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Степени отображений между гомотопическими пространственными формами[1]

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Аннотация

Пусть $\mathcal{G}$ — семейство периодических групп периода 2 или 4, а $\bar{\Sigma}^m$ — гомотопическая m-пространственная форма где $\pi_1(\bar{\Sigma}^m) \in \mathcal{G}$. Для $m = 3$ мы изучаем множество степеней отображения $D(\bar{\Sigma}_1^m, \bar{\Sigma}_2^m)$ из $\bar{\Sigma}_1^m$ в $\bar{\Sigma}_2^m$.

Ключевые слова: Гомотопические сферические пространственные формы, степень отображения

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Mapping degrees between homotopy space forms[2]

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Abstract

Let $\mathcal{G}$ be the family of periodic groups of period either 2 or 4, and $\bar{\Sigma}^m$ be a homotopy m-space form where $\pi_1(\bar{\Sigma}^m) \in \mathcal{G}$. For $m = 3$, we study the set $D(\bar{\Sigma}_1^m, \bar{\Sigma}_2^m)$ of degrees of the maps from $\bar{\Sigma}_1^m$ to $\bar{\Sigma}_2^m$.

Keywords: Homotopy spherical space forms, mapping degrees

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1. Introduction

Let M and N be two closed connected n-manifolds. The study of mapping degrees of maps from M to N is a classical problem in the classification of manifolds. When M and N are 3-manifolds with spherical geometry, the set $D(M, N)$ of mapping degrees from M to N has been determined in [20]. It is natural to study the same problem when the spaces involved are orbit spaces of homotopy spheres, i.e., homotopy spherical space forms.

A space Σ^m is called a *homotopy m-sphere*, if Σ^m is a CW-complex which has the same homotopy type of the sphere S^m. If G is a finite group which acts freely on Σ^m then the quotient Σ^m/G of Σ^m by the free action of the finite group G is called a *homotopy m-spherical space form*, or alternatively *a homotopy m-space form*, and it will be denoted by $\bar{\Sigma}^m$.

We recall the classification of the homotopy types of the homotopy m-space forms as well as the set of self-homotopy equivalences of each homotopy space form. In what follows, we will not distinguish the homotopy type of the space form $\bar{\Sigma}^m = \Sigma^m/G$ whether the homotopy sphere Σ^m is a finite CW-complex, an infinite CW-complex, or the sphere itself.

The following is well known:

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Proposition 1. *Let G be a finite group acting freely on a homotopy m-sphere Σ^m.*

1. *If m is even and G is non-trivial then G is isomorphic to $\mathbb{Z}_2$. Furthermore $H^m(\Sigma^m, \mathbb{Z}) = \mathbb{Z}$ and $H^m(\Sigma^m/G, \mathbb{Z}) = \mathbb{Z}_2$.*

2. *If m is odd then G is a finite periodic group. Furthermore $H^m(\Sigma^m/G, \mathbb{Z}) = \mathbb{Z}$.*

Given two homotopy m-space forms $\bar{\Sigma}_1^m$, $\bar{\Sigma}_2^m$, which are the orbit spaces of Σ_1^m, Σ_2^m, respectively, we have the notion of degree of a map, as a result of the Proposition 1. More precisely, if the cohomology of the domain is $\mathbb{Z}$ then the degree is an integer otherwise it is an element of $\mathbb{Z}_2$. Therefore for m odd it is always an element of $\mathbb{Z}$ and for m even it is an element of $\mathbb{Z}_2$, except in the case where the domain is a homotopy sphere, in which case it is again an element of $\mathbb{Z}$.

For a given finite periodic group G of period an even positive integer n and an odd positive integer m such that n divides $m+1$, the classification of the homotopy types of the homotopy m-space forms is described in [27, Theorem 1.8], which in turn refers to the earlier references [5] and [23]. More precisely, the set of homotopy classes of homotopy m-space forms is in one-to-one correspondence with the equivalence classes of the invertible elements of the cyclic group $\mathbb{Z}_{|G|}$, where two invertible elements k_1, k_2 are related if and only if there is an automorphism $\phi : G \to G$ such that the induced automorphism $\phi^{\#m} : H^m(G, \mathbb{Z}) \to H^m(G, \mathbb{Z})$ satisfies either $\phi^{\#m}(k_1) = k_2$ or $\phi^{\#m}(k_1) = -k_2$. This correspondence was later established using two-stage Postnikov tower, see [6], which is the approach we use in this work.

The classification of finite groups which act freely in a homotopy sphere was obtained by Suzuki-Zassenhaus. (Table I in Section 3)

1.1. Even dimensional homotopy space forms

In the case of an even dimensional space form $\bar{\Sigma}^{2m}$ we have only two possibilities, namely, either $\bar{\Sigma}^{2m}$ has the homotopy type of the $2m$-sphere or it is the quotient of an even dimensional homotopy sphere by a free action of the group $\mathbb{Z}_2$. In the former case we have $H^{2m}(\bar{\Sigma}^{2m}, \mathbb{Z}) \cong \mathbb{Z}$ and in the latter case $H^{2m}(\bar{\Sigma}^{2m}, \mathbb{Z}) \cong \mathbb{Z}_2$. Now we can easily describe the possible degrees for even dimensional homotopy space forms.

(I) If $\bar{\Sigma}_1^{2m}$, $\bar{\Sigma}_2^{2m}$ are homotopy spheres then this reduces to the classical case.

(II) If $\bar{\Sigma}_1^{2m}$ is a homotopy space form for $G = \mathbb{Z}_2$ and $\bar{\Sigma}_2^{2m}$ is a homotopy sphere then $Hom(\mathbb{Z}_2, \{1\})$ contains only the trivial homomorphism. So in this case the degree will lie in $H^{2m}(\bar{\Sigma}_1^{2m}, \mathbb{Z}) = \mathbb{Z}_2$ and using Hopf's theorem about the correspondence between this latter group and $[\bar{\Sigma}_1^{2m}, S^{2m}]$, both elements of $\mathbb{Z}_2$ can be realized as the degree of maps.

(III) If $\bar{\Sigma}_1^{2m}$ is a homotopy sphere and $\bar{\Sigma}_2^{2m}$ is a homotopy space form for $G = \mathbb{Z}_2$, then $Hom(\{1\}, \mathbb{Z}_2)$ contains only the trivial homomorphism. So in this case the degree will lie in $H^{2m}(\bar{\Sigma}_1^{2m}, \mathbb{Z}) = \mathbb{Z}$ and the ones which are realizable are all the even integers. It is because $H^{2m}(\bar{\Sigma}_2^{2m}, \tilde{\mathbb{Z}}) = \mathbb{Z}$, where $\tilde{\mathbb{Z}}$ is the orientation local coefficient system, $H^{2m}(\bar{\Sigma}_1^{2m}, \mathbb{Z}) = \mathbb{Z}$ and any map $f : \bar{\Sigma}_1^{2m} \to \bar{\Sigma}_2^{2m}$ factors through Σ_2, the universal covering of $\bar{\Sigma}_2^{2m}$. Then we use the formula (1) given in the next section.

(IV) If $\bar{\Sigma}_1^{2m}$ and $\bar{\Sigma}_2^{2m}$ are homotopy space forms for $G = \mathbb{Z}_2$ then $Hom(\mathbb{Z}_2, \mathbb{Z}_2)$ contains two homomorphisms, the identity and the trivial homomorphism. In case of the identity, the degree is an integer and the one's which are realizable are congruent to 1 (mod 2) since a homotopy equivalence realizes this degree 1. In case the homomorphism is the trivial homomorphism, the degree lies in $\mathbb{Z}_2$ and it is $\bar{0}$ which can be realized by the constant map.

2. The sets $D(\bar\Sigma_1^m, \bar\Sigma_2^m)$ and $[\bar\Sigma_1^m, \bar\Sigma_2^m]$

Let $\bar\Sigma_1^m$, $\bar\Sigma_2^m$ be two homotopy m-spherical space forms. From now on we will consider only those values of m that are odd. The works of P. Olum [22], [23] can be used to describe the possible degrees that can be realized by maps and to describe the set of homotopy classes of maps between two such space forms. We will first discuss the degree.

It follows from [22] and [23] that we have the following:

Proposition 2. *Let $\bar\Sigma_1^m$ and $\bar\Sigma_2^m$ be two homotopy m-space forms with $G_1 = \pi_1(\bar\Sigma_1^m)$ and $G_2 = \pi_1(\bar\Sigma_2^m)$. Then*

1. *Given any homomorphism $\varphi : G_1 \to G_2$, there is a map $f : \bar\Sigma_1^m \to \bar\Sigma_2^m$ such that $f_\# = \varphi$.*
2. *If two maps $f, g : \bar\Sigma_1^m \to \bar\Sigma_2^m$ induce the same homomorphism $\varphi : G_1 \to G_2$, then their degree are congruent module $|G_2|$.*
3. *If $f : \bar\Sigma_1^m \to \bar\Sigma_2^m$ is a map, then there is a map of degree d for any integer d with $d \equiv \deg(f) \mod |G_2|$.*
4. *Two maps from $\bar\Sigma_1^m$ to $\bar\Sigma_2^m$ are homotopic if and only if $f_{1\#} = f_{2\#}$ and they have the same degree.*

The above results show that in order to find all possible degrees between the two space forms it suffices to find for a given homomorphism $\varphi : G_1 \to G_2$, the degree of one map which induces φ on the fundamental group. We write $\overline{\deg}(\varphi) \in \mathbb{Z}_{|G_2|}$. The goal of this section is to provide a method to compute such degree for a given homomorphism φ.

Let G be a finite group which acts freely on a homotopy m-sphere Σ^m. Then the orbit space $\bar\Sigma^m = \Sigma^m/G$ can be assigned an invertible element $k \in \mathbb{Z}_{|G|} = H^{m+1}(G, \mathbb{Z})$, which is the Postnikov invariant determining the fibration

$$K(\mathbb{Z}, m) \to E \to K(G, 1) \xrightarrow{k} K(\mathbb{Z}, m+1).$$

Lemma 1. (**Fundamental Lemma**) *Let $\bar\Sigma_1^m$ and $\bar\Sigma_2^m$ be two homotopy m-space forms with $G_1 = \pi_1(\bar\Sigma_1^m)$ and $G_2 = \pi_1(\bar\Sigma_2^m)$. If $\varphi : G_1 \to G_2$ is a homomorphism, then the degree $\overline{\deg}(\varphi)$ is determined by*

$$d|G_2| = |G_1|\overline{\deg}(\varphi) \in \mathbb{Z}, \quad dk_1 = \varphi^*(k_2) \in \mathbb{Z}_{|G_1|} = H^{m+1}(G_1, \mathbb{Z}), \tag{1}$$

where k_1, k_2 are respectively the Postnikov invariants determining $\bar\Sigma_1^m$ and $\bar\Sigma_2^m$.

Proof. For $i = 1, 2$, from the theory of Postnikov tower, the space E_i is a total space of a fibration $K(\mathbb{Z}, m) \hookrightarrow E_i \to K(G_i, 1)$ having as fibre $K(\mathbb{Z}, m)$ and the base $K(G_i, 1)$. Such a fibration is classified by $k_i \in H^{m+1}(G_i, \mathbb{Z}) = \mathbb{Z}_{|G_i|} = [K(G_i, 1), K(\mathbb{Z}, m+1)]$. Let $f : \bar\Sigma_1^m \to \bar\Sigma_2^m$ be a map such that the induced homomorphism by f on the fundamental group of the spaces is φ. We have following commutative diagram:

$$\begin{array}{ccc} K(\mathbb{Z}, m) & \xrightarrow{\Omega(f')} & K(\mathbb{Z}, m) \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} \\ E_1 & \xrightarrow{\hat f} & E_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ K(G_1, 1) & \xrightarrow{\varphi} & K(G_2, 1) \\ \downarrow{\scriptstyle k_1} & & \downarrow{\scriptstyle k_2} \\ K(\mathbb{Z}, m+1) & \xrightarrow{f'} & K(\mathbb{Z}, m+1). \end{array} \tag{2}$$

Consider the induced commutative diagram in H^{m+1}, we have

$$\begin{array}{ccc} H^{m+1}(K(G_1,1)) & \xleftarrow{\varphi} & H^{m+1}(K(G_2,1)) \\ \uparrow k_1^* & & \uparrow k_2^* \\ H^{m+1}(K(\mathbb{Z},m+1)) & \xleftarrow{f'^*} & H^{m+1}(K(\mathbb{Z},m+1)). \end{array}$$

Since $H^{m+1}(K(\mathbb{Z},m+1)) = \mathbb{Z}$, we may write f'^* as multiplication by d. It follows that $\Omega(f')^* : H^m(K(\mathbb{Z},m)) \to H^m(K(\mathbb{Z},m))$ is also multiplication by d. Note that Σ_i^m has the homotopy type of S^m. By the construction of Postnikov tower, the space E_i can be obtained from $\bar{\Sigma}_i^m$ by attaching some cells with dimension at least $m+2$. Thus, $H^m(E_i) = H^m(\bar{\Sigma}_i^m)$. Consider the cohomology spectral sequence of the fibration $K(\mathbb{Z},m) \hookrightarrow E_i \to K(G_i,1)$. Then

$$H^m(E_i) = \oplus_{p+q=m} E_\infty^{p,q} = \oplus_{p+q=m} E_{m+2}^{p,q} = \ker d_{m+1} \oplus \operatorname{coker} d_{m+1}.$$

Observe that $d_{m+1} : H^m(K(\mathbb{Z},m)) \to H^{m+1}(K(G_i,1)) = \mathbb{Z}_{|G_i|}$. Since $H^m(E_i) = \mathbb{Z}$, the differential d_{m+1} must be surjective and therefore its kernel is $\mathbb{Z}$. It follows that the inclusion $\iota_i : K(\mathbb{Z},m) \to E_i$ induces a homomorphism ι_i^*, which is actually multiplication by $|G_i|$. Using the induced homomorphism on H^m of (2), we obtain that $d|G_2| = |G_1|\deg(f) \in \mathbb{Z}$. (Note all cohomologies involved here are integral.) □

Corollary 1. *Let G be a finite group which acts freely on a homotopy m-sphere Σ^m. If $\varphi : G \to G$ is an endomorphism, then the degree $\overline{\deg}(\varphi) = \deg(\varphi^* : H^{m+1}(G,\mathbb{Z}) \to H^{m+1}(G,\mathbb{Z}))$.*

Corollary 2. *Let G be a finite group which acts freely on a homotopy m-sphere Σ_1^m, Σ_2^m. Then $\overline{\deg}(id) = k_1^{-1}k_2$, where k_1, k_2 are respectively the Postnikov invariants of Σ_1^m/G and Σ_2^m/G.*

These corollaries tell us that the degree $\overline{\deg}(\varphi)$ coming from self-map is independent of the choice of space form Σ^m. The degrees of maps from Σ_1^m/G to Σ_2^m/G coincide with the degrees for self-maps by multiplying an invertible element $k_1^{-1}k_2$. Thus, we have obtained all degrees for $\varphi : G_1 \to G_2$ as long as both G_1 and G_2 are the fundamental groups of 3-dimensional spherical manifolds, by using [20].

3. Groups of period either 2 or 4

Let $\mathcal{G}$ be the family of all finite periodic groups of period 2 or 4. In this section we summarize the description of the groups $G \in \mathcal{G}$.

Recall from [1, p. 154] we have the table below which provides the Suzuki-Zassenhaus classification of all finite periodic groups.

Family	Definition	Conditions
(I)	$\mathbb{Z}_a \rtimes_\alpha \mathbb{Z}_b$	$(a,b)=1$
(II)	$\mathbb{Z}_a \rtimes_\beta (\mathbb{Z}_b \times \mathcal{Q}_{2^i})$	$(a,b)=(ab,2)=1$
(III)	$\mathbb{Z}_a \rtimes_\gamma (\mathbb{Z}_b \times T_i)$	$(a,b)=(ab,6)=1$
(IV)	$\mathbb{Z}_a \rtimes_\tau (\mathbb{Z}_b \times O_i^*)$	$(a,b)=(ab,6)=1$
(V)	$(\mathbb{Z}_a \rtimes_\alpha \mathbb{Z}_b) \times SL_2(\mathbb{F}_p)$	$(a,b)=(a,p(p^2-1))=1$
(VI)	$\mathbb{Z}_a \rtimes_\mu (\mathbb{Z}_b \times TL_2(\mathbb{F}_p))$	$(a,b)=(ab,p(p^2-1))=1 \quad p \neq 2$

Table I

For the definition of a periodic group, a period and the period of G, see [1, Chapter IV, section 6 Definition 6.1]. The only finite group which has least period 1 is the trivial group. We will consider only non-trivial groups. The following result is well known.

Lemma 2. *The only finite groups which have least period 2 are the non-trivial finite cyclic groups. These groups appear in the family* (I) *of Table I.*

A proof of Lemma 2 can be obtained by describing the least period of the elements of the table above.

Let us recall the following result about period of a semi-direct product of two finite periodic groups.

Proposition 3. *([8, Proposition 2.1]) Let $\mathbb{Z}_a$ be a cyclic group, G a finite group, $\alpha : G \to (\mathbb{Z}_a)^*$ an action and $(|G|, a) = 1$, where $|G|$ denotes the order of G. If G is periodic with period (the least period) $2d$ then the semi-direct product $\mathbb{Z}_a \rtimes_\alpha G$ is also a periodic finite group with period (the least period) $2[\ell(\alpha), d] = 2 \cdot \mathrm{l.c.m.}(\ell(\alpha), d)$. Here, $\ell(\alpha) = \mathrm{l.c.m.}\{|\alpha(g)| \mid g \in G\}$.*

Now we consider the groups which have period 4. First let us consider the groups G in Table II below:

group	condition	presentation	normal form	
$\mathbb{Z}_n$		$\{c \mid c^n = 1\}$	c^s,	$0 \leqslant s < n$
D^*_{4n}	$2 \vert n$	$\{b, a \mid a^2 = b^n = (ab)^2, a^4 = 1\}$	b^s, $b^s a$,	$0 \leqslant s < 2n$
O^*_{48}		$\{b, a \mid a^2 = b^3 = (ab)^4, a^4 = 1\}$		
I^*_{120}		$\{b, a \mid a^2 = b^3 = (ab)^5, a^4 = 1\}$		
$T'_{8\cdot 3^q}$		$\{b, a, w \mid a^2 = b^2 = (ab)^2, a^4 = w^{3^q} = 1,$ $wa = bw, wb = abw\}$	$b^s w^t$, $b^s a w^t$,	$0 \leqslant s < 4,\ 0 \leqslant t < 3^q$
$D'_{n\cdot 2^q}$	$2 \not\vert n, q \geq 1$	$\{u, w \mid u^n = w^{2^q} = 1, uwu = w\}$	$u^s w^t$	$0 \leqslant s < n,\ 0 \leqslant t < 2^q$

Table II

Remark 1. *The only groups in Table II which are not the fundamental groups of any closed* 3*-manifold are the Dihedral groups $D'_{2(2n+1)}$.*

Consider all groups of the form $(\mathbb{Z}/m \rtimes_\alpha \mathbb{Z}/n) \times G$ and $\mathbb{Z}_m \rtimes_\alpha G$ where G belongs to Table II. Each pair (m, n), $(m, |G|)$ and $(n, |G|)$ consists of relatively prime integers, and the image of $\alpha : G \to Aut(\mathbb{Z}_m)$ is either the trivial subgroup or the subgroup $\{\pm Id\}$. This follows by analyzing the Suzuki-Zassenhaus classification, Table II, and Proposition 3. Observe that the families (I) - (VI) given by Table I are not mutually disjoint. But whenever two such subfamilies have non-empty intersection, we have enough information about the intersection which helps to the study of our problem.

Lemma 3. *Let G belong to the family* (I)*. Then it has period* 4 *if and only if the order of α is two.*

Proof. This follows immediately from Proposition 3. □

Observe that the direct product of two finite cyclic groups of relatively prime orders is again a cyclic group. Now we consider the family (V). Recall from [1, Chapter IV, page 149] that $SL_2(F_p)$ is isomorphic to

1. the symmetric group on 3 letters for $p = 2$;
2. the semi-direct product of Q_8 by $\mathbb{Z}_3$ with the action $i \to j, j \to k, k \to i$ (which is the binary tetrahedral group) for $p = 3$;
3. the binary icosahedral group I^* of order 120 for $p = 5$. It has a presentation given by $\langle r, s | r^2 = s^3 = (rs)^5 \rangle$, (see [1, page 151]).

Lemma 4. *Let G belong to the family* (V)*. For $G = SL_2(F_p)$, where p is a prime, the period of G is* 4 *if and only if p is either* 2, 3 *or* 5*. Furthermore, the group $(\mathbb{Z}_a \rtimes_\alpha \mathbb{Z}_b) \times SL_2(\mathbb{F}_p)$ where $(a, b) = (a, p(p^2 - 1)) = 1$ has period* 4 *if and only if p is either* 2, 3 *or* 5 *and $\ell(\alpha) = 2$ if $a \neq 1$.*

Proof. The first part follows from the comment after [10, Theorem 1.2] which says that the period of these groups are the l.c.m. of 4 and $p-1$. Therefore we have only three values of p such that l.c.m. of 4 and $p-1$ is 4, which are $p = 2, 3, 5$. For the second part, since the period of $SL_2(F_p)$ cannot be 2, it is necessary to have period 4. The rest of the proof follows from Lemma 3 and Proposition 3. □

Remark 2. *As we can see the only new groups in this family* (V) *of period* 4 *with respect to the previous families, are I^* and the direct product of I^* with a group of period either* 2 *or* 4 *of the family* (I).

Now we consider the family (VI). In the Table I, the case $p = 2$ for this family is not considered. The reason is that if we perform the construction of $TSL_2(F_p)$ given in [1] we have an extension

$$1 \to SL_2(F_2) \to TSL_2(F_2) \to \mathbb{Z}_2 \to 1,$$

where $SL_2(F_2) \cong S_3$ and consequently $TSL_2(F_p)$ has order 12. There are 5 groups of order 12 where 3 of them are not abelian. They are: the dyclic D^*_{12} (which is already in the family (II)); the alternating group A_4; the dihedral group $D'_{12} \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ (which is already in the family (I)). But A_4 does not contain S_3 as a normal subgroup.

Lemma 5. *Let G belong to the family* (VI). *The group $TSL_2(F_p)$ has period* 4 *if and only if p equals* 3. *Furthermore, the group $\mathbb{Z}_a \rtimes_\mu (\mathbb{Z}_b \times TL_2(\mathbb{F}_p))$ where $(a, b) = (ab, p(p^2-1)) = 1$, $p \neq 2$ has period* 4 *if and only if p is* 3 *and $\mu(\alpha) = 2$ if $a \neq 1$.*

Proof. The first part follows from [11, Corollary 2.3], since $p-1$ cannot be greater than 3. For the second part, we use Proposition 3. □

Remark 3. *By [1, Chapter IV] the group $TSL_2(F_3)$ is isomorphic to O^*. Therefore the groups of period* 4 *which belong to the family* (VI) *already appear in a previous family.*

For the remaining 3 families, we have the following result.

Lemma 6. *If G belongs to one of the families family* (II), (III) *or* (IV), *then the group $\mathbb{Z}_a \rtimes_\theta (\mathbb{Z}_b \times G)$ has period* 4 *if and only if $\ell(\theta) = 2$ if $a \neq 1$.*

Proof. Again we use Proposition 3 and the fact that the groups $\mathcal{Q}_{2^i}$, T_i and O^*_i already have period 4 □

Remark 4. *We should point out that the groups of period* 4 *that appear in Lemma 6 cannot be the fundamental group of any spherical* 3*-manifold.*

4. Integral cohomology ring of the periodic groups of period 4

For $G = \mathbb{Z}_n$, the cyclic group of order n, we have that G has period 2 and its integral cohomology ring $H^*(G, \mathbb{Z}) = \mathbb{Z}[x_2]/\langle nx_2 \rangle$ is the quotient of the polynomial ring over $\mathbb{Z}$ in one generator x_2 of dimension 2 module the ideal generated by nx_2 (see e.g. [2, p.114]). Note that H^0 is $\mathbb{Z}$ and not $\mathbb{Z}_n$.

Next, we describe the integral cohomology ring of periodic groups with least period 4. The additive group structure of $H^*(G, \mathbb{Z})$ is quite straightforward. Since G is a group of period 4 we have $H^0(G, \mathbb{Z}) = \mathbb{Z}$, $H^1(G, \mathbb{Z}) = H^3(G, \mathbb{Z}) = 0$, $H^2(G, \mathbb{Z}) = H_1(G, \mathbb{Z}) = G_{ab}$, and $H^4(G, \mathbb{Z}) = \mathbb{Z}_{|G|}$. Because of the periodicity, it follows that $H^0(G, \mathbb{Z}) = \mathbb{Z}$, $H^{1+4k}(G, \mathbb{Z}) = H^{3+4k}(G, \mathbb{Z}) = 0$, $H^{2+4k}(G, \mathbb{Z}) = G_{ab}$, and $H^{4k}(G, \mathbb{Z}) = \mathbb{Z}_{|G|}$. Furthermore, the cup product with a generator of $\mathbb{Z}_{|G|} \cong H^4(G, \mathbb{Z})$ defines an isomorphism $H^m(G, \mathbb{Z}) \to H^{m+4}(G, \mathbb{Z})$ if $m \neq 0$. Therefore to compute the cup product of any two elements, it suffices to compute the cup product of any two elements of H^2.

Remark 5. *Due to periodicity, the task of finding the mapping degree for spherical space forms of dimension $4n+3$, for groups of period 4, should follow easily from the case of space forms of dimension 3.*

(I) Let G be in the family (I) of the form $\mathbb{Z}_a \rtimes_\theta \mathbb{Z}_b$ where $\theta(\iota_b)$ has order 2 if $a \neq 1$. We will use [13, Proposition 3.1] and [8]. So we obtain

$H^0(G,\mathbb{Z}) = \mathbb{Z}$, $H^n(G,\mathbb{Z}) = 0$ for n odd, $H^{2+4k}(G,\mathbb{Z}) = \mathbb{Z}_b$ $k \geq 0$, $H^{4k}(G,\mathbb{Z}) = \mathbb{Z}_{ab}$ $k > 0$. If ι_b, ι_{ab} are generators of $\mathbb{Z}_b$, $\mathbb{Z}_{ab}$, respectively, then we have: $b\iota_b = 0$ and $\iota_b^2 = a\iota_{ab}$. In this family we have included the Dihedral groups D'_{n2^q}, where n is odd.

(II) For the family (II), observe that the group is of the form $\mathcal{Q}_{2^i}$ or $\mathbb{Z}_a \rtimes_\beta \mathcal{Q}_{2^i}$. The integral cohomology of $\mathcal{Q}_{2^i}$ is given by $H^2(\mathcal{Q}_{2^i},\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $H^4(\mathcal{Q}_{2^i},\mathbb{Z}) \cong \mathbb{Z}_{2^i}$. One should get the multiplicative structure from [1] or from [8], or from [29]. From [29, Theorem 3.7] we have:

Proposition 4. *The cohomology ring $H^*(Q_{4n};\mathbb{Z})$ has the following presentation:*

$$\begin{array}{ll} \mathbb{Z}[\gamma_2,\gamma_2',\alpha_4]/(2\gamma_2 = 2\gamma_2' = 4n\alpha_4, \gamma_2^2 = 0, \gamma_2\gamma_2' = \gamma_2'^2 = 2n\alpha_4), & \text{if } n = 4m \\ \mathbb{Z}[\gamma_2',\alpha_4]/(4\gamma_2' = 0 = 4n\alpha_4, \gamma_2'^2 = n\alpha_4) & \text{if } n = 4m+1 \\ \mathbb{Z}[\gamma_2,\gamma_2',\alpha_4]/(2\gamma_2 = 2\gamma_2' = 4n\alpha_4, \gamma_2^2 = 0 = \gamma_2'^2, \gamma_2\gamma_2' = 2n\alpha_4) & \text{if } n = 4m+2 \\ \mathbb{Z}[\gamma_2',\alpha_4]/(4\gamma_2' = 0 = 4n\alpha_4, \gamma_2'^2 = 3n\alpha_4), & \text{if } n = 4m+3 \end{array}$$

(III) The tetrahedral groups ([29, Theorem 4.4]). The ring structure of the group cohomology is given by

$$H^*(T^*_{24},\mathbb{Z}) \approx \mathbb{Z}[\gamma_2,\alpha_4]/(\gamma_2^2 = 8\alpha_4, 3\gamma_2 = 0 = 24\alpha_4).$$

The general case $T'_{8.3^k}$. The ring structure is given by (but [29] only gives the additive structure):

$$H^*(T^*_{8.3^k},\mathbb{Z}) \approx \mathbb{Z}[\gamma_2,\alpha_4]/(\gamma_2^2 = 8\alpha_4, 3^k\gamma_2 = 0 = 8\cdot 3^k\alpha_4).$$

(IV) The octahedral group [29, Thorem 4.10]. The ring structure of the group cohomology is given by

$$H^*(O^*_{48},\mathbb{Z}) \approx \mathbb{Z}[\gamma_2,\alpha_4]/(\gamma_2^2 = 24\alpha_4, 2\gamma_2 = 0 = 48\alpha_4).$$

(V) The only case to be considered is I^*, see [29, Theorem 4.17]. In this case $H^m(I^*,\mathbb{Z})$ is $\mathbb{Z}$ for $m=0$, $\mathbb{Z}_{120}$ for $m = 4k$ with $k \in N > 0$, and zero otherwise. So the cohomology ring is a polynomial algebra on a generator of dimension 4. On the other hand there is no group of the form $G \rtimes_\alpha I^*$ with α of order 2. This follows because there is no epimorphism $I^* \to \mathbb{Z}_2$ since besides the trivial group and the entire group the only normal subgroup of I^* is $\mathbb{Z}_2$, which is the center.

There is no need for the case **(VI)** since the groups already belong to the previous families.

4.1. The cohomology ring of G, the space forms of dimensions $2k+1$ and $4k+3$, and degree

Let $G \in \mathcal{G}$. If G has period 2 then it is cyclic and we have homotopy $2k+1$-space forms for all positive k. For the remaining groups in $\mathcal{G}$, which are the ones with period 4, we have homotopy $4n+3$-space forms. The description of the cohomology ring of the space forms $\bar\Sigma^{2n+1} = \Sigma^{2n+1}/G$, which for n even include the cases of period 4, are quite simple. The additive group structure is $H^i(\bar\Sigma^{2n+1},\mathbb{Z}) = H^i(G,\mathbb{Z})$ for $0 \leq i \leq 2n$, $H^{2n+1}(\bar\Sigma^{2n+1},\mathbb{Z}) = \mathbb{Z}$, and $H^i(\bar\Sigma^{2n+1},\mathbb{Z}) = 0$ for $i > 2n+1$. The ring structure follows promptly from the ring structure of the cohomology ring $H^*(G,\mathbb{Z})$ and dimensional reason. The ring structure of the cohomology ring $H^*(G,\mathbb{Z})$ was given in the previous section. Now comes the main useful result which relates the degree with the homomorphism induced in cohomology.

For $G \in \mathcal{G}$ we have:

Proposition 5. *Let q be the period of G. If the induced homomorphism on $H^q(G,\mathbb{Z})$ is multiplication by d, then the degree obtained among all maps from $\bar{\Sigma}_1(kd+d-1)$ to $\bar{\Sigma}_2(kd+d-1)$ are all the integers congruente to d^{k+1} mod the cardinality of G_2, as in Olum.*

Proof. This follows from the comparison of the cohomology of the group and that of the homotopy space form. □

5. Classification of the maps between homotopy space forms Σ_1^3, Σ_2^3 for groups in $\mathcal{G}$

We first consider homotopy space forms where the associated fundamental groups belong to the Table II. Following [20], we will focus on surjective homomorphisms between two groups and at least one of the groups is of the form D'_{2n} for n odd, i.e. we consider surjective homomorphisms $\phi : G \to D'_{2m}$(Dihedral group) and $\phi : D'_{2m} \to G$. Such a homomorphism induces a homomorphism in cohomology at dimension 4, i.e. a homomorphism $\phi^* : \mathbb{Z}_{2m} \to \mathbb{Z}_{|G|}$ and a homomorphism $\phi^* : \mathbb{Z}_{|G|} \to \mathbb{Z}_{2m}$, respectively, which we must compute in order to determine the degrees. The case where ϕ is an isomorphism has been computed, see [8].

Now we will describe all surjective homomorphisms which are either in $Hom(G, D'_{2m})$ or in $Hom(D'_{2m}, G)$, and m is odd. If $G = D'_{2m}$ then a surjective homomorphisms is an isomorphism and this case is known (see [8]). Thus, we divide into three cases: a) $Hom(D_{2^q m_1}, D_{2m_2})$, for m_1, m_2 odd, $m_1 > m_2$ and $q \geq 1$; b) $Hom(D_{2m}, G)$ for G not dihedral; c) $Hom(G, D_{2m})$ for G not dihedral.

Lemma 7. *Let φ be a surjective homomorphism from G to D'_{2m} with G in Table* II *and m odd. Then $G = D^*_{4n}$ or $D'_{m'\cdot 2^q}$, and $\varphi = \eta'\psi\eta''$, where $\eta'' \in Aut(G)$ and $\eta' \in Aut(D'_{2m})$. Moreover,*

1. *if $G = D^*_{4n}$, then $m|n$ and ψ is given by $b \mapsto u$, $a \mapsto w$;*
2. *if $G = D'_{m'\cdot 2^q}$, then $m|m'$ and ψ is given by $u' \mapsto u$, $w' \mapsto w$.*

Proof. The results in [20, Sec. 4] about quotient groups in Table II tell us that there are two possibilities: either $G = D^*_{4n}$ with $m|n$ or $G = D'_{m'\cdot 2^q}$ with $m|m'$. In the first case, the corresponding normal subgroup of D^*_{4n} is $\langle b^m \rangle$. We have our ψ. The second case follows from [20, Theorem 4.12], where the argument still works in the situation $q = 1$. □

Lemma 8. *Let φ be a surjective homomorphism from D'_{2m} to G with G in Table* II *and m odd. Then $G = \mathbb{Z}_2$ or $D'_{2m'}$, and $\varphi = \eta'\psi\eta''$, where $\eta'' \in Aut(D'_{2m})$ and $\eta' \in Aut(G)$. Moreover,*

1. *if $G = \mathbb{Z}_2 = \langle c \mid c^2 = 1\rangle$, then ψ is given by $u \mapsto 1$, $w \mapsto c$;*
2. *if $G = D'_{2m'}$, then $m'|m$ and ψ is given by $u \mapsto u'$, $w \mapsto w'$.*

Proof. By [20, Lemma 2.11], the quotient group of D'_{2m} is either $\mathbb{Z}_2$ or $D'_{2m'}$ with $m'|m$. If $G = \mathbb{Z}_2$, then the commutator $\langle u \rangle$ must be sent to 1. Hence, we obtain the first case. The second case is obtained by [20, Theorem 4.12]. The Theorems in [20, Sec. 4] show that there is no more. □

Now we proceed to compute the induced homomorphisms on H^4 for all the homomorphisms described above. Using the functor EXT or the abelianization, one may determine the induced homomorphism on H^2 for a homomorphism between two groups. Using certain short exact sequences associated to the groups G_1, G_2, for a given homomorphism $\varphi : G_1 \to G_2$ such that the homomorphism induces a homomorphism of short exact sequence, we will consider the map induced between the spectral sequences associated to the correspondents short exact sequence. This approach will be used to determine the induced homomorphism on H^4.

First consider the following well known example. Let $\phi : \mathbb{Z}_m \to \mathbb{Z}_n$ be a surjective homomorphism which send the generator ι_m to $k\iota_n$ where k is relatively prime with n (for example

$k=1$) and n divides m. Then the induced homomorphism $\phi^*: H^2(\mathbb{Z}_n,\mathbb{Z})=\mathbb{Z}_n \to H^2(\mathbb{Z}_m,\mathbb{Z})=\mathbb{Z}_m$ send the generator ι_n to $(km/n)\iota_m$ (in particular if $k=1$ then ι_n is mapped to $(m/n)\iota_m$. Now the induced $\phi^*: H^4(\mathbb{Z}_n,\mathbb{Z})=\mathbb{Z}_n \to H^4(\mathbb{Z}_m,\mathbb{Z})=\mathbb{Z}_m$ is the square, i.e. ι_n to $(km/n)^2\iota_m$. In order to compute the induced homomorphisms on H^4 for other homomorphisms between two groups we need the following two lemmas.

Let us consider abelian groups of the form $\mathbb{Z}_m \oplus \mathbb{Z}_n$ with m and n relatively prime. Call ι_m, ι_n generators of $\mathbb{Z}_m, \mathbb{Z}_n$, respectively. Let ι_{mn} denote one generator of the cyclic group $\mathbb{Z}_{mn}$. We identify the two groups by the isomorphism $\phi: \mathbb{Z}_m \oplus \mathbb{Z}_n \to \mathbb{Z}_{mn}$ defined by $\phi(\iota_m)=n\iota_{mn}$ and $\phi(\iota_n)=m\iota_{mn}$.

Lemma 9. *Consider the groups $\mathbb{Z}_{m_i} \oplus \mathbb{Z}_{n_i}$ with $(m_i,n_i)=1$ for $i=1,2$. Call $\iota_{m_i}, \iota_{n_i}, \iota_{m_in_i}$ generators of $\mathbb{Z}_{m_i}, \mathbb{Z}_{n_i}, \mathbb{Z}_{m_in_i}$ respectively. Denote by $\phi_i: \mathbb{Z}_{m_i} \oplus \mathbb{Z}_{n_i} \to \mathbb{Z}_{m_in_i}$ the isomorphism defined by $\phi_i(\iota_{m_i})=n_i\iota_{m_in_i}$ and $\phi_i(\iota_{n_i})=m_i\iota_{m_in_i}$, for $i=1,2$. Let $\varphi=(\varphi_1,\varphi_2): \mathbb{Z}_{m_1}\oplus\mathbb{Z}_{n_1} \to \mathbb{Z}_{m_2}\oplus\mathbb{Z}_{n_2}$ be a homomorphism where φ_i is multiplication by d_i for $i=1,2$. The homomorphism $\tilde{\varphi} = \phi\circ\varphi\circ\phi^{-1}: \mathbb{Z}_{m_1n_1} \to \mathbb{Z}_{m_2n_2}$ is multiplication by $d=(n_2d_1+m_2d_2)(m_1+n_1)^{-1}$.*

Proof. The element $m_1 \oplus n_1$ is invertible in $\mathbb{Z}_{m_1}\oplus\mathbb{Z}_{n_1}$. Since $\tilde{\varphi}((n_1+m_1)\iota_{m_1n_1})=(d_1n_2+d_2m_2)\iota_{m_2n_2}$ the result follows. □

Let $G=A\rtimes B$ where the orders of A and B are relatively prime, and p is a period of all three groups $A, B, A\rtimes B$.

Lemma 10. *Let $G_i = A_i \rtimes B_i$ where A_i, B_i have orders relatively prime, for $i=1,2$, and $\theta: G_1 \to G_2$ a homomorphism such that $\theta(A_1)\subset A_2$, and denote by $\bar\theta$ the induced homomorphism on the quotient $G_1/A_1 \to G_2/A_2$. If A_i, B_i and G_i have period p, for $i=1,2$ and the induced homomorphisms on cohomology $\theta|_{A_1}^{\#}: H^p(A_2,\mathbb{Z})=\mathbb{Z}_{|A_2|} \to H^p(A_1,\mathbb{Z})=\mathbb{Z}_{|A_1|}$, and $\bar\theta^{\#}: H^p(B_2,\mathbb{Z})=\mathbb{Z}_{|B_2|} \to H^p(B_1,\mathbb{Z})=\mathbb{Z}_{|B_1|}$, are multiplication by d_1, d_2, respectively, then the homomorphism induced on cohomology $\theta^{\#}: H^p(G_2,\mathbb{Z})=\mathbb{Z}_{|G_2|} \to H^p(G_1,\mathbb{Z})=\mathbb{Z}_{|G_1|}$ is multiplication by $d=(d_1|B_1|+d_2|A_1|)(|A_2|+|B_2|)^{-1}$.*

Proof. Consider the homomorphism of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_1 & \longrightarrow & G_1 & \longrightarrow & B_1 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \theta|_{A_1}} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \bar\theta} & & \\ 1 & \longrightarrow & A_2 & \longrightarrow & G_2 & \longrightarrow & B_2 & \longrightarrow & 1. \end{array} \qquad (3)$$

The Lyndon-Hochschild-Serre spectral sequence associated to these short exact sequences are very simple since the groups (subgroup and quotient of a given short exact sequence) have order relatively prime. Namely the E_2 page of the spectral sequence in cohomology have all terms $E_2^{p,q}=0$ if $pq\neq 0$. The spectral sequence collapse and $E_\infty^{r,0}=H^r(B_i,\mathbb{Z})$, $E_\infty^{0,s}=H^0(B_i,H^s(A_i,\mathbb{Z}))$. For $r=s=p$ we have $E_\infty^{p,0}=H^p(B_i,\mathbb{Z})=\mathbb{Z}_{|B_i|}$ and $E_\infty^{0,p}=H^0(B_i,H^p(A_i,\mathbb{Z}))=\mathbb{Z}_{|A_i|}$ since p is the period and the action of the local coefficient system is trivial. So we have a short exact sequence

$$1 \to \mathbb{Z}_{|B_i|} \to \mathbb{Z}_{|G_i|} \to \mathbb{Z}_{|A_i|} \to 1,$$

as well a homomorphism of short exact sequence induced by the homomorphism θ. Then we apply Lemma 10. □

Now we make use of Lemmas 7, 8, 9 and 10 above to compute all the induced homomorphisms on H^4 of the above homomorphisms. The homomorphisms are divided in the following three cases.

Degree for Case I: Consider the homomorphism $\varphi \in Hom(D'_{2^qm_1}, D'_{2m_2})$ defined by $u_1 \to u_2$, $w_1 \to w_2$, where w_1 has order 2^q, w_2 has order 2, u_i has order m_i, $i=1,2$. Then we get a homomorphism of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & [D'_{2^q m_1}, D'_{2^q m_1}] = \mathbb{Z}_{m_1} & \longrightarrow & D'_{2^q m_1} & \longrightarrow & \mathbb{Z}_{2^q} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \varphi & & \downarrow & & \\ 1 & \longrightarrow & [D'_{2m_2}, D'_{2m_2}] = \mathbb{Z}_{m_2} & \longrightarrow & D'_{2m_2} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1. \end{array} \tag{4}$$

Now we can apply Lemma 10 for the diagram above. Since the induced map on the quotient $\mathbb{Z}_{2^q} \to \mathbb{Z}_2$ is the homomorphism sending w_1 to w_2, the induced in dimension on H^2 is multiplication by 2^{q-1} and on H^4 is multiplication by 2^{2q-2}, which is degree 0 except when $q = 1$, which is degree 1. The degree of the homomorphisms on H^4 when restricted to the kernels of the examples above is multiplication by $(m_2/m_1)^2$. Therefore the induced homomorphism $H^4(D'_{2m_2}, \mathbb{Z}) \to H^4(D'_{2^q m_1}, \mathbb{Z})$ is multiplication by $2^{2q-2}(m_2/m_1)^2$. This complete the calculation.

Degree for Case II: Consider the unique surjective homomorphism $D'_{2n} \to \mathbb{Z}_2$ which is the Abelianization. The induced homomorphism $H^2(\mathbb{Z}_2, \mathbb{Z}) \to H^2(D'_{2n}, \mathbb{Z}) = \mathbb{Z}_2$ is the identity. Therefore the induced homomorphism on H^4 is the homomorphism which sends the generator $\iota_2^2 \in H^4(\mathbb{Z}_2, \mathbb{Z})$ to the square of the generator of $H^2(D'_{2n}, \mathbb{Z})$ which is $n\iota_4$, using the ring structure from section 4. This complete the calculation.

Degree for Case III: Consider the homomorphism $\phi : D^*_{4m} \to D'_{2n}$ defined by $a \mapsto w$, $b \mapsto u$. Then we decompose D^*_{4m} as $\mathbb{Z}_l \rtimes Q_{2^r}$ where l is odd, so $r \geq 3$. We also have the decomposition $D'_{m_2} = \mathbb{Z}_{m_2} \rtimes \mathbb{Z}_2$ and we have a homomorphism of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_l & \longrightarrow & D^*_{4m} & \longrightarrow & Q_{2^q} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \varphi & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}_{m_2} & \longrightarrow & D'_{2m_2} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1. \end{array} \tag{5}$$

Using the same strategy as in Case I, we have a homomorphism induced on the spectral sequences of the corresponding short exact sequences. It remains to determine the induced homomorphism of the surjective homomorphism $Q_{2^r} \to \mathbb{Z}_2$ in cohomology. In dimension 2, this is an inclusion of $\mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Now we should identify the image of the generator $\iota_2 \in H^2(\mathbb{Z}_2, \mathbb{Z})$ in $H^2(Q_{2^r}, \mathbb{Z}) \cong \mathbb{Z}_2[\bar{\gamma}_2] \oplus \mathbb{Z}_2[\bar{\gamma}'_2]$, where γ_2, γ'_2 are given by Proposition 4 and $\bar{\gamma}_2, \bar{\gamma}'_2$ are the projection on the abelianization of $\bar{\gamma}_2, \bar{\gamma}'_2$. From the proof of Proposition 4 follows that the image of ι_2 is γ'_2. So it suffices to take the square of this element using the ring structure and the result follows. Again from Propositon 4 we have two cases. The first case is for $m = 4l$. Then we will see that the degree will be multiplication by $2m$. The case for $m = 4l + 2$, the square of any element of H^2 is trivial. So follows that the degree is 0, and this complete the proof of the result.

6. Degree from space forms associated to two arbitrary groups of period 4

In this section we show how to compute the mapping degree between two space forms where the fundamental groups of the space forms are of the form $\mathbb{Z}_{m_1} \rtimes_{\alpha_1} G_1$ and $\mathbb{Z}_{m_2} \rtimes_{\alpha_2} G_2$, where G_i belong to the Table II, and α_i is either trivial or the image $\alpha_i(G_i)$ is isomorphic to $\mathbb{Z}_2$. We will assume that m_i is relatively prime to $|G_i|$.

Given a surjective homomorphism $\phi : \mathbb{Z}_{m_1} \rtimes_{\alpha_1} G_1 \to \mathbb{Z}_{m_2} \rtimes_{\alpha_2} G_2$ the calculation of the degree of this general case can be reduced to the case given by the Lemma 10 above.

Lemma 11. *Given a surjective homomorphism $\phi : \mathbb{Z}_{m_1} \rtimes_{\alpha_1} G_1 \to \mathbb{Z}_{m_2} \rtimes_{\alpha_2} G_2$, let $H_1 = \phi(\mathbb{Z}_{m_1})$ and $H_2 = \phi(G_1)$. Then we have:*

1. *H_1 is a normal subgroups of $\mathbb{Z}_{m_2} \rtimes_{\alpha_2} G_2$;*

2. the subgroup $\langle H_1, H_2\rangle = \mathbb{Z}_{m_2} \rtimes_{\alpha_2} G_2$*;*

3. $H_1 \cap H_2 = \{1\}$*;*

4. $\mathbb{Z}_{m_2} \rtimes_{\alpha_2} G_2 = H_1 \rtimes H_2$*, where* H_1 *is cyclic and* H_2 *belong to the Table* II.

Proof. Note that the image of a cyclic group is again cyclic. The image of a group G_1 is isomorphic to a quotient of G_1 by a normal subgroup which in turn is also periodic. By inspection we see that the image of G_1 is again in Table II. □

Now we can state the main result:

Theorem 1. *Given a surjective homomorphism* $\phi : \mathbb{Z}_{m_1} \rtimes_{\alpha_1} G_1 \to \mathbb{Z}_{m_2} \rtimes_{\alpha_2} G_2$*, the degree is the product of the degrees of* $\phi_1 = \phi|_{\mathbb{Z}_{m_1}}$ *with the degree of* $\phi_2 = \phi|_{G_1}$*.*

Proof. Follows from the Lemma 10 and Lemma 11. □

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