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Дифференциальные включения с производными в среднем, имеющие асферические правые части

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Аннотация

На плоском n-мерном торе изучаются стохастические дифференциальные включения с производными в среднем, у которых правые части имеют, вообще говоря, не выпуклые (асферические) значения. Выделен подкласс таких включений, для которых существует последовательность ε-аппроксимаций, поточечно сходящаяся к измеримому по Борелю селектору. На этой основе получена теорема существования решения.

Ключевые слова: производные в среднем; дифференциальные включения; асферические правые части; поточечная сходимость; существование решений

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Differential inclusions with mean derivatives, having aspherical right-hand sides

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Abstract

On flat n-dimensional torus we study stochastic differential inclusions with mean derivatives, for which the right-hand sides have, generally speaking, not convex (aspherical) values. A subclass of such inclusions is distinguished for which there exists a sequence of ε-approximations, converging point-wise to a Borel measurable selector. On this base a solution existence theorem is obtained.

Keywords: mean derivatives; differential inclusions; aspherical right-hand sides; point-wise convergence; solution existence

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1. Introduction

The concept of mean derivatives was introduced by E. Nelson in the 60-s years of 20th century (see [1, 2, 3]) for the needs of the so-called stochastic mechanics constructed by him (a variant of quantum mechanics). Then it turned out that equations and inclusions with mean derivatives naturally arose in many branches of mathematical physics, economics and other sciences.

In this article, differential inclusions with mean derivatives, for which the right-hand sides are, generally speaking, have non-convex values of points. These are mappings that are aspherical in all dimensions from 1 to $n-1$ (see exact definitions below). This class of mappings was the first time described by A.D. Myshkis in 1954 in [4]. In [5] and in [6] for such mappings the topological characteristics of the type of topological index and Lefschetz number were constructed. Later (in the 1980s) this class was independently rediscovered by the group of Polish mathematicians led by Lech Górniewicz, and named "mappings whose values at each point have the so-called uv^k-property for $k = 1, \dots, n$" (see, for example, [7]). It is important that for such upper semicontinuous mappings, there exist the so-called ε-approximations (a special case here is the well-known construction of ε-approximations for mappings with convex values).

Here we study ε-approximations for such mappings from the point of view of existence of their sequences with the property of point-wise convergence to a Borel-measurable selector of the set valued mapping. For a subclass of mappings with this property on the flat n-dimensional torus an existence of solution theorem is proved for differential inclusions with mean derivatives.

It should be pointed out, that no unform but only point-wise convergence of ε-approximations of the right-hand sides of the ordinary differential inclusion, gives nothing useful for the investigation of those inclusions. But this paper shows that in the case of stochastic differential inclusions the point-wise convergence of ε-approximations of the right-hand sides is a powerful machinery for proving the existence of solution theorems.

Preliminaries from the theory of set-valued mappings can be found in [8, 9, 10], and the necessary information on stochastic analysis – in [11, 12].

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2. A brief introduction to the theory of set-valued mappings

The set-valued mapping F from the set X to the set Y is the rule that associates the nonempty set $F(x) \subset Y$ to each point $x \in X$; $F(x)$ is called the value of F at x.

To distinguish the set-valued maps from the single-valued ones, we introduce notation $F : X \multimap Y$ for the set-valued mapping F, acting from X to Y, and for single-valued mappings we keep the standard notation $f : X \to Y$.

If X and Y are metric spaces, for set-valued mappings there are several continuity analogues that turn into ordinary continuity in the case of single-valued mappings (here we do not consider the description of these properties for set-valued mappings of topological spaces, referring the reader, for example, to [9]). In this article we use only upper semicontinuity.

Definition 1. *The set-valued mapping F is called upper semicontinuous at the point $x \in X$ if for each $\varepsilon > 0$ there is a neighborhood $U(x)$ of the point x such that from $x' \in U(x)$ it follows that $F(x')$ belongs to ε -neighborhood of the set $F(x)$. F is called upper semicontinuous if it is upper semicontinuous at each point of the set X.*

A set-valued mapping is called closed if its graph is a closed set in $X \times Y$. It follows, e.g., from [9, 1.2.29] that every upper semicontinuous mapping with bounded closed values of metric spaces is closed.

An important technical role in the study of set-valued mappings play the single-valued mappings approximating them in some sense.

Definition 2. *For a given $\varepsilon > 0$ the continuous single-valued mapping $f_\varepsilon : X \to Y$ is called the ε-approximation of the set-valued mapping $F : X \multimap Y$ if the graph of f as a set in $X \times Y$ belongs to the ε-neighborhood of the graph of F.*

For the following classes of upper semicontinuous set-valued mappings in finite-dimensional spaces the existence of ε-approximations for any $\varepsilon > 0$ is shown:

— mappings with convex closed values;

— the so-called mappings that are aspherical in all dimensions from 1 to $n-1$ and are slightly aspherical in dimensions n (see the history of research and references in §1).

3. ε-approximations for upper semicontinuous mappings with aspherical values

We give an exact definition of set-valued mappings with aspherical values, following [4, 5, 6].

Everywhere below, we consider a set-valued upper semicontinuous mapping $F : X \multimap E$ from an n -dimensional compact polyhedron X lying in some Euclidean space, to E or to the polyhedron X itself. Assumption that X is a polyhedron does not restrict the generality. In particular, $F : X \multimap X$ and $F : E \multimap E$ can be considered as such mappings.

By the symbol $O(A, r)$ we denote the r-neighborhood of the set A, the symbol $d(A)$ means the diameter of A.

It is easy to see that from the definition of upper semicontinuity or closeness of the mapping F it follows that for any $\varepsilon > 0$ and $\beta > 0$ there exists a number $\alpha(\varepsilon, \beta)$ such that in β -neighborhood $O(T, \beta)$ of an arbitrary set T with the diameter smaller than α, there exists a point x_0 called *satellite* of the set T, such that $O(F(x_0), \varepsilon) \supset F(T)$.

Definition 3. *The map $F : X \multimap X$ is called aspherical in dimension k, if in each neighborhood $O(F(x), \varepsilon)$ of each value $F(x)$ there exists a neighborhood of $Q(x, \varepsilon, k)$ containing $\delta = \delta(\varepsilon)$ neighborhood $F(x)$ (δ is independent from x) such that $\pi_k(Q) = 0$, where $\pi_k(Q)$ is the k-th homotopy group of Q.*

Everywhere below we assume that F is aspherical in dimensions $k = 0, 1, \ldots, n-1$. Recall that $\pi_0(Q) = 0$ means that Q is linearly connected. We describe the construction of ε -approximations for such set-valued mappings F that are upper semicontinuous and have closed values, by modifying the approach of [5].

Let μ be a real number such that $O(F(x), \mu)$ lies in aspherical in dimension $n-1$ neighborhood of $F(x)$, μ is independent of x. We construct a sequence

$$\mu > \varepsilon_{2n+1} > \varepsilon_{2n} > \delta(\varepsilon_{2n}) > \varepsilon_{2n-1} > \cdots > \varepsilon_2 > \delta(\varepsilon_2) > \varepsilon_1 \tag{1}$$

where $\delta(\varepsilon_i)$ is the number defining the $\delta(\varepsilon_i)$-neighborhood of the value $F(x)$ that is contained in $\bigcap_{k=0}^{n} Q(x, \varepsilon_i, k)$. Then we construct the sequence $\{\beta_i\}_1^{n+1}$ and the number α_0 such that

$$0 < \beta_k < \frac{1}{4}\beta_{k+1}; \qquad \beta_k + \alpha_0 < \alpha(\varepsilon_{2k+1} - \varepsilon_{2k}, \beta_{k+1}) \tag{2}$$

where $\alpha(\varepsilon, \beta)$ is introduced above in this section. Obviously, such a sequence can be constructed by starting with the largest indices.

Now we define a triangulation of X such that the diameter of each simplex would be less than $d < min(\alpha_0, \alpha(\varepsilon_1, \beta_1))$. To each 0-dimensional simplex T_i^0 we associate a point $f(T_i^0) \in F(T_i^0)$. For each 1 -dimensional simplex T_i^1 we get that $d(T_i^1) < \alpha(\varepsilon_1, \beta_1)$. Thus, for the satellite x_i^1 it holds that $x_i^1 \in O(T_i^1, \beta_1)$ and $F(T_i^1) \subset O(F(x_i^1), \varepsilon_1)$. Hence, the following inclusions

$$f(T_{i_1}^0) \cup f(T_{i_2}^0) \subset F(T_i^1) \subset O(F(x_i^1), \varepsilon_1) \tag{3}$$

and

$$O(F(x_i^1), \varepsilon_1) \subset O(F(x_i^1), \delta(\varepsilon_2)) \subset Q(x_i^1, \varepsilon_2, 0) \subset O(F(x_i^1), \varepsilon_2) \tag{4}$$

hold, where $T_{i_1}^0$ and $T_{i_2}^0$ are the edges of T_i^1. Since Q is aspherical in dimension 0, f can be extended on T_i^1 as continuous mapping and

$$f(T_i^1) \subset Q(x_i^1, \varepsilon_2, 0) \subset O(F(x_i^1), \varepsilon_2). \tag{5}$$

Let T_i^2 be a 2-dimensional simplex with 1-dimensional edges $T_{i_1}^1$, $T_{i_2}^1$ and $T_{i_3}^1$. Let $x_{i_1}^1$, $x_{i_2}^1$ and $x_{i_3}^1$ be the satellites corresponding to these edges. They form the set $\tilde{T}_i^1$ for which

$$d(\tilde{T}_i^1) < 2\beta_1 + \alpha_0 < \alpha(\varepsilon_3 - \varepsilon_2, \beta_2).$$

There is a satellite x_i^2 of the set $\tilde{T}_i^1$ such that,

$$x_i^2 \in O(\tilde{T}_i^1, \beta_2) \quad \text{and} \quad F(\tilde{T}_i^1) \subset O(F(x_i^2), \varepsilon_2).$$

Taking into account (4) and (5) we derive

$$\bigcup_{j=1,2,3} f(T_{i_j}^1) \subset O(F(\tilde{T}_i^1), \varepsilon_2) \subset O(F(x_i^2), \varepsilon_2). \tag{6}$$

By (1) we have inclusions

$$O(F(x_i^2), \varepsilon_3) \subset O(F(x_i^2), \delta(\varepsilon_4)) \subset Q(x_i^2, \varepsilon_4, 1) \subset O(F(x_i^2, \varepsilon_4). \tag{7}$$

Since $\pi_2 Q(x_i^2, \varepsilon_4, 1) = 0$, we can extend f from the boundary of the simplex T_i^2 to the whole simplex as continuous mapping. In addition, we obtain that

$$f(T_i^2) \subset Q(x_i^2, \varepsilon_4, 1) \subset O(F(x_i^2), \varepsilon_4). \tag{8}$$

And so on. In the last step, we extend f from $(n-1)$-skeleton of X to the entire X as a continuous mapping. By the construction the graph of f lies in ε_{2n+1}-neighborhood of the graph of F.

Theorem 1. *For F as above there exists a sequence $f^{(k)}$ of continuous ε_{2n+1}^k-approximations of the type described above, $\varepsilon_{2n+1}^k \to 0$ for $k \to \infty$, such that for any point x of some countable everywhere dense subset $\Xi \subset X$ there is an integer K such that for any $k > K$ the inclusion $f^{(k)}(x) \in F(x)$ and $f^{(k+l)}(x) = f^{(k)}(x)$ for any integer $l > 0$.*

Proof. By the construction, for each x from 0-dimensional skeleton of X for f constructed above, we set the value of $f(x) \in F(x)$. Now consider the sequence of barycentric partitions of X. We denote by X_0^k the 0-dimensional skeleton of the k-th partitions. At each $k+1$ -th step for $x \in X_0^k$ we save the value $f^{(k+1)}(x) = f^{(k)}(x)$ and introduce an arbitrary value $f^{(k+1)}(x) \in F(x)$ for $x \in X_0^{(k+1)} \backslash X_0^{(k)}$. Then we construct continuous $f^{(k+1)}$ on all X in the same way, as above. The limit set Ξ in $X_0^{(k)}$ for $k \to \infty$ is the desired set. $\square$

Corollary 1. *In the notation of Theorem1 the sequence $f^{(k)}$ on Ξ converges point-wise to the selector f of F so that for any $x \in \Xi$ the values of $f^{(k)}(x)$ stabilize starting from some number $K(x)$. From the point-wise convergence it follows that f is Borel measurable on Ξ.*

Among the set-valued mappings with aspherical values there is a subclass of mappings that have sequences of ε-approximations that converge point-wise to some Borel selector on the whole polyhedron. For example, this includes upper semicontinuous mappings with values aspherical in dimensions $k = 0, 1, \dots, n-1$ only at a finite number points (which we include in the number of vertices of simplexes), and in the rest points the value are closed and convex. Existence of point-wise convergent ε-approximations for upper semicontinuous set-valued mappings with convex values in two different cases are proved in [13, 14].

For the convenience of references in the text, we write down the property of those mappings, which we will consider below.

Condition 1. *We assume that the considered mappings with aspherical values have sequences of ε-approximations, which converge point-wise to a certain Borel selector on the whole polyhedron.*

4. Preliminary Information on Mean Derivatives

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, T]$ defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is L_1-random element for all t. Denote by $\mathcal{N}_t^\xi$ the completed with sets of measure 0 σ-subalgebra of σ-algebra $\mathcal{F}$ generated by the preimages of Borel sets in $\mathbb{R}^n$ under the map $\xi(t)$ (following Nelson, see, for example, [1, 2, 3], we call $\mathcal{N}_t^\xi$ the "present" for the process $\xi(t)$). For convenience, we denote the conditional expectation of $\xi(t)$ relative to $\mathcal{N}_t^\xi$ by $E_t^\xi(\cdot)$.

The usual ("unconditional") mathematical expectation we denote by E.

Strictly speaking, almost sure (a.s.) the sample trajectories of $\xi(t)$ are not differentiable for almost all t. So, the classic derivatives $\xi(t)$ exist only in the sense of generalized functions. To avoid using generalized functions, by following Nelson (see, for example, [1, 2, 3]) we give

Definition 4. *The forward mean derivative $D\xi(t)$ of the process $\xi(t)$ at time instant t is an L_1-random element of the form*

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\Big(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t}\Big) \tag{9}$$

where the limit is assumed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$, and $\Delta t \to +0$ means that Δt tends to 0 $\Delta t > 0$.

From the properties of conditional mathematical expectation (see [11]) it follows that $D\xi(t)$ can be represented as a superposition of $\xi(t)$ swith the Borel measurable vector field (regression)

$$a(t,x) = \lim_{\Delta t \to +0} E\Big(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t} | \xi(t) = x\Big) \tag{10}$$

on $\mathbb{R}^n$. This means that $D\xi(t) = a(t, \xi(t))$.

Following [15] (see also [12]), we introduce the mean derivative D_2 of the process $\xi(t)$ by the following formula

$$D_2\xi(t) = \lim_{\triangle t \to +0} E_t^\xi\left(\frac{(\xi(t+\triangle t) - \xi(t))(\xi(t+\triangle t) - \xi(t))^*}{\triangle t}\right), \tag{11}$$

where $(\xi(t+\triangle t) - \xi(t))$ is considered as a column (vector in $\mathbb{R}^n$) and $(\xi(t+\triangle t) - \xi(t))^*$ - as a row (transposed or conjugate vector). As above, the limit exists in $L_1(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 5. *D_2 is called the quadratic mean derivative.*

It is shown that the quadratic mean derivative takes values in the space of symmetric positive semi-definite $(n \times n)$ matrices $\overline{S}_+(n)$.

5. Differential inclusions with aspherical right-hand sides

In this section we consider differential inclusions with mean derivatives on the flat n-dimensional torus $\mathbb{T}^n$. On the one hand, T^n is a compact polyhedron, i.e. on it the constructions from §3 are well-defined. On the other hand, Riemannian metric on $\mathbb{T}^n$ is inherited from $\mathbb{R}^n$ on factorizing with respect to integer lattice. This makes possible applying the technique developed for $\mathbb{R}^n$.

Consider on $\mathbb{T}^n$ a set-valued vector field $\mathbf{a}(t,x)$ and a set-valued $(2,0)$-tensor field $\boldsymbol{\alpha}(t,x)$ having closed uniformly bounded values, aspherical in dimensions $k = 0, 1, \dots, n-1$ values and satisfy Condition 1. In addition, for $\boldsymbol{\alpha}(t,x)$ we assume that it takes values in symmetric positive definite bilinear forms ($(2,0)$ tensors). The differential inclusion with those fields is the system of the form

$$\begin{cases} D\xi(t) \in \mathbf{a}(t,\xi(t)), \\ D_2\xi(t) \in \boldsymbol{\alpha}(t,\xi(t)). \end{cases} \tag{12}$$

Definition 6. *The inclusion* (12) *has a solution with the initial condition $\xi_0 \in \mathbb{T}^n$ if there exists a probability space and a random process $\xi(t)$ given on it and taking values in $\mathbb{T}^n$, such that $\xi(0) = \xi_0$ and a.s. $\xi(t)$ satisfies inclusion* (12)*.*

For simplicity, we deal only with determinate initial conditions.

Theorem 2. *Under the assumptions made above, for any initial condition $\xi(0) \in \mathbb{T}^n$ inclusion* (12) *has a solution.*

Proof. Choose a sequence of positive numbers $\varepsilon_k \to 0$ such that the corresponding continuous ε_k-approximations of $a_k(t,x)$ and of $\alpha_k(t,x)$ converge point-wise to Borel selectors $a(t,x)$ and $\alpha(t,x)$ of set-valued fields $\mathbf{a}(t,x)$ and $\boldsymbol{\alpha}(t,x)$, respectively. Moreover, it is easy to see that $\alpha_k(t,x)$ are symmetric and positive definite. Then by [12, Lemma 8.40] there exist continuous matrix fields $A_k(t,x)$ such that $\alpha_k(t,x) = A_k(t,x)A_k^*(t,x)$, where $A_k^*(t,x)$ is the transposed matrix $A(t,x)$. In this case, the sequence $A_k(t,x)$ converges point-wise to the matrix field $A(t,x)$.

Consider the sequence of stochastic differential equations in Ito form

$$\xi_k(t) = \xi_0 + \int_0^t a_k(s,\xi_k(s))ds + \int_0^t A_k(s,\xi_k(s))dw(s). \tag{13}$$

Since the coefficients in (13) are continuous and also, by construction, are uniformly bounded, by [16, Theorem III.2.4] all these equations have weak solutions $\xi_k(t)$. The corresponding measures μ_k on the space $(C^0([0,T],\mathbb{T}^n),\mathfrak{C})$, where $\mathfrak{C}$ is the σ-algebra of cylindrical sets, are weakly compact by [16, Corollary to Lemma III.2.2], hence we can choose a subsequence that weakly converges to some measure μ. For convenience, we assume that the sequence μ_k itself converges weakly to μ. We shall show that the coordinate process $\xi(t)$ on the probability space $(C^0([0,T],\mathbb{T}^n),\mathfrak{C},\mu)$ is the solution we are looking for. Recall that the coordinate process is defined by the equality $\xi(t,x(\cdot)) = x(t)$.

Denote by λ the normalized Lebesgue measure on $[0,T]$.

Since $a_i(t,x(\cdot))$ point-wise converges as $i \to \infty$ to $a(t,x(\cdot))$, this convergence takes place a.s. for each measure $\lambda \times \mu_k$. Choose $\delta > 0$. By the Egorov theorem (see, e.g. , [17]) for any k there is a subset $\tilde{K}_\delta^k \subset [0,T] \times C^0([0,T],\mathbb{T}^n)$ such that $(\lambda \times \mu_k)(\tilde{K}_\delta^k) > 1-\delta$, and the sequence $a_i(t,x(\cdot))$ converges to $a(t,x(\cdot))$ uniformly on $\tilde{K}_\delta^k$. We introduce $\tilde{K}_\delta = \bigcup\limits_{i=0}^{\infty} \tilde{K}_\delta^k$. The sequence $a_i(t,x(\cdot))$ converges to $a(t,x(\cdot))$ uniformly on $\tilde{K}_\delta$ and $(\lambda \times \nu_k)(\tilde{K}_\delta) > 1-\delta$ for all $k = 0, \dots, \infty$.

Note that $a(t,x(\cdot))$ is continuous on the set of full measure $\lambda \times \mu$ on $[0,T] \times C^0([0,T],\mathbb{T}^n)$. Indeed, consider the sequence $\delta_k \to 0$ and the corresponding sequence $\tilde{K}_{\delta_k}$ from the Egorov theorem. From the above construction it follows that $a(t,x(\cdot))$ is the uniform limit of the sequence of continuous functions on each $\tilde{K}_{\delta_k}$. Thus it is continuous on every $\tilde{K}_{\delta_k}$ and so on every final union $\bigcup\limits_{i=1}^{n}\tilde{K}_{\delta_i}$. It is obvious that $\lim\limits_{n\to\infty}(\lambda\times\mu)(\bigcup\limits_{i=1}^{n}\tilde{K}_{\delta_i})=(\lambda\times\mu)([0,T]\times C^0([0,T],\mathbb{T}^n)$.

Let $g_t(x(\cdot))$ be bounded (set $|g_t(x(\cdot))|<\Theta$ for all $x(\cdot)\in C^0([0,T],\mathbb{T}^n)$ and continuous, $\mathcal{N}_t^{\xi}$-measurable function on $C^0([0,T],\mathbb{T}^n)$, where $\mathcal{N}_t^{\xi}$ is the "present" of the coordinate process, see §4).

From uniform convergence (see above) to $\tilde{K}_\delta$ for all k and the boundedness of g_t it follows that for sufficiently large k

$$\left\|\int_{\tilde{K}_\delta}\left(\int_t^{t+\Delta t}(a_k(\tau,x(\cdot))-a(\tau,x(\cdot)))d\tau\right)g_t(x(\cdot))d\mu_k\right\|<\delta.$$

Since $(\lambda\times\mu_k)(\tilde{K}_\delta)>1-\delta$ for all k, $\|a_k(t,x(\cdot))-a(t,x(\cdot))<Q$ for all k and $|g_t(x(\cdot))|<\Theta$ (see above), we obtain

$$\left\|\int_{C^0([0,T],\mathbb{T}^n)\setminus\tilde{K}_\delta}\left(\int_t^{t+\Delta t}(a_k(\tau,x(\cdot))-a(\tau,x(\cdot)))d\tau\right)g_t(x(\cdot))d\mu_k\right\|<2Q\Theta\delta.$$

From the fact that δ is an arbitrary positive number, it follows that

$$\lim_{k\to\infty}\int_{C^0([0,T],\mathbb{T}^n)}\left(\int_t^{t+\Delta t}a_k(\tau,x(\cdot))d\tau-\int_t^{t+\Delta t}a(\tau,x(\cdot))d\tau\right)g_t(x(\cdot))d\mu_k=0.$$

From the weak convergence of μ_k to μ it follows that

$$\lim_{k\to\infty}\int\limits_{C^0([0,T],\mathbb{T}^n)}\left(\int_t^{t+\Delta t}a(\tau,x(\cdot))d\tau\right)g_t(x(\cdot))d\mu_k=$$
$$\int\limits_{C^0([0,T],\mathbb{T}^n)}\left(\int_t^{t+\Delta t}a(\tau,x(\cdot))d\tau\right)g_t(x(\cdot))d\mu. \tag{14}$$

It's obvious that

$$\lim_{i\to\infty}\int\limits_{C^0([0,T],\mathbb{T}^n)}(x(t+\Delta t)-x(t))d\mu_k=\int\limits_{C^0([0,T],\mathbb{T}^n)}(x(t+\Delta t)-x(t))d\mu. \tag{15}$$

Note that

$$\int_{C^0([0,T],\mathbb{T}^n)}\left(\Big[x(t+\Delta t)-x(t)\Big]-\int_t^{t+\Delta t}a_k(\tau,x(\cdot))d\tau\right)g_t(x(\cdot))d\mu_k=0 \tag{16}$$

since

$$\int\limits_{C^0([0,T],\mathbb{T}^n)}[x(t+\Delta t)-x(t)]g_t(x(\cdot))d\mu_k=E\left[(\xi_k(t+\Delta t)-\xi_k(t))g_t(\xi_k(t))\right],$$

$$\int\limits_{C^0([0,T],\mathbb{T}^n)}\left(\int_t^{t+\Delta t}a_k(\tau,x(\cdot))d\tau\right)g_t(x(\cdot))d\mu_k=E\Big[\left(\int_t^{t+\Delta t}a_k(\tau,\xi_k(\tau))d\tau\right)g_t(\xi_k(t))\Big]$$

and $\xi_k(t)$ is a solution (13).

Formulae (14), (15) and (16) yield the equality

$$\int\limits_{C^0([0,T],\mathbb{T}^n)} \left(\Big[x(t+\Delta t) - x(t) \Big] - \int\limits_t^{t+\Delta t} a(s,x(\cdot))ds \right) g_t(x(\cdot))d\mu = 0.$$

Since g_t is an arbitrary continuous bounded function, measurable with respect to $\mathcal{N}_t^\xi$, the latter relation is equivalent to

$$E_t^\xi \left(\Big[\xi(t+\Delta t) - \xi(t) \Big] - \int\limits_t^{t+\Delta t} a(s,\xi(\cdot))ds \right) = 0. \tag{17}$$

So from (17) it follows that

$$D\xi(t) = a(t,\xi(\cdot)) \in \mathbf{a}(t,\xi(\cdot)). \tag{18}$$

Now we turn to $A_k(t,x(\cdot))$. Recall that $\alpha_k(t,x(\cdot)) = A_k(t,x(\cdot))A_k^*(t,x(\cdot))$ point-wise converges to $\alpha(t,x(\cdot))$, a Borel measurable selector of $\boldsymbol{\alpha}(t,x(\cdot))$. Absolutely similar to the above arguments, it is easy to show that

$$\int\limits_{C^0([0,T],\mathbb{T}^n)} \left(\Big[(x(t+\Delta t) - x(t))(x(t+\Delta t) - x(t))^* \Big] - \int\limits_t^{t+\Delta t} \alpha(s,x(\cdot))ds \right) g_t(x(\cdot))d\mu = 0 \tag{19}$$

with the same g_t as above. Relation (19) is equivalent to

$$E_t^\xi \left(\Big[(\xi(t+\Delta t) - \xi(t))(\xi(t+\Delta t) - \xi(t))^* \Big] - \int\limits_t^{t+\Delta t} \alpha(s,\xi(\cdot)))ds \right) = 0$$

which obviously implies that

$$D_2\xi(t) = \alpha(t,\xi(\cdot)) \in \boldsymbol{\alpha}(t,\xi(\cdot)). \tag{20}$$

Equalities (18) and (20) complete the proof. □

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