frac{R_0}{3}}t} + \tau e^{-\sqrt{\frac{R_0}{3}}t}, \\ R_0 = 0, &\qquad b(t) = -k\, t^2 + \sigma\, t + \tau, \\ R_0 < 0, &\qquad b(t) = \frac{6\,k}{R_0} + \sigma \cos\sqrt{\frac{-R_0}{3}}t + \tau \sin\sqrt{\frac{-R_0}{3}}t. \end{aligned} \tag{53}$$

If we now replace $R = R_0$ into trace equation (40) and 00 equation (41), we obtain a system of equation defined by the coefficient f_0

$$-2\,U + R_0\,W = R_0 - 4\,\Lambda, \qquad \frac{1}{2}\,U + R_{00}\,W \quad = \Lambda - G_{00}, \tag{54}$$

where $U = f_0\,\mathcal{H}(R_0)\,\mathcal{G}(R_0)$ and $W = f_0\,\frac{\partial}{\partial R}\Big(\mathcal{H}(R)\,\mathcal{G}(R)\Big)|_{R=R_0}$.

After elimination of U from (54) we obtain, for more detail see [21],

$$(R_0 + 4R_{00})\,(W + 1) = 0. \tag{55}$$

In the case when $R_0 + 4R_{00} = 0$, the following conditions on the parameters σ and τ hold:

$$\begin{aligned} &R_0 > 0, \qquad 9\,k^2 = R_0^2\,\sigma\,\tau,\\ &R_0 = 0, \qquad \sigma^2 + 4\,k\,\tau = 0,\\ &R_0 < 0, \qquad 36\,k^2 = R_0^2\,(\sigma^2 + \tau^2). \end{aligned} \tag{56}$$

Finally, we get the following theorem.

Theorem 6. *Let $R = R_0 = constant$. Then, we have solutions in the following cases:*

1. *If $R_0 > 0$ then for $k = 0$ there is a solution with constant Hubble parameter, for $k = +1$ the solution is $a(t) = \sqrt{\frac{12}{R_0}}\cosh\frac{1}{2}\left(\sqrt{\frac{R_0}{3}}t + \varphi\right)$ and for $k = -1$ it is $a(t) = = \sqrt{\frac{12}{R_0}}\left|\sinh\frac{1}{2}\left(\sqrt{\frac{R_0}{3}}t + \varphi\right)\right|$, where $\sigma + \tau = \frac{6}{R_0}\cosh\varphi$ and $\sigma - \tau = \frac{6}{R_0}\sinh\varphi$.*
2. *If $R_0 = 0$ then for $k = 0$ the solution is $a(t) = \sqrt{\tau} = const$ and for $k = -1$ the solution is $a(t) = |t + \frac{\sigma}{2}|$.*
3. *If $R_0 < 0$ then for $k = -1$ the solution is $a(t) = \sqrt{\frac{-12}{R_0}}\left|\cos\frac{1}{2}\left(\sqrt{-\frac{R_0}{3}}t - \varphi\right)\right|$, where $\sigma = \frac{-6}{R_0}\cos\varphi$ and $\tau = \frac{-6}{R_0}\sin\varphi$.*

Proof. The proof directly follows from (53) and conditions (56), for more details see [26, 21]. □

Remark 2. *The second case of the equation (55) gives the solution*

$$W = -1, \qquad U = 2\Lambda - R_0. \tag{57}$$

Equations (54) have common solution in f_0 iff the following condition is satisfied,

$$\mathcal{H}(R_0)\,\mathcal{G}(R_0) - (R_0 - 2\Lambda)\frac{\partial}{\partial R}\big(\mathcal{H}(R)\,\mathcal{G}(R)\big)|_{R=R_0} = 0. \tag{58}$$

4.5. Case 5: $\mathcal{H}(R) = (R + R_0)^m$, $\mathcal{G}(R) = (R + R_0)^m$.

Here we consider the action (14) given by $\mathcal{H} = \mathcal{G} = (R + R_0)^m$, where R_0 and m are real constants and scaling factor $a(t)$ is of the form

$$a(t) = A\,t^n e^{-\frac{\gamma}{12}t^2}, \tag{59}$$

for more details see [28, 29]

Let us consider the ansatz in the form

$$\Box(R+R_0)^m = r(R+R_0)^m + s, \tag{60}$$

where R_0, r, s, m are n real constants. In the case when $s=0$, the anzatz gives the following system of equations:

$$\begin{aligned}
0 &= -648mn^2(2n-1)^2(2m-3n+1),\\
0 &= -324n(2n-1)\left(-\gamma m + 6\gamma mn^2 - 4\gamma mn - mnR_0 + mR_0 + 2n^2r - nr\right),\\
0 &= 18n(2n-1)\left(8\gamma^2m^2 - 13\gamma^2m + 12\gamma^2mn - 3\gamma mR_0 + 24\gamma nr + 6\gamma r - 6rR_0\right),\\
0 &= -2\gamma^3m - 24\gamma^3mn^2 - 14\gamma^3mn + 6\gamma^2mnR_0 + 2\gamma^2mR_0 + 72\gamma^2n^2r + 12\gamma^2nr\\
&\qquad - 24\gamma nrR_0 + 3\gamma^2r - 6\gamma rR_0 + 3rR_0^2,\\
0 &= -\gamma^2\left(4\gamma^2m^2 + \gamma^2m + 18\gamma^2mn - 3\gamma mR_0 - 24\gamma nr - 6\gamma r + 6rR_0\right),\\
0 &= -\gamma^4(r-\gamma m).
\end{aligned}$$

This system has five solutions:

1. $r=m\gamma$, $n=0$, $R_0=\gamma$, $m=\frac{1}{2}$
2. $r=m\gamma$, $n=0$, $R_0=\frac{\gamma}{3}$, $m=1$
3. $r=m\gamma$, $n=\frac{1}{2}$, $R_0=\frac{4}{3}\gamma$, $m=1$
4. $r=m\gamma$, $n=\frac{1}{2}$, $R_0=3\gamma$, $m=-\frac{1}{4}$
5. $r=m\gamma$, $n=\frac{2m+1}{3}$, $R_0=\frac{7}{3}\gamma$, $m=\frac{1}{2}$.

The case 2. is considered in the Subsection 3 and case 3. is known and considered in ([33]).

Let us consider now the case 1: $n=0$, $m=\frac{1}{2}$, then the anzatz is reduced to $\Box\sqrt{R+\gamma} = \frac{1}{2}\gamma\sqrt{R+\gamma}$, and we have the following consequence: $\mathcal{F}(\Box)\sqrt{R+\gamma} = \mathcal{F}(\frac{1}{2}\gamma)\sqrt{R+\gamma}$.

Trace equation (40) and 00 equation (41) are linear in R and they define two systems of equations in $\mathcal{F}(\frac{\gamma}{2})$ and $\mathcal{F}'(\frac{\gamma}{2})$:

$$\gamma + 4\Lambda - \gamma\mathcal{F}(\frac{\gamma}{2}) - \frac{\gamma^2}{3}\mathcal{F}'(\frac{\gamma}{2}) = 0, \quad -\frac{\gamma^2}{3} - \frac{\gamma^2}{3}\mathcal{F}(\frac{\gamma}{2}) + \frac{\gamma^3}{3}\mathcal{F}'(\frac{\gamma}{2}) = 0. \tag{61}$$

$$-\Lambda + \frac{\gamma}{2}\mathcal{F}(\frac{\gamma}{2}) + \frac{\gamma^2}{6}\mathcal{F}'(\frac{\gamma}{2}) = 0, \quad \gamma^2 + \gamma^2\mathcal{F}(\frac{\gamma}{2}) - \gamma^3\mathcal{F}'(\frac{\gamma}{2}) = 0. \tag{62}$$

The solutions of these systems are the same for $\gamma = -2\Lambda$ and equal to:

$$\mathcal{F}(\frac{\gamma}{2}) = -1, \qquad \mathcal{F}'(\frac{\gamma}{2}) = 0. \tag{63}$$

It is clear from (63) that considered case is a generalization of the Case 3. Here we allow that p, q are rational numbers. In this case we have $p+q=1$, and similarly as in Theorem 5 there exist a unique solution in $\mathcal{F}(\frac{\gamma}{2})$ and $\mathcal{F}'(\frac{\gamma}{2})$.

Taking the similar analysis it was shown that in the Case 4. there are no solutions which satisfy EOM.

Finally, in the Case 5. the parameters are: $r=\frac{\gamma}{2}$, $n=\frac{2}{3}$, $R_0=\frac{7}{3}\gamma$, $m=\frac{1}{2}$. Since, the action is the same as in the 3. and similar calculations, using trace and 00 equations give

$$\mathcal{F}(\frac{\gamma}{2}) = -1, \qquad \mathcal{F}'(\frac{\gamma}{2}) = 0, \qquad \Lambda = -\frac{7}{6}\gamma. \tag{64}$$

Let us remark that in (64) the solution could be given in terms of $\mathcal{F}(\frac{\gamma}{2})$ and $\mathcal{F}'(\frac{\gamma}{2})$, with a constrain which connects cosmological constant Λ, and γ. Similar case was considered in the Case 3 for $p = q = 1$.

4.6. Case 5: $\mathcal{H}(R) = \mathcal{G}(R) = \sqrt{R - 2\,\Lambda}$.

In this nonlocal model (for more details see [29, 28, 30]) we can rewrite the action (14) in more compact form

$$S = \frac{1}{16\pi G} \int \sqrt{R - 2\Lambda}\, F(\Box)\, \sqrt{R - 2\Lambda}\, \sqrt{-g}\; d^4x, \tag{65}$$

where $F(\Box) = 1 + \sum_{n=1}^{\infty} f_n\, \Box^n$.

The related Friedmann equations (9) are

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p}) + \frac{\Lambda}{3}\,, \qquad \frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G}{3}\bar{\rho} + \frac{\Lambda}{3}\,, \tag{66}$$

where $\bar{\rho}$ and $\bar{p}$ are analogs of the energy density and pressure of the dark side of the universe, respectively. Denote the corresponding equation of state as $\bar{p}(t) = \bar{w}(t)\, \bar{\rho}(t)$.

In this case we consider several subcases defining by different scaling factors. We mention three of them, and emphasize the first one since this case is the most important.

4.6.1. Cosmological solution $a(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14}t^2}\,,\ k = 0$.

Theorem 7. *In the case of nonlocality $\mathcal{H}(R) = \mathcal{G}(R) = \sqrt{R - 2\,\Lambda}$ with scaling factor of the form $a(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14}t^2}$ and $k = 0$ holds:*

$$R(t) = \frac{4}{3}t^{-2} + \frac{22}{7}\Lambda + \frac{12}{49}\Lambda^2 t^2, \qquad H(t) = \frac{2}{3}t^{-1} + \frac{1}{7}\Lambda t. \tag{67}$$

There exists a cosmological solution for ansatz $\Box\sqrt{R - 2\Lambda} = -\frac{3}{7}\Lambda\sqrt{R - 2\Lambda}$, and conditions

$$\mathcal{F}\Big(-\frac{3}{7}\Lambda\Big) = -1\,, \qquad \mathcal{F}'\Big(-\frac{3}{7}\Lambda\Big) = 0\,, \qquad \Lambda \neq 0.$$

Let us remark that in this case we have,

$$R_{00} = \frac{2}{3}\,t^{-2} - \Lambda - \frac{3}{49}\Lambda^2 t^2\,, \qquad G_{00} = \frac{4}{3}\,t^{-2} + \frac{4}{7}\Lambda + \frac{3}{49}\Lambda^2 t^2\,, \tag{68}$$

and

$$\bar{\rho}(t) = \frac{1}{12\pi G}\,\big(\frac{2}{t^{-2}} + \frac{9}{98}\Lambda^2 t^2 - \frac{9}{14}\Lambda\big)\,, \qquad \bar{p}(t) = -\frac{\Lambda}{56\pi G}\big(\frac{3}{7}\Lambda t^2 - 1\big). \tag{69}$$

This cosmological solution for $a(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14}t^2}$ can be viewed as a product of $t^{\frac{2}{3}}$ factor, related to the matter dominated case in Einstein's gravity, and $e^{\frac{\Lambda}{14}t^2}$ which is related to an acceleration. Moreover, the Hubble parameter consists of two terms: $\frac{2}{3}t^{-1}$ is just $H(t)$ in Einstein's theory of gravity for the universe dominated by matter, which play leading role for small t; the second term $\frac{1}{7}\Lambda t$ corresponds to an acceleration for $\Lambda > 0$, which is dominant role for larger times. Time dependent expansion acceleration is given by

$$\ddot{a}(t) = \big(-\frac{2}{9}t^{-2} + \frac{\Lambda}{3} + \frac{\Lambda^2}{49}t^2\big)\, a(t). \tag{70}$$

Also, according to expressions (69) follows that $\bar{w}(t) \to -1$ when $t \to \infty$, what corresponds to an analog of Λ dark energy dominance in the standard cosmological model. Therefore, one can say that nonlocal gravity model (65) with cosmological solution $a(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14}t^2}$ describes some effects usually attributed to the dark matter and dark energy. This solution is invariant under transformation $t \to -t$ and singular at cosmic time $t = 0$. Namely, $R(t)$, $H(t)$ and $\bar{\rho}(t)$ tend to $+\infty$ when $t \to 0$, while $\bar{\rho}(0)$ is finite.

Taking the above Planck results for t_0 and H_0 in the second formula of (67) one obtains $\Lambda = 1,05 \cdot 10^{-35}\, s^{-2}$ (in $c = 1$ units). This is close to $\Lambda = 0.98 \cdot 10^{-35} s^{-2}$ calculated by standard formula $\Lambda = 3H_0^2\Omega_\Lambda$. From the same formula (67) one can also find time (t_m) for which the Hubble parameter has minimum value H_m, i.e. $t_m = 21,1 \cdot 10^9$ yr and $H_m = 61,72$ km/s/Mpc.

From (70) one can find that beginning of the universe expansion acceleration was at $t_a = 7,84 \cdot 10^9$ yr, or in other words at 5.96 billion years ago.

The first of Friedmann equations, $\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\bar{\rho} + \frac{\Lambda}{3}$, combined with expression (67) for the Hubble parameter, gives the critical energy density ρ_c and the energy density of the dark matter $\bar{\rho}$ for the solution $a(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14}t^2}$:

$$\rho_c = \frac{3}{8\pi G} H_0^2 = 8,51 \cdot 10^{-30} \frac{g}{cm^3} \tag{71}$$

$$\bar{\rho} = \Big(\frac{4}{9} t_0^{-2} - \frac{\Lambda}{7} + \frac{\Lambda^2}{49} t_0^2\Big)\frac{3}{8\pi G} = 2,26 \cdot 10^{-30}\, \frac{g}{cm^3}. \tag{72}$$

From above formula, it follows that $\bar{\Omega} = \frac{\bar{\rho}}{\rho_c} = 0,265$, and since Ω_v for the visible matter is approximatively $\Omega_v = 0,05$, then the value of $\bar{\Omega}_\Lambda = 1 - \bar{\Omega} - \Omega_v = 0,685$ coincides with the value obtained from Planck 2018 mission.

4.6.2. Another cosmological solution.

In this model we consider different scale factors and different type of the universe. Results are given in the following theorem.

Theorem 8. *(i1) In the case of nonlocality $\mathcal{H}(R) = \mathcal{G}(R) = \sqrt{R - 2\,\Lambda}$ with scaling factor of the form $a(t) = A\, e^{\frac{\Lambda}{6}t^2}$ and $k = 0$ holds:*

$$R(t) = 2\Lambda(1 + \frac{2}{3}\,\Lambda\, t^2), \qquad H(t) = \frac{1}{3}\,\Lambda\, t. \tag{73}$$

There exists a cosmological solution for ansatz $\Box\sqrt{R - 2\Lambda} = -\Lambda\sqrt{R - 2\Lambda}$, and conditions

$$\mathcal{F}(-\Lambda) = \sum_{n=1}^{+\infty} f_n(-\Lambda)^n = -1\,, \quad \mathcal{F}'(-\Lambda) = \sum_{n=1}^{+\infty} f_n\, n(-\Lambda)^{n-1} = 0. \tag{74}$$

(i2) In the case of nonlocality $\mathcal{H}(R) = \mathcal{G}(R) = \sqrt{R - 2\,\Lambda}$ with scaling factor of the form $a(t) = A\, e^{\pm\sqrt{\frac{\Lambda}{6}}\,t}$ and $k = \pm 1$ holds:

$$R(t) = \frac{6\,k}{A^2} e^{\mp\sqrt{\frac{2}{3}\Lambda}\,t} + 2\Lambda, \qquad H = \pm\sqrt{\frac{\Lambda}{6}}\,. \tag{75}$$

There exists a cosmological solution for ansatz $\Box\sqrt{R - 2\Lambda} = \frac{\Lambda}{3}\sqrt{R - 2\Lambda}$, and conditions

$$\mathcal{F}\big(\frac{\Lambda}{3}\big) = -1\,, \qquad \mathcal{F}'\big(\frac{\Lambda}{3}\big) = 0. \tag{76}$$

Remark 3. *(i1) In this case one can calculate R_{00} and G_{00},*

$$R_{00} = -\frac{\Lambda^2}{3}\,t^2 - \Lambda\,, \quad G_{00} = \frac{\Lambda^2}{3}\,t^2\,. \tag{77}$$

Also, we have

$$\bar\rho(t) = \frac{\Lambda}{8\pi G}\big(\frac{\Lambda}{3}t^2 - 1\big)\,, \quad \bar p(t) = -\frac{\Lambda}{24\pi G}\big(\Lambda t^2 - 1\big). \tag{78}$$

Solution $a(t) = A\,e^{\frac{\Lambda}{6}t^2}$ is nonsingular with $R(0) = 2\Lambda$ and $H(0) = 0$. There is acceleration expansion $\ddot a(t) = \big(\frac{\Lambda}{3} + \frac{\Lambda^2}{9}\,t^2\big)\,a(t)$ which is positive and increasing with time.

(i2) Similarly to the previous case we find

$$R_{00} = -\frac{\Lambda}{2}, \qquad G_{00} = \frac{3k}{A^2}e^{\mp\sqrt{\frac{2}{3}\Lambda}\,t} + \frac{\Lambda}{2}. \tag{79}$$

$\bar\rho$ and $\bar p$ in this case are

$$\bar\rho(t) = \frac{1}{8\pi G}\,\big(-\frac{\Lambda}{2} + \frac{3\,k}{A^2}\,e^{\mp\sqrt{\frac{2}{3}\Lambda}\,t}\big)\,, \qquad \bar p(t) = \frac{1}{8\pi G}\,\big(\frac{\Lambda}{2} - \frac{k}{A^2}\,e^{\mp\sqrt{\frac{2}{3}\Lambda}\,t}\big). \tag{80}$$

We have two solutions: (1) $a(t) = A\,e^{\sqrt{\frac{\Lambda}{6}}\,t}$ and (2) $a(t) = A\,e^{-\sqrt{\frac{\Lambda}{6}}\,t}$, for both $k = +1$ and $k = -1$. They are similar to the de Sitter solution $a(t) = A\,e^{\pm\sqrt{\frac{\Lambda}{3}}\,t}$, $k = 0$, but have time dependent $R(t), \bar\rho(t)$ and $\bar p(t)$. When $t \to +\infty$, parameter $\bar w(t) \to -1$ in the case (1) and $\bar w(t) \to -\frac{1}{3}$ for solution (2).

5. Perturbations

This section is based on the papers [27, 30, 21, 48].

5.1. Conformal time.

In this section we are searching the cosmological spatially flat de Sitter which can be written as

$$ds^2 = -dt^2 + a_0^2 e^{2Ht} d\vec x^{\,2} = -dt^2 + a(t)^2 d\vec x^{\,2}, \tag{81}$$

where H is constant, t the cosmic time and $\vec x$ is the 3-dimensional vector. The last equality shows that this metric is a particular case of a spatially flat FLRW metric.

It is more convenient to work with the conformal time τ in perturbation theory, which is defined by $a\,d\tau = dt$. Then the general FLRW metric (81) transforms to

$$ds^2 = a(\tau)^2(-d\tau^2 + d\vec x^2), \tag{82}$$

and connection between cosmic and conformal time is given by

$$\tau = -\frac{1}{a_0 H}e^{-Ht} \;\Rightarrow\; a(\tau) = -\frac{1}{H\tau}.$$

So when t goes from past to future infinity, τ goes from $-\infty$ to 0_-. $t = 0$ corresponds to $\tau = -\frac{1}{a_0 H}$.

5.2. Perturbations.

The variation of the metric is as usual

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,. \tag{83}$$

where bars denote the back ground metric, and we will use this convention for background quantities.

Perturbations of equations of motion (37) around the de Sitter vacuum are

$$-m^2\,\delta\,G^\mu_\nu + (\bar{R}^\mu_\nu - \bar{K}^\mu_\nu)\,v(\Box)\,\delta R = 0\,, \tag{84}$$

where $m^2 = 1 + f_0(\bar{\mathcal{H}}'\bar{\mathcal{G}} + \bar{\mathcal{G}}'\bar{H})$ and $v(\Box) = -\big((\bar{\mathcal{H}}''\bar{\mathcal{G}} + \bar{G}''\bar{\mathcal{H}})f_0 - 2\bar{\mathcal{H}}\bar{\mathcal{G}}'\mathcal{F}(\Box)\big)$, where $'$ denotes derivative with respect to R. Since the variation of the $\Box$ acting on a scalar function is a pure differential operator and all background curvatures are constants, we have

$$(\delta\Box)f = [-h^{\mu\nu}(\partial_\mu\partial_\nu - \bar{\Gamma}^\rho_{\mu\nu}\partial_\rho) - \bar{g}^{\mu\nu}\gamma^\rho_{\mu\nu}\partial_\rho]f. \tag{85}$$

Here $\Gamma^\rho_{\mu\nu}$ denotes the Christoffel symbol. If we start with (83), then we get

$$\delta g^{\mu\nu} = -h^{\mu\nu},\;\; \delta\Gamma^\rho_{\mu\nu} = \gamma^\rho_{\mu\nu} = \frac{1}{2}(\bar{\nabla}_\mu h^\rho_\nu + \bar{\nabla}_\nu h^\rho_\mu - \bar{g}^{\rho\sigma}\bar{\nabla}_\sigma h_{\mu\nu}). \tag{86}$$

So, for f be a constant we get $(\delta\Box)f = 0$. The same is true for K^μ_ν

$$(\delta K^\mu_\nu)f = [-h^{\mu\sigma}(\partial_\sigma\partial_\nu - \bar{\Gamma}^\rho_{\sigma\nu}\partial_\rho) - \bar{g}^{\mu\sigma}\gamma^\rho_{\sigma\nu}\partial_\rho - \delta^\mu_\nu\delta\Box]f, \tag{87}$$

which is zero in the case that f is a constant. Hence all the corresponding terms vanish.

Taking the trace of (40) one gets

$$[m^2 + (\bar{R} + 3\,\Box)\,v(\Box)]\,\delta R = \mathcal{U}(\Box)\,\delta R = 0. \tag{88}$$

This is a homogeneous equation on δR. The general method of solving it is to use the method of Weierstrass factorization

$$\mathcal{U}(\Box)\delta R = \prod_i(\Box - \omega_i^2)\,e^{\gamma(\Box)}\delta R = 0\,, \tag{89}$$

where ω_i^2 are the roots of the equation $\mathcal{U}(\omega^2) = 0$ and since $\gamma(\Box)$ is taken to be an entire function and $e^{\gamma(\omega^2)}$ has no roots. Let assume that there are no multiple roots. Such roots complicate the story, but can be treated analogously, see [40]. Then we can solve (89) for each ω_i separately

$$(\Box - \omega_i^2)\delta R = 0. \tag{90}$$

The latter differential equation can be written explicitly as

$$\left(\partial_\tau^2 - \frac{2}{\tau}\partial_\tau + k^2 + \frac{\omega_i^2}{H^2\tau^2}\right)\delta R = 0, \tag{91}$$

where we have taken the de Sitter form of the background. The solution yields

$$\delta R_i = (-k\tau)^{3/2}\left(C_{1i}J_{\nu_i}(-k\tau) + C_{2i}Y_{\nu_i}(-k\tau)\right), \tag{92}$$

where J, Y are the Bessel functions of the first and second kinds, respectively, with $\nu_i = \sqrt{\frac{9}{4} - \frac{\omega_i^2}{H^2}}$ and $C_{1i,2i}$ are the integration constants.

For small values of τ which correspond to large cosmic times t the Bessel functions have the following asymptotic behavior

$$\begin{aligned} J_\nu(z) &\sim z^{\mathrm{Re}\,\nu}, \\ Y_\nu(z) &\sim z^{-|\mathrm{Re}\,\nu|} \text{ for } \mathrm{Re}\,\nu \neq 0, \\ Y_\nu(z) &\sim \ln z \text{ for } \mathrm{Re}\,\nu = 0. \end{aligned}$$

From this we conclude that δR_i are bounded,

$$|\mathrm{Re}\,\nu_i| < \frac{3}{2}. \tag{93}$$

The general solution for δR is

$$\delta R = \sum_i \delta R_i. \tag{94}$$

where each δR_i has its arbitrary integration constants, chosen in such manner that δR is a real.

5.3. Scalar perturbation and Bardeen potentials.

Since the behavior of vector and tensor classical perturbations remain the same as in GR, we will focus on scalar classical perturbations.

The metric for the scalar perturbations around a FLRW background is given by

$$ds^2 = a(\tau)^2 \left[-(1+2\phi)d\tau^2 - 2\partial_i\beta d\tau dx^i + ((1-2\psi)\delta_{ij} + 2\partial_i\partial_j\gamma)dx^i dx^j\right]. \tag{95}$$

where ϕ, β, ψ and γ are scalar functions.

From 4 scalar modes only 2 are gauge invariant. The convenient gauge invariant variables, known also as Bardeen potentials, are introduced as

$$\Phi = \phi - \frac{1}{a}(a\vartheta)' = \phi - \dot{\chi}, \qquad \Psi = \psi + \mathcal{H}\vartheta = \psi + H\chi, \tag{96}$$

where $\chi = a\beta + a^2\dot{\gamma}$, $\vartheta = \beta + \gamma'$, $\mathcal{H}(\tau) = a'/a$. The prime denotes the differentiation with respect to the conformal time τ and the dot with the respect to the cosmic time t.

Now, we want to determine the Bardeen potentials introduced by (96). To do this we need two equations: the first one is given by the formulation of δR in terms of Φ and Ψ accounting that the time behavior of δR itself is found,

$$\widetilde{\delta R} = \frac{2}{a^2}\left[k^2(\Phi - 2\Psi) - 3\frac{a'}{a}\Phi' - 6\frac{a''}{a}\Phi - 3\Psi'' - 9\frac{a'}{a}\Psi'\right], \tag{97}$$

where $\widetilde{\delta R} = \delta R - \bar{R}'(\beta + \gamma')$ is a convenient gauge invariant analog of δR. This expression is valid for any a while its de Sitter form is

$$\widetilde{\delta R} = -6H^2\left(4\Phi - \tau(\Phi' + 3\Psi') + \tau^2\Psi''\right) + 2\tau^2 H^2 k^2 (\Phi - 2\Psi). \tag{98}$$

The second equation can be obtained from the system of equations (84). We find the $i \neq j$ component of the system (84) which becomes

$$-m^2(\Phi - \Psi) + v(\Box)\widetilde{\delta R} = 0. \tag{99}$$

Now, we write down the $(0i)$ equation of the system (84), which is

$$2m^2(\Psi' + \mathcal{H}\Phi) + (v(\Box)\widetilde{\delta R})' - \mathcal{H}v(\Box)\widetilde{\delta R} = 0, \tag{100}$$

and finally, we deduce the (00) equation of system (84) which yields

$$-2m^2(k^2\Psi+3\mathcal{H}\Psi'+3\mathcal{H}^2\Phi)-3\mathcal{H}(v(\Box)\widetilde{\delta R})'-\left(k^2-\frac{3}{\tau^2}\right)v(\Box)\widetilde{\delta R}=0, \tag{101}$$

where the last term is proportional to $1/\tau^2$ as a consequence that the background is a de Sitter space. If we multiply (99) by k^2, (100) by $3\,\mathcal{H}$ and summing these results with (101) and taking in account that for the de Sitter space-time $\mathcal{H}^2=1/\tau^2$ we finally get

$$-m^2\,k^2\,(\Phi+\Psi)=0, \tag{102}$$

which is our second equation. Obviously, it simplifies the succeeding computations considerably. Since $\widetilde{\delta R}$ is given by (94), now from expression (99) easily follows

$$2\,\Phi=\frac{1}{m^2}\sum_i v(\omega_i^2)\,\delta R_i. \tag{103}$$

Let us remark, $\widetilde{\delta R}$ coincides with δR if $\bar{R}$ is a constant or on a more general basis in the longitudinal (Newtonian) gauge $\beta=\gamma=0$. If we take into account (93), we see that Bardeen potentials are zero when $|\mathrm{Re}\,\nu_i|<3/2$ for each i. If $\mathrm{Re}\,\nu_i=0$ for some i then the perturbations become frozen. At last, in the case that for at least one i we have $|\mathrm{Re}\,\nu_i|>3/2$ then perturbations grow. This is in perfect agreement with [9]. In that reference a more general class of solutions was studied which asymptote to the de Sitter background at late times while the nonlocality is given by $R\mathcal{F}(\Box)R$. In our case when $\mathcal{H}(R)=\mathcal{G}(R)=R$ we obtained exactly the same result as in Section 4 of [6].

Let us consider special case $m^2=0$, then it is not possible to find out the Bardeen potentials, separately. The compatibility of system (88,99) implies that either $\delta R=0$ or there is a root of $v(\Box)$. All equations coming from (84) since all of them do not carry information about individual Bardeen potentials if $m^2=0$. Physically this means that effectively the Einstein-Hilbert term vanishes and one reduces the number of propagating degrees of freedom.

5.4. Stability for constant curvature background.

The main result of Section 5.2 implies the natural question of stability of the de Sitter vacuum solution of eq. (93). It turns out that ν depends on the structure of the nonlocal operator $\mathcal{U}(\Box)$ such that

$$\nu=\sqrt{\frac{9}{4}-\frac{\omega^2}{H^2}}, \tag{104}$$

and $\mathcal{U}(\omega^2)=0$.

Moreover, the system does not lead to ghosts demands that there is no more than one such a root ω^2 for the operator $\mathcal{U}$, for more details see [27].

Firstly, let us make the analysis of nonlocality given by [21]

$$\mathcal{H}(R)=R^p,\quad \mathcal{G}(R)=R^q, \tag{105}$$

for some nonzero p and q. From the following modification of equation (40) we have,

$$R-2\,\Lambda+f_0\,R^{p+q}\,(2-p-q)=0. \tag{106}$$

This equation can be solved w.r.t R in general for $p,\ q$ integers and $-3\le p+q\le 4$. It is necessary to analyze $\mathcal{U}$ to see whether the stability condition can be reached. Indeed, $\mathcal{U}$ is analytic by construction but a compatibility condition must be fulfilled

$$1+R^{p+q-1}\,(p+q)\,(2-p-q)f_0=-\omega^2 e^{\gamma(0)}. \tag{107}$$

It is obvious that as long as ω^2 is real it should be at least positive in order to satisfy (93). The constrain (107) clearly shows that ω^2 is real and therefore reduces to the following necessary inequality

$$1 + R^{p+q-1}(p+q)(2-p-q)f_0 < 0. \tag{108}$$

Satisfactory solution of previous relation is a necessary stability condition. Obviously we have two special cases, namely $p+q=0$ and $p+q=2$, and in both cases there is no stable solutions.

In a general situation we have to understand equation (106) together with the inequality (108). One can simplify (108) using the background equation (106) to

$$R\,(p+q-1) \;>\; 2\,\Lambda\,(p+q). \tag{109}$$

In the case $p+q=1$ one can have a stable solution for a negative Λ. The latter condition is possible as long as $\lambda f_0 < 0$.

In an attempt to solve the system (106) and (108) one can rewrite it as

$$1 - s + u = 0, \qquad 1 + uz < 0, \tag{110}$$

where $s = \frac{2\Lambda}{R}$, $z = p+q$, $u = f_0 R^{z-1}(2-z)$. This latter system looks simple but unfortunately does not provide new interesting solutions from the physical point of view.

5.5. Perturbation of Minkowski space.

Let us recall that in 2018 the gravitational waves were experimentally discovered following results of GR, see [1]. In this subsection we want to investigate if our nonlocal modified gravity predict gravitational waves and their explicit description.

In GR the equations of gravitational waves were obtained as perturbations of EOM for Minkowski metric, i.e. in the form

$$\Box\,\psi_{\mu\nu} = 0, \qquad \nabla_\mu \psi^{\mu\nu} = 0, \tag{111}$$

where $\psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,h$, $h_{\mu\nu} = \delta g_{\mu\nu}$, $h = g^{\mu\nu}h_{\mu\nu}$ and $|h_{\mu\mu}| \ll 1$. These equations are similar to expansion of electromagnetic waves with Lorentz condition. In this subsection we assume that metric $g_{\mu\nu}$ is Minkowski metric, given by $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}\,(-1,1,1,1)$.

The covariant derivative is equal to the partial derivative and d'Alambert operator is given by

$$\Box = -\partial^2_{tt} + \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz}.$$

The perturbations of EOM (37) up to the linear degree of Minkowski metric are in the following form

$$\begin{aligned} &-\frac{1}{2}\left(g_{\mu\nu}\mathcal{H}(R)\mathcal{F}(\Box)\mathcal{G}(R) - g_{\mu\nu}\mathcal{H}'(R)f_0\mathcal{G}(R)\delta R - h_{\mu\nu}f_0\mathcal{G}(R)\mathcal{H}(R)\right) \\ &+ \delta R_{\mu\nu}W - K_{\mu\nu}\delta W + (\delta R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\delta R + h_{\mu\nu}\Lambda), \end{aligned} \tag{112}$$

where $\delta W = -2\mathcal{G}'(R)\mathcal{H}'(R)\mathcal{F}(\Box)\delta R - f_0(\mathcal{G}''(R)\mathcal{H}(R) + \mathcal{H}''(R)\mathcal{G}(R))\delta R$ and where the variation of curvature tensor is $\delta R = -K_{\mu\nu}h^{\mu\nu}$, $\delta R_{\mu\nu} = \nabla_\lambda \gamma^\lambda_{\mu\nu} - \nabla_\mu \gamma^\lambda_{\lambda\nu}$.

If we use tensor $\psi_{\mu\nu}$ (112) becomes

$$\begin{aligned} &\left(\frac{1}{2}g_{\mu\nu}\mathcal{G}'(0)\mathcal{H}(0) - 2\mathcal{G}'(0)\mathcal{H}'(0)K_{\mu\nu}\right)\mathcal{F}(\Box)\delta R \\ &+ \left(\frac{1}{2}g_{\mu\nu}f_0\mathcal{H}'(0)\mathcal{G}(0) - f_0(\mathcal{G}''(0)\mathcal{H}(0) + \mathcal{H}''(0)\mathcal{G}(0))K_{\mu\nu} + \frac{1}{2}\right)\delta R \\ &- h_{\mu\nu}\left(\Lambda - \frac{1}{2}f_0\mathcal{G}(0)\mathcal{H}(0)\right) + \delta R_{\mu\nu}\left(f_0(\mathcal{G}\mathcal{H})'(0) + 1\right) = 0, \\ &\delta R = \nabla_\mu\nabla_\nu\psi^{\mu\nu} + \frac{1}{2}\Box(g_{\mu\nu}\psi^{\mu\nu}). \end{aligned} \tag{113}$$

Let us remark that if the equations (111) are satisfied then variations of δR and $\delta R_{\mu\nu}$ are vanishing and previous equation is reduced to

$$\Lambda = \frac{1}{2}\, f_0 \mathcal{G}(0)\, \mathcal{H}(0), \tag{114}$$

and gravitational waves in nonlocal modified gravity with nonlocality given by $\mathcal{H}\,\mathcal{F}(\Box)\,\mathcal{G}$ have the same behavior as in the GR. So, we prove the following theorem.

Theorem 9. *Let M be a Minkowski manifold with nonlocality given by $\mathcal{H}(R)\,\mathcal{F}(\Box)\,\mathcal{G}(R)$, then for $\Lambda = \frac{1}{2}\, f_0\mathcal{G}(0)\,\mathcal{H}(0)$, the equations of gravitational waves are:*

$$\Box\, \psi_{\mu\nu} = 0, \qquad \nabla_\mu \psi^{\mu\nu} = 0. \tag{115}$$

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