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ПМП, (ко)присоединённое представление и нормальные геодезические левоинвариантных (суб)финслеровых метрик на группах Ли[1]

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Аннотация

С помощью основ теории групп и алгебр Ли, их (ко)присоединенных представлений и принципа максимума Понтрягина для задачи оптимального быстродействия даны независимое обоснование методов геодезического векторного поля поиска геодезических левоинвариантных (суб)финслеровых метрик на группах Ли и поиска соответствующих локально оптимальных управлений в (суб)римановом случае, а также несколько их применений.

Ключевые слова: алгебра Ли, группа Ли, (ко)присоединенное представление, левоинвариантная (суб)риманова метрика, левоинвариантная (суб)финслерова метрика, математический маятник, нормальная геодезическая, оптимальное управление.

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PMP, (co)adjoint representation, and normal geodesics, of left-invariant (sub-)Finsler metric on Lie groups

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Abstract

On the ground of origins of the theory of Lie groups and Lie algebras, their (co)adjoint representations, and the Pontryagin maximum principle for the time-optimal problem are given an independent foundation for methods of geodesic vector field to search for normal geodesics of left-invariant (sub-)Finsler metrics on Lie groups and to look for the corresponding locally optimal controls in (sub-)Riemannian case, as well as some their applications.

Keywords: (co)adjoint representation, left-invariant (sub-)Finsler metric, left-invariant (sub-)Riemannian metric, Lie algebra, Lie group, mathematical pendulum, normal geodesic, optimal control.

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1. Introduction

After Gromov's 1980s papers, homogeneous sub-Finsler manifolds, in particular, sub-Riemannian manifolds were actively studied [1], [15], [22], [26]. Their investigation is based on the Rashevsky–Chow theorem which states that any two points of a connected manifold can be joined by a piecewise smooth curve tangent to a given totally nonholonomic distribution [14], [20].

1) Every homogeneous manifold with intrinsic metric is the quotient space G/H of some connected Lie group G by its compact subgroup H, equipped with G-invariant Finsler or sub-Finsler metric d; in particular, it may be Riemannian or sub-Riemannian metric [3], [4], [5];

2) moreover, according to a form of metric d, there exists a left-invariant Finsler, sub-Finsler, Riemannian or sub-Riemannian metric ρ on G such that the canonical projection $(G,\rho)\rightarrow(G/H,d)$ is a submetry [5], [2], [18].

The search for geodesics of homogeneous (sub-)Finsler manifolds are reduced to the case of Lie groups with left-invariant (sub-)Finsler metrics.

The shortest arcs on Lie groups with left-invariant (sub)-Finsler metrics are optimal trajectories of the corresponding left-invariant time-optimal problem on Lie groups [3]. This permits to apply the Pontryagin maximum principle (PMP) for their search [13]. By this method, in [7] are found all geodesics and shortest arcs of an arbitrary sub-Finsler metric on the three-dimensional Heisenberg group.

In [8] is proposed a search method of normal geodesics on Lie groups with left-invariant sub-Riemannian metrics. The method is applicable to Lie groups with left-invariant Riemannian metrics, since all their geodesics are normal.

In this paper, to find geodesics of left-invariant (sub-)Finsler metrics on Lie groups and corresponding locally optimal controls in (sub-)Riemannian case we use the geodesic vector field method (Theorems 7, 8) and an improved version of method from [8], applying (co)adjoint representations. The version is based on differential equations from Theorem 9 for controls, using only the structure constants of Lie algebras of Lie groups.

An interesting feature of these two methods in (sub-)Riemannian case is that locally optimal controls on Lie algebras of Lie groups for geodesics and corresponding geodesic vector fields on Lie groups (their integral curves are geodesics, i.e., locally optimal trajectories) can be determined independently of each other. Moreover, controls on different Lie algebras could be solutions of the same mathematical pendulum equation (see sections 6–8).

Analogues of Theorems 4 and 7 (but for the last theorem is only along one geodesic) are proved in the book [22] on the basis of more complicated concepts and apparatus. Apparently, other researchers did not apply PMP *for the time-optimal problem* to find geodesics of left-invariant metrics on Lie groups.

2. Preliminaries

The left and the right shifts $l_g : h \in G \to g \cdot h$, $r_g : h \in G \to h \cdot g$, $g, h \in G$, of a Lie group $(G, \cdot)$ by an element g are diffeomorphisms with the inverse shifts $l_{g^{-1}}$, $r_{g^{-1}}$, and their differentials $(dl_g)_h : T_hG \to T_{gh}G$ and $(dr_g)_h : T_hG \to T_{hg}G$ are linear isomorphisms of tangent vector spaces to G at corresponding points.

There exist an open neighborhoods U of zero in the Lie algebra $\mathfrak{g} = T_eG$ of the Lie group G and W of unit e in G such that $\exp : U \to W$ is a diffeomorphism. If $\dim G = n$ then after introduction of arbitrary Cartesian coordinates $(x_1, \dots, x_n)$ with zero origin 0 in $\mathfrak{g}$, it is naturally identified with $\mathbb{R}^n$. Then $\exp^{-1} : W \to U \subset \mathbb{R}^n$ is a local chart (a coordinate system) on G in the neighborhood W of the point $e \in G$. This coordinate system in W is called *a coordinate system of the first kind.*

The group $\mathrm{GL}(n) = \mathrm{GL}(n, \mathbb{R})$ of all nondegenerate real squared $(n \times n)$-matrices is a Lie group relative to the global map that associates to each matrix $g \in \mathrm{GL}(n)$ its elements g_{ij}, $i, j = 1, \dots, n$. Obviously, for every $g \in G$ the mapping $I(g) : G \to G$ such that

$$I(g)(h) = g \cdot h \cdot g^{-1} = (l_g \circ r_{g^{-1}})(h) = (r_{g^{-1}} \circ l_g)(h)$$

is an automorphism of the Lie group $(G, \cdot)$, $I(g)(e) = e$, and the differential

$$(dI(g))_e := dl_g \circ dr_{g^{-1}} : T_eG \to T_eG$$

is a nondegenerate linear map (i.e. an element of the Lie group $\mathrm{GL}(n)$ relative to some vector basis in T_eG, if $\dim G = n$), denoted with $\mathrm{Ad}(g)$. The calculation rule for the differential of composition gives

$$\mathrm{Ad}(g_1 \cdot g_2) = (dI(g_1 \cdot g_2))_e = (d(I(g_1) \circ I(g_2)))_e =$$

$$(dI(g_1))_e \circ (dI(g_2))_e = \mathrm{Ad}(g_1) \circ \mathrm{Ad}(g_2),$$

i.e., $\mathrm{Ad} : G \to \mathrm{GL}(n)$ is a homomorphism of Lie groups, called the *adjoint representation of the Lie group G.*

3. Theoretical results

Definition 1. *Let* $(\mathfrak{l},[\cdot,\cdot])$ *be a Lie algebra;* $\mathfrak{p},\mathfrak{q}\subset\mathfrak{l}$ *are nonzero vector subspaces. By definition,*

$$[\mathfrak{p},\mathfrak{q}]=\{[v,w]:v\in\mathfrak{p},w\in\mathfrak{q}\}.$$

If $\dim\mathfrak{p}\geq 2$ *then by definition,*

$$\mathfrak{p}^1=\mathfrak{p},\quad \mathfrak{p}^{k+1}=[\mathfrak{p},\mathfrak{p}^k],\quad \mathfrak{p}_m=\sum_{k=1}^{m}\mathfrak{p}^k.$$

The vector subspace $\mathfrak{p}\subset\mathfrak{l}$ *generates the Lie algebra* $(\mathfrak{l},[\cdot,\cdot])$*, if* $\mathfrak{l}=\mathfrak{p}_m$ *for some natural number* m*; the smallest number* $m:=s$ *with such property is called the generation degree (of the algebra* $(\mathfrak{l},[\cdot,\cdot])$ *by the subspace* $\mathfrak{p}$*).*

It is clear that subsets from Definition 1 are vector subspaces of $\mathfrak{l}$.

Let $\{e_1,\dots,e_r\}$ be any basis of the vector subspace $\mathfrak{p}\subset\mathfrak{g}$, generating the Lie algebra $(\mathfrak{g},[\cdot,\cdot])$ of a Lie group $(G,\cdot)$. One can prove the following special case of the Rashevsky-Chow theorem.

Theorem 1. *Let* $(G,\cdot)$ *be a connected Lie group and a vector subspace* $\mathfrak{p}\subset\mathfrak{g}$ *generates Lie algebra* $(\mathfrak{g},[\cdot,\cdot])$*. Then the control system*

$$\dot{g}=dl_g(u),\quad u\in\mathfrak{p},\tag{1}$$

is controllable (attainable) by means of piecewise constant controls

$$u=u(t)\in\mathfrak{p},\quad 0\leq t\leq T,\tag{2}$$

where $u(t)=\pm e_j$*,* $j=1,\dots,r$*, in the constancy segments of the control. In other words, for any elements* $g_0,g_1\in G$ *there exists a piecewise constant control (2) of this type such that* $g(T)=g_1$ *for solution of the Cauchy problem*

$$\dot{g}(t)=dl_{g(t)}(u(t)),\quad g(0)=g_0.$$

Every left-invariant (sub-)Finsler metric $d=d_F$ on a connected Lie group G with Lie algebra $(\mathfrak{g},[\cdot,\cdot])$ is defined by a subspace $\mathfrak{p}\subset\mathfrak{g}$, generating $\mathfrak{g}$, and some norm F on $\mathfrak{p}$. A distance $d(g,h)$ for $g,h\in G$ is defined as the infimum of lengths $\int_0^T|\dot{g}(t)|dt$ of piecewise smooth paths $g=g(t)$, $0\leq t\leq T$, such that $dl_{g(t)^{-1}}(\dot{g}(t))\in\mathfrak{p}$ and $g(0)=g$, $g(T)=h$; T is not fixed, $|\dot{g}(t)|=F(dl_{g(t)^{-1}}(\dot{g}(t)))$. The existence of such paths and, consequently, the finiteness of d are guaranteed by Theorem 1. Obviously, all three metric properties for d are fulfilled. If $\mathfrak{p}=\mathfrak{g}$ then d is a left-invariant Finsler metric on G; if $F(v)=\sqrt{\langle v,v\rangle}$, $v\in\mathfrak{p}$, where $\langle\cdot,\cdot\rangle$ is some scalar product on $\mathfrak{g}$, then d is a left-invariant sub-Riemannian metric on G, and d is a left-invariant Riemannian metric, if additionally $\mathfrak{p}=\mathfrak{g}$.

The following statements were proved in [4]. The space (G,d) is a locally compact and complete. Then in consequence of S.E. Cohn–Vossen theorem [12] the space (G,d) is a *geodesic* space, i.e. for any elements $g,h\in G$ there exists a shortest arc $c=c(t)$, $0\leq t\leq T$, in (G,d), which joins them. This means that c is a continuous curve in G, whose length in the metric space (G,d) is equal to $d(g,h)$. Therefore we can assume that c is parameterized by arc length, i.e. $T=d(g,h)$ and $d(c(t_1),c(t_2))=t_2-t_1$ if $0\leq t_1\leq t_2\leq d(g,h)$. Then $c=c(t)$, $0\leq t\leq d(g,h)$, is a Lipschitz curve relative to the smooth structure of the Lie group G. Therefore this curve is absolutely continuous. Then in consequence of well–known theorem from mathematical analysis, there exists a measurable, almost everywhere defined derivative function $\dot{c}(t)$, $0\leq t\leq d(g,h)$, and $c(t)=c(0)+\int_0^t\dot{c}(\tau)d\tau$, $0\leq t\leq T$.

Theorem 2. *[3] Every shortest arc $g = g(t)$, $0 \leq t \leq T = d(g_0, g_1)$, in (G, d) with $g(0) = g_0$, $g(T) = g_1$, is a solution of the time-optimal problem for the control system (1) with compact control region*

$$U = \{u \in \mathfrak{p} : F(u) \leq 1\}$$

and indicated endpoints.

In consequence of Theorem 2, one can apply the Pontryagin maximum principle [13] for the time-optimal problem from Theorem 2 and a covector function $\psi = \psi(t) \in T^*_{g(t)}$ to find shortest arcs on the Lie group G with left-invariant sub-Finsler metric d. The function ψ can be considered as a left-invariant 1-form on $(G, \cdot)$ and therefore it is natural to identify it with a covector function $\psi(t) \in \mathfrak{g}^* = T^*_e G$. Then every optimal trajectory $g(t)$, $0 \leq t \leq T$, is determined by some mesurable optimal control $\overline{u} = \overline{u}(t) \in U$, $0 \leq t \leq T$. Moreover, for some non-vanishing absolutely continuous function $\psi = \psi(t)$, $0 \leq t \leq T$, we have

$$H = H(g, \psi, u) = \psi(dl_g(u)) = \psi(u), \tag{3}$$

$$\dot{g} = \frac{\partial H}{\partial \psi}, \quad \dot{\psi} = -\frac{\partial H}{\partial g}, \tag{4}$$

$$H(\tau) := H(g(\tau), \psi(\tau), \overline{u}(\tau)) = \psi(\tau)(\overline{u}(\tau)) = \max_{u \in U} \psi(\tau)(u) \tag{5}$$

for almost all $\tau \in [0, T]$.

Definition 2. *Later on, an extremal for the problem from Theorem* 2 is called a parameterized curve $g = g(t)$, $t \in \mathbb{R}$, satisfying PMP for the time-optimal problem.

Remark 1. *For every extremal, $H(t) = \mathrm{const} := M_0 \geq 0$, $t \in \mathbb{R}$, [1], [13].*

Definition 3. *An extremal is called normal (abnormal), if $M_0 > 0$ ($M_0 = 0$). Every normal extremal is parameterized by arc length; proportionally changing $\psi = \psi(t)$, $t \in \mathbb{R}$, if it is necessary, one can assume that $M_0 = 1$. Every normal extremal for a left-invariant (sub-)Riemannian metric on a Lie group is a geodesic, i.e. a locally shortest curve [23].*

Theorem 3. *[8] The Hamiltonian system for the function H on the Lie group $G = \mathrm{GL}(n)$ with the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n)$ has a form*

$$g' = g \cdot u, \quad g \in G, \quad u \in \mathfrak{g}, \tag{6}$$

$$\psi(v)' = \psi([u, v]), \quad g \in G, \quad u, v \in \mathfrak{g}. \tag{7}$$

Proof. Each element $g \in G = \mathrm{GL}(n) \subset \mathbb{R}^{n^2}$ is defined by its standard matrix coordinates g_{ij}, $i, j = 1, \ldots, n$, and ψ is defined by its components $\psi_{ij} = \psi(e_{ij})$, $i, j = 1, \ldots, n$, where $e_{ij} \in \mathfrak{g}$ is a matrix having 1 in the ith row and the jth column and 0 in all other places.

In consequence of (3),

$$H(g, \psi, u) = \sum_{i,j=1}^{n} \psi_{ij} \left(\sum_{l=1}^{n} g_{il} u_{lj} \right) = \sum_{l,j=1}^{n} (g^T \psi)_{lj} u_{lj}. \tag{8}$$

The variables g_{ij}, ψ_{ij} must satisfy the Hamiltonian system of equations

$$g'_{ij} = \frac{\partial H}{\partial \psi_{ij}}(g, \psi, u) = \sum_{l=1}^{n} g_{il} u_{lj} = (gu)_{ij}, \tag{9}$$

$$\psi'_{ij} = -\frac{\partial H}{\partial g_{ij}} = -\sum_{m=1}^{n} \psi_{im} u_{jm} = -(\psi u^T)_{ij}. \tag{10}$$

The formula (9) is a special case of the formula (6). It is clear that

$$\psi(v) = \psi(gv) = \sum_{i,j=1}^{n} \psi_{ij} \left(\sum_{l=1}^{n} g_{il} v_{lj} \right).$$

On the ground of formulae (9) and (10) we get from here that

$$(\psi(v))' = \sum_{i,j=1}^{n} \psi'_{ij} \left(\sum_{l=1}^{n} g_{il} v_{lj} \right) + \sum_{i,j=1}^{n} \psi_{ij} \left(\sum_{l=1}^{n} g'_{il} v_{lj} \right) =$$

$$-\sum_{i,j=1}^{n} \left(\sum_{m=1}^{n} \psi_{im} u_{jm} \sum_{l=1}^{n} g_{il} v_{lj} \right) + \sum_{i,j=1}^{n} \psi_{ij} \left(\sum_{l,m=1}^{n} g_{im} u_{ml} v_{lj} \right) =$$

$$-\sum_{i,j=1}^{n} \psi_{ij} \left(\sum_{l=1}^{n} g_{il} (vu)_{lj} \right) + \sum_{i,j=1}^{n} \psi_{ij} \left(\sum_{l=1}^{n} g_{il} (uv)_{lj} \right) =$$

$$\sum_{i,j=1}^{n} \psi_{ij} (g[u,v])_{ij} = \psi([u,v]),$$

which proves the formula (7). □

THEOREM 4. *[8] The Hamiltonian system for the function H on a Lie group G with Lie algebra $\mathfrak{g}$ has a form*

$$\dot{g} = dl_g(u), \quad g \in G, \quad u \in \mathfrak{g}, \tag{11}$$

$$\psi(v)' = \psi([u,v]), \quad g \in G, \quad u, v \in \mathfrak{g}. \tag{12}$$

PROOF. In consequence of Theorem 3, Theorem 4 holds for every matrix Lie group and for every Lie group $(G, \cdot)$, because it is known that $(G, \cdot)$ is locally isomorphic to some connected Lie subgroup (may be, virtual) of the Lie group $\mathrm{GL}(n) \subset \mathbb{R}^{n^2}$. □

It follows from Theorem 4, especially from (12), and Remark 1 that

THEOREM 5. *If* $\dim G = 3$, $\dim \mathfrak{p} \geq 2$ *in Theorem* 2 then every extremal of the problem from Theorem 2 is normal.

The following lemma holds.

LEMMA 1. *[16] Let $g = g(t)$, $t \in (a,b)$, be a smooth path in the Lie group G. Then*

$$(g(t)^{-1})' = -g(t)^{-1} g'(t) g(t)^{-1}. \tag{13}$$

PROOF. Differentiating the identity $g(t)g(t)^{-1} = e$ by t, we get

$$0 = (g(t)g(t)^{-1})' = g'(t)g(t)^{-1} + g(t)(g(t)^{-1})',$$

whence the equality (13) follows immediately. □

Theorem 6. *[16] Let* $\psi \in \mathfrak{g}^* = T_e^*G$ *be a covector,*

$$\mathrm{Ad}^*\psi(g) := (\mathrm{Ad}\, g)^*(\psi) = \psi \circ Ad(g), \quad g \in G,$$

an action of the coadjoint representation of the Lie group G *on* ψ*. Then*

$$(d(\mathrm{Ad}^*\psi)(w))(v) = ((\mathrm{Ad}\, g_0)^*(\psi))([u,v]),$$

if

$$u, v \in \mathfrak{g}, \quad w = dl_{g_0}(u) \in T_{g_0}G, \quad g_0 \in G.$$

Proof. In the case of a matrix Lie group G,

$$\mathrm{Ad}(g)(v) = gvg^{-1}, \quad dl_g(u) = gu, \quad u, v \in \mathfrak{g}, \quad g \in G.$$

We choose a smooth path $g = g(t)$, $t \in (-\varepsilon, \varepsilon)$, in the Lie group G such that $g(0) = g_0$, $g'(0) = w$. Then by Lemma 1,

$$(d(\mathrm{Ad}^*\psi)(w))(v) = (\psi(g(t)vg(t)^{-1}))'(0) = \psi((g(t)vg(t)^{-1})'(0)) =$$

$$\psi(g'(0)vg_0^{-1} + g_0v(g(t)^{-1})'(0)) = \psi(g_0uvg_0^{-1} - g_0v(g_0^{-1}g'(0)g_0^{-1})) =$$

$$\psi(g_0uvg_0^{-1} - g_0v(g_0^{-1}g_0ug_0^{-1})) = \psi(g_0uvg_0^{-1} - g_0vug_0^{-1}) =$$

$$\psi(g_0[u,v]g_0^{-1}) = ((\mathrm{Ad}\, g_0)^*(\psi))([u,v]),$$

as required. □

It follows from Theorems 4 and 6 that

Theorem 7. *1. Any normal extremal* $g = g(t) : \mathbb{R} \to G$ *(parameterized by arc length and with origin* $e \in G$*), of left-invariant (sub-)Finsler metric* d *on a Lie group* G*, defined by a norm* F *on the subspace* $\mathfrak{p} \subset \mathfrak{g}$ *with closed unit ball* U*, is a Lipschitz integral curve of the following vector field*

$$v(g) = dl_g(u(g)), \quad u(g) = \psi_0(\mathrm{Ad}(g)(w(g)))w(g), \quad w(g) \in U,$$

$$\psi_0(\mathrm{Ad}(g)(w(g))) = \max_{w \in U} \psi_0(\mathrm{Ad}(g)(w)),$$

where $\psi_0 \in \mathfrak{g}^*$ *is some fixed covector with* $\max_{v \in U} \psi_0(v) = 1$*.*

2. (Conservation law) In addition, $\psi(t)(g(t)^{-1}g'(t)) \equiv 1$ *for all* $t \in \mathbb{R}$*, where* $\psi(t) := (\mathrm{Ad}\, g(t))^*(\psi_0)$*.*

Remark 2. *Every extremal with origin* g_0 *is obtained by the left shift* l_{g_0} *from some extremal with origin* e*.*

Remark 3. *In (sub-)Riemannian case, the vector* $u(g)$ *is characterized by condition* $\langle u(g), v\rangle = = \psi_0(Ad(g)(v))$ *for all* $v \in \mathfrak{p}$*. In Riemannian case, every extremal is a normal geodesic, and we can assume that* ψ_0 *is an unit vector in* $(\mathfrak{p} = \mathfrak{g}, \langle\cdot,\cdot\rangle)$*, setting* $\psi_0(v) = \langle\psi_0, v\rangle$*,* $v \in \mathfrak{g}$*. Moreover,* $\dot{g}(0) = \psi_0$*.*

Theorem 8. *If* $v(g_0) \neq 0$*,* $g_0 \in G$*, then an integral curve of the vector field* $v(g), g \in G$*, with origin* g_0 *is a normal extremal parameterized proportionally to arc length with the proportionality factor* $|dl_{g_0^{-1}}(v(g_0))|$*.*

Proof. Let $g(t)$, $t \in \mathbb{R}$, be an integral curve under consideration and set $\gamma = \gamma(t) = g_0^{-1}g(t)$, $t \in \mathbb{R}$. Then γ is an integral curve of vector field $dl_{g_0^{-1}}(v(g))$, $g \in G$, with origin e. Hence

$$\dot{\gamma}(t) = dl_{g_0^{-1}}(\dot{g}(t)) = dl_{g_0^{-1}}(dl_{g(t)}(u(g(t)))) = dl_{\gamma(t)}(u(g(t))). \tag{14}$$

In addition,

$$\mathrm{Ad}(g(t))^* = \mathrm{Ad}(g_0 \cdot \gamma(t))^* = \mathrm{Ad}(\gamma(t))^* \circ \mathrm{Ad}(g_0)^*. \tag{15}$$

By definition,

$$u(g(t)) = \mathrm{Ad}(g(t))^*(\psi_0)(w(g(t)))w(g(t)),$$
$$\mathrm{Ad}(g(t))^*(\psi_0)(w(g(t))) = \max_{w\in U} \mathrm{Ad}(g(t))^*(\psi_0)(w),$$

that by (15) can be rewriten as

$$u(g(t)) = \mathrm{Ad}(\gamma(t))^*(\psi_0')(w(g(t))),$$
$$\mathrm{Ad}(\gamma(t))^*(\psi_0')(w(g(t))) = \max_{w\in U} \mathrm{Ad}(\gamma(t))^*(\psi_0')(w),$$

where $\psi_0' = \mathrm{Ad}(g_0)^*(\psi_0)$. As a result of this and (14), we see that $u(g(t))$ plays a role of $u(\gamma(t))$ for constant covector ψ_0' (instead of ψ_0). Due to point 2 of Theorem 7 the curve $\gamma(t)$ is a normal extremal parameterized proportionally to arc length with the proportionality factor $|dl_{g_0^{-1}}(v(g_0))|$. Then its left shift $g(t) = g_0 \cdot \gamma(t)$ also has this property. □

Remark 4. *Theorem* 8 holds for left-invariant Riemannian metrics on (connected) Lie groups. In this case, $v(g_0) \neq 0$ for all $g_0 \in G$.

Let us choose a basis $\{e_1, \dots, e_n\}$ in $\mathfrak{g}$, assuming that $\{e_1, \dots, e_r\}$ is an orthonormal basis for the scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$ in case of left-invariant (sub-)Riemannian metric. Define a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, considering $\{e_1, \dots, e_n\}$ as its orthonormal basis. Then each covector $\psi \in \mathfrak{g}^*$ can be considered as a vector in $\mathfrak{g}$, setting $\psi(v) = \langle\psi, v\rangle$ for every $v \in \mathfrak{g}$. If $\psi = \sum_{i=1}^n \psi_i e_i$, $v = \sum_{k=1}^n v_k e_k$, then $\psi(v) = \psi \cdot v$, where ψ and v are corresponding vector-row and vector-column, $\cdot$ is the matrix multiplication. If $l : \mathfrak{g} \to \mathfrak{g}$ is a linear map, then we denote by (l) its matrix in the basis $\{e_1, \dots, e_n\}$.

If $g(t)$, $t \in \mathbb{R}$, is a normal geodesic of a left-invariant (sub-)Riemannian metric d on a Lie group G, then $u(g(t))$ is the orthogonal projection onto $\mathfrak{p}$ of the vector $(\mathrm{Ad}\, g(t))^*(\psi_0)$ in the notation of Theorem 7 for the scalar product $\langle\cdot,\cdot\rangle$ introduced above on $\mathfrak{g}$. This fact and formula (12) imply

Theorem 9. *Every normal parameterized by arc length geodesic of left-invariant (sub-) Riemannian metric on a Lie group G issued from the unit is a solution of the following system of differential equations*

$$\dot{g}(t) = dl_{g(t)}(u(t)),\ u(t) = \sum_{i=1}^{r} \psi_i(t)e_i,\ |u(0)| = 1,\ \dot{\psi}_j(t) = \sum_{k=1}^{n}\sum_{i=1}^{r} c_{ij}^k \psi_i(t)\psi_k(t), \tag{16}$$

where $j = 1, \dots, n$, c_{ij}^k are structure constants of Lie algebra $\mathfrak{g}$ in its basis $\{e_1, ..., e_n\}$. In Riemannian case, $r = n$.

Corollary 1.

$$|\dot{g}(t)| = |u(t)| \equiv 1, \quad t \in \mathbb{R}. \tag{17}$$

Proof. The first equality in (17) is a consequence of the first equality in (16) and left invariance of the scalar product. Therefore, due to the equality $|u(0)| = 1$, it suffices to prove that $\frac{d}{dt}\langle u(t), u(t)\rangle = 0$. Now by (16),

$$\frac{d}{dt}\langle u(t), u(t)\rangle = \left(\sum_{j=1}^{r} \psi_j^2(t)\right)' = 2\sum_{j=1}^{r} \psi_j(t)\psi_j'(t) = \sum_{k=1}^{n}\sum_{i,j=1}^{r} c_{ij}^k \psi_i(t)\psi_j(t)\psi_k(t),$$

which is zero by the skew symmetry of c_{ij}^k with respect to subscripts. □

Remark 5. *In fact, the same equations for $\dot{\psi}_j(t)$ from (16) in a different interpretation were obtained in [21] as "normal equations". Their derivation there uses more complicated concepts and techniques.*

4. Lie groups all of whose left-invariant Riemannian metrics have constant negative curvature

The only Lie groups which do not admit left-invariant sub-Finsler metrics are commutative Lie groups and Lie groups G_n, $n \geq 2$, consisting of parallel translations and homotheties (without rotations) of Euclidean space E^{n-1} [5], [17]. Up to isomorphisms, Lie groups G_n can be described as connected Lie groups every whose left-invariant Riemannian metric has constant negative sectional curvature [24].

The group G_n, $n \geq 2$, is isomorphic to the group of real block matrices

$$g = (y, x) := \begin{pmatrix} xE_{n-1} & y^T \\ 0 & 1 \end{pmatrix}, \tag{18}$$

where E_{n-1} is unit matrix of order $n-1$, y^T is a transposed $(n-1)-$vector–row y, 0 is a zero $(n-1)-$vector–row, $x > 0$.

It is clear that in vector notation the group operations have a form

$$(y_1, x_1) \cdot (y_2, x_2) = x_1(y_2, x_2) + (y_1, 0), \quad (y, x)^{-1} = x^{-1}(-y, 1). \tag{19}$$

Let E_{ij}, $i, j = 1, \dots, n$, be a $(n \times n)$-matrix having 1 in the ith row and the jth column and 0 in all other places. Matrices

$$e_i = E_{in},\ i = 1, \dots, n-1, \quad e_n = \sum_{k=1}^{n-1} E_{kk} \tag{20}$$

constitute a basis of Lie algebra $\mathfrak{g}_n$ of the Lie group G_n. In addition,

$$[e_i, e_j] = 0,\ i, j = 1, \dots, n-1; \quad [e_n, e_i] = e_i,\ i = 1, \dots, n-1,$$

so all nonzero structure constants in the basis $\{e_1, \dots, e_n\}$ are equal to

$$c^i_{ni} = -c^i_{in} = 1, \quad i = 1, \dots, n-1. \tag{21}$$

Let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathfrak{g}_n$ with the orthonormal basis $e_1, \dots, e_n$. Then we get left-invariant Riemannian metric d on the Lie group G_n of constant sectional curvature -1 [24].

On the ground of Theorem 9 and (21), $\psi_i = \psi_i(t)$, $i = 1, \dots, n$, are solutions of the Cauchy problem

$$\begin{cases} \dot{\psi}_i(t) = \psi_i(t)\psi_n(t),\ i = 1, \dots, n-1, \quad \dot{\psi}_n(t) = -\sum\limits_{i=1}^{n-1} \psi_i^2(t); \\ \psi_i(0) = \varphi_i,\ i = 1, \dots, n, \quad \sum\limits_{i=1}^{n} \varphi_i^2 = 1. \end{cases} \tag{22}$$

It follows from (22) that

$$\ddot{\psi}_n(t) = -2\psi_n(t) \sum_{i=1}^{n-1} \psi_i^2(t) = 2\psi_n(t)\dot{\psi}_n(t) = \left(\psi_n^2\right)^{\cdot}(t),$$

whence on the ground of initial data of the Cauchy problem (22), it follows that

$$\dot{\psi}_n(t) = \psi_n^2(t) - 1, \quad \psi_n(0) = \varphi_n.$$

Solving this Cauchy problem, we find that

$$\psi_n(t) = \frac{\varphi_n \cosh t - \sinh t}{\cosh t - \varphi_n \sinh t}.$$

Then on the base of (22), for $i = 1, \ldots, n-1$,

$$\ln|\psi_i(t)| = \int_0^t \frac{\varphi_n \cosh\tau - \sinh\tau}{\cosh\tau - \varphi_n \sinh\tau} d\tau + \ln|\varphi_i| = -\ln|\cosh t - \varphi_n \sinh t| + \ln|\varphi_i|,$$

if $\varphi_i \neq 0$, so

$$\psi_i(t) = \frac{\varphi_i}{\cosh t - \varphi_n \sinh t}, \quad i = 1, \ldots, n-1,$$

and these formulae are true also when $\varphi_i = 0$.

Consequently, on the ground of (16),

$$u(t) = \frac{1}{\cosh t - \varphi_n \sinh t}\left(\sum_{i=1}^{n-1} \varphi_i e_i + (\varphi_n \cosh t - \sinh t)\, e_n\right). \tag{23}$$

If $g \in G_n$ is defined by formula (18), $u = \sum_{i=1}^{n} u_i e_i \in \mathfrak{g}_n$, then

$$gu = \begin{pmatrix} (xu_n)E_{n-1} & v \\ 0 & 0 \end{pmatrix}, \quad v = (xu_1, \ldots, xu_{n-1})^T. \tag{24}$$

Therefore on the base of Theorem 9 and (23) in the notation (18), the corresponding parameterized by arc length normal geodesic $g = g(t)$, $t \in \mathbb{R}$, of the space (G_n, d) with $g(0) = e$ is a solution of the Cauchy problem

$$\begin{cases} \dot{x}(t) = \frac{\varphi_n \cosh t - \sinh t}{\cosh t - \varphi_n \sinh t} x(t), \ \dot{y}_i(t) = \frac{\varphi_i}{\cosh t - \varphi_n \sinh t} x(t), \quad i = 1, \ldots, n-1, \\ x(0) = 1, \quad y_i(0) = 0, \ i = 1, \ldots, n-1. \end{cases} \tag{25}$$

Solving the problem, we find

$$x(t) = \frac{1}{\cosh t - \varphi_n \sinh t}, \quad y_i(t) = \int_0^t \frac{\varphi_i dt}{(\cosh t - \varphi_n \sinh t)^2} = \frac{\varphi_i \sinh t}{\cosh t - \varphi_n \sinh t}. \tag{26}$$

This implies that

$$x(t) = e^{\pm t}, \quad y_i(t) \equiv 0, \quad i = 1, \ldots, n-1, \quad \text{if} \quad \varphi_n = \pm 1. \tag{27}$$

Let $\varphi_n^2 < 1$. Let us show that for any $t \in \mathbb{R}$, the equality

$$\sum_{i=1}^{n-1} (y_i(t) - a_i)^2 + x^2(t) = \sum_{i=1}^{n-1} a_i^2 + 1 \tag{28}$$

holds, where a_i, $i = 1, \dots, n-1$, are real numbers such that

$$\sum_{i=1}^{n-1} a_i \varphi_i = \varphi_n. \tag{29}$$

We introduce a function $f(t) = \sum\limits_{i=1}^{n-1} (y_i(t) - a_i)^2 + x^2(t)$. Due to initial data (25), $f(0) = \sum\limits_{i=1}^{n-1} a_i^2 + 1$. On the ground of (25), (26) and last equation in (22), we get

$$\frac{1}{2} f'(t) = \sum_{i=1}^{n-1} (y_i(t) - a_i)\dot{y}_i(t) + x(t)\dot{x}(t) = \sum_{i=1}^{n-1} \left(\frac{\varphi_i \sinh t}{\cosh t - \varphi_n \sinh t} - a_i \right) \varphi_i +$$

$$\frac{\varphi_n \cosh t - \sinh t}{\cosh t - \varphi_n \sinh t} = \frac{\sinh t \left(\sum\limits_{i=1}^{n-1} \varphi_i^2 - 1 \right) + \varphi_n \cosh t}{\cosh t - \varphi_n \sinh t} - \sum_{i=1}^{n-1} a_i \varphi_i =$$

$$\varphi_n - \sum_{i=1}^{n-1} a_i \varphi_i = 0.$$

Consequently, $f(t) \equiv f(0)$ and the equality (28) is proved.

It is easy to check that the equality (29) holds for

$$a_i = \varphi_i \varphi_n / (1 - \varphi_n^2), \quad i = 1, \dots, n-1; \quad \text{moreover} \quad \sum_{i=1}^{n-1} a_i^2 + 1 = \frac{1}{1 - \varphi_n^2}. \tag{30}$$

These numbers a_i are obtained as halves of sums of limits $y_i(t)$ when $t \to +\infty$ and $t \to -\infty$, which are equal to $\varphi_i/(1 - \varphi_n)$ and $-\varphi_i/(1 + \varphi_n)$ respectively.

Formulae (19) show that the group G_n is a simply transitive isometry group of the famous Poincare's model of the Lobachevsky space L^n in the half space $\mathbb{R}^n_+$ with metric $ds^2 = (\sum_{k=1}^{n-1} dy_k^2 + dx^2)/x^2$.

The above results, including formulae (26), (27), (30), show that geodesics of the space L^n in this model, passing through the point $(0, \dots, 0, 1)$, are semi-straights or semi-circles (with centers $(a_1, \dots, a_{n-1}, 0)$ and radii $1/\sqrt{1 - \varphi_n^2}$, (30)), orthogonal to the hyperplane $\mathbb{R}^{n-1} \times \{0\}$. Since all other geodesics are obtained by left shifts on the group, in other words, by indicated parallel translations and homotheties of this model, then also all straights and semi-circles, orthogonal to the hyperplane $\mathbb{R}^{n-1} \times \{0\}$, are geodesics of the space L^n.

We got a well-known description of geodesics in this Poincare's model.

Now let us look what the vector field method gives us for the problem.

Every vector $\psi \in \mathfrak{g}_n$ can be considered as a covector $\mathfrak{g}^*$, setting $\psi(v) = \langle \psi, v \rangle$ for $v \in \mathfrak{g}_n$. Then any (co)vector ψ_0 from Theorem 7 has a form

$$\psi_0 = \sum_{i=1}^{n} \varphi_i e_i, \quad \sum_{i=1}^{n} \varphi_i^2 = 1.$$

Let $w = \sum\limits_{i=1}^{n} w_i e_i \in \mathfrak{g}_n$, $g \in G_n$ is defined by formula (18). It is easy to see that

$$\mathrm{Ad}(g)(w) = gwg^{-1} = \sum_{i=1}^{n-1} (w_i x - w_n y_i) e_i + w_n e_n,$$

$$\langle\psi_0, \mathrm{Ad}(g)(w)\rangle = \sum_{i=1}^{n-1}(w_i x - w_n y_i)\varphi_i + w_n\varphi_n = x\sum_{i=1}^{n-1}\varphi_i w_i + \left(\varphi_n - \sum_{i=1}^{n-1}\varphi_i y_i\right) w_n.$$

It is clear that

$$u(g) = x\sum_{i=1}^{n-1}\varphi_i e_i + \left(\varphi_n - \sum_{i=1}^{n-1}\varphi_i y_i\right) e_n,$$

$$v(g) = gu(g) = x\sum_{i=1}^{n} u_i e_i = x^2\sum_{i=1}^{n-1}\varphi_i e_i + x\left(\varphi_n - \sum_{i=1}^{n-1}\varphi_i y_i\right) e_n.$$

Thus geodesic $g = g(t)$, $t \in \mathbb{R}$, with $g(0) = e$ is a solution of the Cauchy problem

$$\begin{cases} \dot{x}(t) = \left(\varphi_n - \sum\limits_{i=1}^{n-1}\varphi_i y_i(t)\right) x(t), \quad \dot{y}_i(t) = \varphi_i x^2(t),\ i = 1,\dots,n-1, \\ x(0) = 1, \quad y_i(0) = 0,\ i = 1,\dots,n-1. \end{cases} \tag{31}$$

Dividing the first equation in (31) by $x(t)$, we get on the left hand side the derivative of the function $\ln x(t) := z(t)$. Differentiating both sides of the resulting equation and using the second equation in (31) and the equality $\sum\limits_{i=1}^{n}\varphi_i^2 = 1$, we get

$$\ddot{z}(t) = -\sum_{i=1}^{n-1}\varphi_i^2 x^2(t) = -(1-\varphi_n^2)e^{2z(t)}, \quad z(0) = 0,\ \dot{z}(0) = \varphi_n.$$

If $\varphi_n = \pm 1$ then $\ddot{z}(t) \equiv 0$ and due to the initial data and the second equation in (31), we get $z(t) = \pm t$, $x(t) = e^{\pm t}$, $y_i(t) \equiv 0$, $i = 1,\dots,n-1$.

Let $0 \le \varphi_n^2 < 1$. Let us multiply both sides of the resulting equation by $2\dot{z}$. Then

$$2\dot{z}\ddot{z} = -(1-\varphi_n^2)e^{2z}2\dot{z}, \quad d(\dot{z})^2 = -(1-\varphi_n^2)e^{2z}d(2z), \quad \dot{z}^2 = -(1-\varphi_n^2)e^{2z} + C.$$

Taking into account the initial conditions for $z(t)$, we get $C = 1$ and $\dot{z}(t)^2 = 1-(1-\varphi_n^2)e^{2z(t)}$. The expression on the right is positive for t sufficiently close to zero. Therefore, with these t, we get

$$\dot{z}(t) = \pm\sqrt{1-(1-\varphi_n^2)e^{2z(t)}},$$

where the sign coincides with the sign of φ_n, if $\varphi_n \neq 0$. Separating variables, we get

$$dt = \frac{\pm dz}{\sqrt{1-(1-\varphi_n^2)e^{2z}}} = \frac{\pm dz}{e^z\sqrt{1-\varphi_n^2}\sqrt{(e^{-2z}/(1-\varphi_n^2))-1}} =$$

$$\frac{\mp d(e^{-z}/\sqrt{1-\varphi_n^2})}{\sqrt{(e^{-2z}/(1-\varphi_n^2))-1}} = \mp d\left(\cosh^{-1}\left(\frac{e^{-z}}{\sqrt{1-\varphi_n^2}}\right)\right),$$

$$\pm\cosh^{-1}\left(\frac{e^{-z}}{\sqrt{1-\varphi_n^2}}\right) = c - t, \quad c = \cosh^{-1}\left(\frac{1}{\sqrt{1-\varphi_n^2}}\right).$$

The applying cosh to the left and right sides of the resulting equality gives

$$\frac{e^{-z(t)}}{\sqrt{1-\varphi_n^2}} = \cosh c\cosh t - \sinh c\sinh t = \frac{\cosh t - \varphi_n\sinh t}{\sqrt{1-\varphi_n^2}}.$$

Consequently, when t is sufficiently close to zero,

$$x(t) = e^{z(t)} = \frac{1}{\cosh t - \varphi_n\sinh t}.$$

Since the right sides of the system of differential equations (31) are real analytic, this equality is true for all $t \in \mathbb{R}$. We obtain from this and the second system in (31) the same solutions $y_i(t)$, $t \in \mathbb{R}$, $i = 1, \dots, n-1$, as in (26).

Using formulae (19) and (26) for $x = x(t)$, $y_i = y_i(t)$, we shall find a formula for distances d between group elements, or, which is the same, between points of the Lobachevsky space in Poincare's model under consideration. We obtain from (26)

$$\frac{1}{x} = \cosh t - \varphi_n \sinh t, \quad x = \frac{\cosh t + \varphi_n \sinh t}{\cosh^2 t - \varphi_n^2 \sinh^2 t} = \frac{\cosh t + \varphi_n \sinh t}{1 + (1-\varphi_n^2)\sinh^2 t},$$

$$\sum_{i=1}^{n-1} (y_i/x)^2 = \sinh^2 t \sum_{i=1}^{n-1} \varphi_i^2 = (1-\varphi_n^2)\sinh^2 t,$$

$$\cosh t + \varphi_n \sinh t = \frac{x}{x^2}\left(x^2 + \sum_{i=1}^{n-1} y_i^2\right) = \frac{1}{x}\left(x^2 + \sum_{i=1}^{n-1} y_i^2\right),$$

$$\cosh t = \frac{1}{2x}\left(1 + x^2 + \sum_{i=1}^{n-1} y_i^2\right),$$

$$d((0,1),(y,x)) = \cosh^{-1}\left[\frac{1}{2x}\left(1 + x^2 + \sum_{i=1}^{n-1} y_i^2\right)\right].$$

Now by (19), the last formula, and left-invariance of metric d,

$$(y_1, x_1)^{-1}(y_2, x_2) = x_1^{-1}(-y_1, 1)(y_2, x_2) = (x_1^{-1}(y_2 - y_1), x_1^{-1}x_2),$$

$$d((y_1,x_1),(y_2,x_2)) = d((0,1),(x_1^{-1}(y_2-y_1), x_1^{-1}x_2)) =$$

$$\cosh^{-1}\left[\frac{x_1}{2x_2}\left(1 + \frac{x_2^2}{x_1^2} + \frac{1}{x_1^2}\sum_{i=1}^{n-1}(y_{2,i} - y_{1,i})^2\right)\right] =$$

$$\cosh^{-1}\left[\frac{1}{2x_1x_2}\left(x_1^2 + x_2^2 + \sum_{i=1}^{n-1}(y_{2,i} - y_{1,i})^2\right)\right] = d((y_1,x_1),(y_2,x_2)). \tag{32}$$

5. The three–dimensional Heisenberg group

This Heisenberg group is a nilpotent Lie group of upper–triangular matrices

$$H = \left\{ h = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}, \ x, y, z \in \mathbb{R}. \tag{33}$$

It is easy to compute that

$$h^{-1} = \begin{pmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}. \tag{34}$$

Clearly, H is naturally diffeomorphic to $\mathbb{R}^3$ and H is a connected Lie group with respect to this differential structure. Matrices

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{35}$$

constitute a basis of Lie algebra $\mathfrak{h}$ of Heisenberg group H. In addition,

$$[e_1, e_2] = e_1e_2 - e_2e_1 = e_3.$$

Hence the vector subspace $\mathfrak{p} \subset \mathfrak{h}$ with basis $\{e_1, e_2\}$ generates $\mathfrak{h}$.

Thus the triple $(H, \mathfrak{h}, \mathfrak{p})$ satisfies all conditions of Theorems 1 and 2.

Let us search for all geodesics of the problem from Theorem 2. They are all normal by Theorem 5, and we can use Theorem 7.

Let us define a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}$ with orthonormal basis $\{e_1, e_2, e_3\}$. Then each vector $\psi \in \mathfrak{h}$ can be considered as a covector from $\mathfrak{h}^*$, if we set $\psi(v) = \langle \psi, v \rangle$ for $v \in \mathfrak{h}$. Then any (co)vector ψ_0 from Theorem 7 has a form

$$\psi_0 = \cos\xi e_1 + \sin\xi e_2 + \beta e_3, \quad \xi, \beta \in \mathbb{R}. \tag{36}$$

Let

$$v = \sum_{k=1}^{2} v_k e_k = \begin{pmatrix} 0 & v_1 & 0 \\ 0 & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad v \in \mathfrak{p},\ v_k \in \mathbb{R},\ k = 1, 2.$$

Using formulae (33), (34), we get

$$Ad(h)(v) = hvh^{-1} = \begin{pmatrix} 0 & v_1 & -yv_1 + xv_2 \\ 0 & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\langle \psi_0, \mathrm{Ad}(h)(v) \rangle = \cos\xi v_1 + \sin\xi v_2 + \beta(-yv_1 + xv_2) =$$

$$(\cos\xi - \beta y)v_1 + (\sin\xi + \beta x)v_2.$$

It is clear that

$$u(h) = (\cos\xi - \beta y)e_1 + (\sin\xi + \beta x)e_2$$

and so a geodesic is an integral curve of the vector field

$$v(h) = hu(h) = (\cos\xi - \beta y)e_1 + (\sin\xi + \beta x)e_2 + x(\sin\xi + \beta x)e_3.$$

Therefore $h(t)$ is a solution of the Cauchy problem

$$\begin{cases} \dot{x} = \cos\xi - \beta y, \\ \dot{y} = \sin\xi + \beta x, \\ \dot{z} = x(\sin\xi + \beta x)(= x\dot{y}) \end{cases} \tag{37}$$

with initial data $x(0) = y(0) = z(0) = 0$.

Let us turn to *the coordinate system* $\tilde{x}, \tilde{y}, \tilde{z}$ *of the first kind* on the Lie group H :

$$\exp\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & z + (xy)/2 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $\tilde{x} = x,\ \tilde{y} = y,\ \tilde{z} = z - (xy)/2$.

It is easy to see that for $\beta = 0$ we get

$$x(t) = t\cos\xi,\ y(t) = t\sin\xi,\ z(t) = \frac{t^2}{2}\cos\xi\sin\xi,\ \tilde{z}(t) \equiv 0,\ t \in \mathbb{R},$$

and geodesic is a 1–parameter subgroup

$$g(t) = \exp(t(\cos\xi e_1 + \sin\xi e_2)),\ t \in \mathbb{R}.$$

If $\beta \neq 0$, the calculations are more difficult:

$$\ddot{x} = -\beta\dot{y} = -\beta(\sin\xi + \beta x) = -\beta^2 x - \beta\sin\xi,$$

$$x(t) = C_1\cos\beta t + C_2\sin\beta t - \frac{\sin\xi}{\beta}.$$

Since $x(0) = 0$, $\dot{x}(0) = \cos\xi$, then $C_1 = (\sin\xi)/\beta$, $C_2 = (\cos\xi)/\beta$,

$$x(t) = \frac{1}{\beta}(\sin\xi\cos\beta t + \cos\xi\sin\beta t - \sin\xi) = \frac{1}{\beta}(\sin(\xi + \beta t) - \sin\xi); \tag{38}$$

$$\ddot{y} = \beta\dot{x} = \beta(\cos\xi - \beta y) = -\beta^2 y + \beta\cos\xi,$$

$$y(t) = C_1\cos\beta t + C_2\sin\beta t + \frac{\cos\xi}{\beta}.$$

Since $y(0) = 0$, $\dot{y}(0) = \sin\xi$, then $C_1 = -(\cos\xi)/\beta$, $C_2 = (\sin\xi)/\beta$,

$$y(t) = \frac{1}{\beta}(-\cos\xi\cos\beta t + \sin\xi\sin\beta t + \cos\xi) = \frac{1}{\beta}(-\cos(\xi + \beta t) + \cos\xi), \tag{39}$$

$$\tilde{z}' = \dot{z} - \frac{(xy)'}{2} = x\dot{y} - \frac{1}{2}(\dot{x}y + x\dot{y}) = \frac{1}{2}(x\dot{y} - \dot{x}y) =$$

$$\frac{1}{2\beta}[(\sin(\xi + \beta t) - \sin\xi)\sin(\xi + \beta t) - \cos(\xi + \beta t)(-\cos(\xi + \beta t) + \cos\xi)] =$$

$$\frac{1}{2\beta}[1 - (\sin\xi\sin(\xi + \beta t) + \cos(\xi + \beta t)\cos\xi)] = \frac{1}{2\beta}(1 - \cos\beta t) = \tilde{z}'.$$

Since $\tilde{z}(0) = 0$ then

$$\tilde{z}(t) = \frac{1}{2\beta}\left(t - \frac{\sin\beta t}{\beta}\right), \quad t \in \mathbb{R}. \tag{40}$$

It follows from equalities (38), (39), (40) that the projection of geodesic $g = g(t)$ onto the plane x, y is *a circle with radius* $1/|\beta|$ *and center* $(1/\beta)(-\sin\xi, \cos\xi)$, $T = 2\pi/|\beta|$ *is a circulation period*, while $\tilde{z}(t)$, $t \in \mathbb{R}$, does not depend on the parameter ξ. Therefore, if we fix $\beta \neq 0$ then for different ξ all geodesic segments $g(\beta, \xi, t), 0 \leq t \leq 2\pi/|\beta|$, start at e and finish at the same point. It follows from the existence of the shortest arcs, Theorem 2, PMP and our calculations that if $\beta = 0$ (respectively, $\beta \neq 0$) then every segment (respectively, of the length less or equal to $T = 2\pi/|\beta|$) of these geodesics is a shortest arc. There is no other geodesic or shortest arc except indicated above and their left shifts.

6. Controls for left-invariant sub-Riemannian metrics on $SO(3)$

It is well known that every two–dimensional vector subspace $\mathfrak{p}$ of Lie algebra $(\mathfrak{so}(3), [\cdot,\cdot])$ of the Lie group $SO(3)$ generates $\mathfrak{so}(3)$. Moreover, there exists a basis $\{e_1, e_2\}$ of the space $\mathfrak{p}$ such that $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$ for the vector $e_3 = [e_1, e_2]$. Let $(\cdot,\cdot)$ be a scalar product on $\mathfrak{so}(3)$ with orthonormal basis $\{e_1, e_2, e_3\}$. Then if a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$ defines a left-invariant sub-Riemannian metric d on the Lie group $G = SO(3)$, then there exists a basis $\{v, w\}$ in $\mathfrak{p}$ that is orthonormal relative to $\langle\cdot,\cdot\rangle$, orthogonal relative to $(\cdot,\cdot)$, and such that $(v, v) = a^2 \leq b^2 = (w, w)$, $[v, w] = (ab)e_3$, where $0 < a \leq b$. Let v, w be new vectors e_1, e_2. Then

$$[e_1, e_2] = (ab)e_3,\ [e_3, e_1] = (b/a)e_2,\ [e_2, e_3] = (a/b)e_1,\ 0 < a \leq b. \tag{41}$$

It follows from (41) that all nonzero structure constants are

$$c_{12}^3 = -c_{21}^3 = ab,\ c_{31}^2 = -c_{13}^2 = b/a,\ c_{23}^1 = -c_{32}^1 = a/b.$$

Let $g(t)$, $t \in \mathbb{R}$, be a geodesic of the space $(SO(3), d)$, parameterized by arc length, and $g(0) = e$. On the ground of Theorem 9,

$$g'(t) = g(t)u(t), \quad u(t) = \psi_1(t)e_1 + \psi_2(t)e_2,$$

where

$$\psi_1'(t) = -ab\psi_2(t)\psi_3(t), \quad \psi_2'(t) = ab\psi_1(t)\psi_3(t), \quad \psi_3'(t) = \frac{a^2 - b^2}{ab}\psi_1(t)\psi_2(t). \tag{42}$$

Since $|u(t)| \equiv 1$ then $\psi_1(t) = \cos\xi(t)$, $\psi_2(t) = \sin\xi(t)$ and (42) is written as

$$-\sin\xi(t)\dot{\xi}(t) = -ab\sin\xi(t)\psi_3(t), \quad \cos\xi(t)\dot{\xi}(t) = ab\cos\xi(t)\psi_3(t),$$

$$\psi_3'(t) = \frac{a^2 - b^2}{ab}\cos\xi(t)\sin\xi(t).$$

Then $\psi_3(t) = \frac{1}{ab}\xi'(t)$ and $\xi = \xi(t)$ is a solution of the differential equation

$$\xi''(t) = \frac{a^2 - b^2}{2}\sin 2\xi(t). \tag{43}$$

If $a = b$ then $\xi''(t) = 0$, $\xi'(t) = \text{const} = \beta$. Then geodesics are obtained from geodesics in the case of $a = b = 1$ with the change the parameter s by the parameter $t = s/a$. Geodesics, shortest arcs, the distance d, the cut locus and conjugate sets for geodesics in the case of $a = b = 1$ are found in papers [9] and [10].

The case $0 < a < b$ is reduced to the case $a^2 - b^2 = -1$ by proportional change of the metric d. Then the variable $\omega(t) := 2\xi(t)$ allows us to rewrite the equation as the mathematical pendulum equation

$$\omega''(t) = -\sin\omega(t). \tag{44}$$

In [11], I.Yu. Beschastnyi and Yu.L. Sachkov studied geodesics of left-invariant sub-Riemannian metrics on the Lie group $SO(3)$ and gave estimates for the cut time and the metric diameter. Under replacement $b^2 - a^2$ by a^2 and ξ by ψ, the equation (43) coincides with the equation (2.4) from their paper, obtained by another method.

7. To search for geodesics of a sub-Riemannian metric on $SH(2)$

The Lie group $SH(2)$ consists of all matrices of a form

$$g = \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}; \quad A = \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}, \quad v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2. \tag{45}$$

It is not difficult to see that

$$g^{-1} = \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}v \\ 0 & 1 \end{pmatrix}. \tag{46}$$

Clearly, matrices

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{47}$$

constitute a basis of Lie algebra $\mathfrak{sh}(2)$. In addition,

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = e_2. \tag{48}$$

Let us define a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{sh}(2)$ with orthonormal basis $\{e_1, e_2, e_3\}$ and the subspace $\mathfrak{p}$ with orthonormal basis $\{e_1, e_2\}$ generating Lie algebra $\mathfrak{sh}(2)$. Thus a left-invariant sub-Riemannian metric d is defined on the Lie group $SH(2)$.

Let us take a (co)vector $\psi_0 = \cos\alpha e_1 + \sin\alpha e_2 + \beta e_3 \in \mathfrak{sh}(2)$. We calculate

$$\psi_g(w) = \langle\psi_g, w\rangle = \langle\psi_0, gwg^{-1}\rangle \quad g \in SH(2),\ w = w_1e_1 + w_2e_2 \in \mathfrak{p}.$$

$$gwg^{-1} = \begin{pmatrix} \cosh\varphi & \sinh\varphi & x \\ \sinh\varphi & \cosh\varphi & y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & w_1 & w_2 \\ w_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cosh\varphi & -\sinh\varphi & -x\cosh\varphi + y\sinh\varphi \\ -\sinh\varphi & \cosh\varphi & x\sinh\varphi - y\cosh\varphi \\ 0 & 0 & 1 \end{pmatrix}$$

$$= w_1e_1 + (-w_1y + w_2\cosh\varphi)e_2 + (-w_1x + w_2\sinh\varphi)e_3,$$

$$\psi_g(v) = w_1\cos\alpha + (-w_1y + w_2\cosh\varphi)\sin\alpha + (-w_1x + w_2\sinh\varphi)\beta =$$

$$w_1(\cos\alpha - y\sin\alpha - \beta x) + w_2(\cosh\varphi\sin\alpha + \beta\sinh\varphi).$$

Therefore,

$$u(g) = (\cos\alpha - y\sin\alpha - \beta x)e_1 + (\sin\alpha\cosh\varphi + \beta\sinh\varphi)e_2,\ v(g) = gu(g) =$$

$$\begin{pmatrix} \cosh\varphi & \sinh\varphi & x \\ \sinh\varphi & \cosh\varphi & y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \cos\alpha - y\sin\alpha - \beta x & \sin\alpha\cosh\varphi + \beta\sinh\varphi \\ \cos\alpha - y\sin\alpha - \beta x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} \sinh\varphi(\cos\alpha - y\sin\alpha - \beta x) & \cosh\varphi(\cos\alpha - y\sin\alpha - \beta x) & \cosh\varphi(\sin\alpha\cosh\varphi + \beta\sinh\varphi) \\ \cosh\varphi(\cos\alpha - y\sin\alpha - \beta x) & \sinh\varphi(\cos\alpha - y\sin\alpha - \beta x) & \sinh\varphi(\sin\alpha\cosh\varphi + \beta\sinh\varphi) \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence integral curves of vector field $v(g)$, $g \in SH(2)$, satisfy the system of differential equations

$$\begin{cases} \dot\varphi = \cos\alpha - y\sin\alpha - \beta x, \\ \dot x = \cosh\varphi(\sin\alpha\cosh\varphi + \beta\sinh\varphi), \\ \dot y = \sinh\varphi(\sin\alpha\cosh\varphi + \beta\sinh\varphi). \end{cases} \tag{49}$$

The geodesic $g(t)$, $t \in \mathbb{R}$, with $g(0) = e$ is a solution of this system with initial data $\varphi(0) = x(0) = y(0) = 0$. In this case, $|u(g(t))| \equiv 1$, i.e.

$$g(t) \in M_1 = \{(\sin\alpha\cosh\varphi + \beta\sinh\varphi)^2 + (\cos\alpha - y\sin\alpha - \beta x)^2 = 1\} \subset SH(2). \tag{50}$$

Therefore there exists a differentiable function $\gamma = \gamma(t)$ such that

$$\cos\frac{\gamma}{2} = \sin\alpha\cosh\varphi + \beta\sinh\varphi, \quad \sin\frac{\gamma}{2} = \cos\alpha - y\sin\alpha - \beta x. \tag{51}$$

Since $\varphi(0) = x(0) = y(0) = 0$, then we can assume that $\gamma(0) = \pi - 2\alpha$.

On the ground of (51) the sistem (49) is written in the form

$$\begin{cases} \dot\varphi = \sin\frac{\gamma}{2}, \\ \dot x = \cos\frac{\gamma}{2}\cosh\varphi, \\ \dot y = \cos\frac{\gamma}{2}\sinh\varphi. \end{cases} \tag{52}$$

Differentiating the first and the second equalities in (51) and using (52), we get

$$-\frac{\dot\gamma}{2}\sin\frac{\gamma}{2}=(\sin\alpha\sinh\varphi+\beta\cosh\varphi)\dot\varphi=\sin\frac{\gamma}{2}\left(\sin\alpha\sinh\varphi+\beta\cosh\varphi\right),$$

$$\frac{\dot\gamma}{2}\cos\frac{\gamma}{2}=-\dot y\sin\alpha-\beta\dot x=-\cos\frac{\gamma}{2}\left(\sin\alpha\sinh\varphi+\beta\cosh\varphi\right),$$

whence

$$\dot\gamma=-2(\sin\alpha\sinh\varphi+\beta\cosh\varphi),\quad \dot\gamma(0)=-2\beta.$$

Consequently, on the ground of the first equality in (51) and (52)

$$\ddot\gamma=-2(\sin\alpha\cosh\varphi+\beta\sinh\varphi)\dot\varphi=-2\cos\frac{\gamma}{2}\sin\frac{\gamma}{2}=-\sin\gamma.$$

We got the mathematical pendulum equation. In paper [19] this equation together with equations (52) are obtained by another method replacing φ with z.

8. To search for geodesics of a sub-Riemannian metric on $SE(2)$

The Lie group $SE(2)$ is isomorphic to the group of matrices of a form

$$\begin{pmatrix} A & v\\ 0 & 1\end{pmatrix};\quad A=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix},\quad v=\begin{pmatrix}x\\ y\end{pmatrix}\in\mathbb{R}^2. \tag{53}$$

The same formula (46) is true.

It is clear that matrices

$$e_1=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix},\quad e_2=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\quad e_3=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} \tag{54}$$

constitute a basis of Lie algebra $\mathfrak{se}(2)$. In addition,

$$[e_1,e_2]=e_3,\quad [e_1,e_3]=-e_2,\quad [e_2,e_3]=0. \tag{55}$$

Let us define a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{se}(2)$ with orthonormal basis $\{e_1,\,e_2,\,e_3\}$ and the subspace $\mathfrak{p}$ with orthonormal basis $\{e_1,\,e_2\}$ generating Lie algebra $\mathfrak{se}(2)$. Thus a left-invariant sub-Riemannian metric d is defined on the Lie group $SE(2)$ (see [6], [25], [27] and other papers).

Let us take a (co)vector $\psi_0=\cos\alpha e_1+\sin\alpha e_2+\beta e_3\in\mathfrak{se}(2)$. We calculate

$$\psi_g(w)=\langle\psi_g,w\rangle=\langle\psi_0,gwg^{-1}\rangle,\quad g\in SH(2),\ w=w_1e_1+w_2e_2\in\mathfrak{p}.$$

$$gwg^{-1}=\begin{pmatrix}\cos\varphi&-\sin\varphi&x\\0&0&1\end{pmatrix}\begin{pmatrix}0&-w_1&w_2\\w_1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}\cos\varphi&\sin\varphi&-x\cos\varphi-y\sin\varphi\\-\sin\varphi&\cos\varphi&x\sin\varphi-y\cos\varphi\\0&0&1\end{pmatrix}=$$

$$w_1e_1+(w_1y+w_2\cos\varphi)e_2+(-w_1x+w_2\sin\varphi)e_3,$$

$$\psi_g(w)=w_1\cos\alpha+(w_1y+w_2\cos\varphi)\sin\alpha+(-w_1x+w_2\sin\varphi)\beta=$$

$$w_1(\cos\alpha+y\sin\alpha-\beta x)+w_2(\sin\alpha\cos\varphi+\beta\sin\varphi).$$

Consequently,

$$u(g)=(\cos\alpha+y\sin\alpha-\beta x)e_1+(\sin\alpha\cos\varphi+\beta\sin\varphi)e_2,\ v(g)=gu(g)=$$

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & x \\ \sin\varphi & \cos\varphi & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\cos\alpha - y\sin\alpha + \beta x & \sin\alpha\cos\varphi + \beta\sin\varphi \\ \cos\alpha + y\sin\alpha - \beta x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} \sin\varphi(\beta x - \cos\alpha - y\sin\alpha) & \cos\varphi(\beta x - \cos\alpha - y\sin\alpha) & \cos\varphi(\sin\alpha\cos\varphi + \beta\sin\varphi) \\ \cos\varphi(\cos\alpha + y\sin\alpha - \beta x) & \sin\varphi(\beta x - \cos\alpha - y\sin\alpha) & \sin\varphi(\sin\alpha\cos\varphi + \beta\sin\varphi) \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence integral curves of vector field $v(g)$, $g \in SE(2)$, satisfy the system of differential equations

$$\begin{cases} \dot\varphi = \cos\alpha + y\sin\alpha - \beta x, \\ \dot x = \cos\varphi(\sin\alpha\cos\varphi + \beta\sin\varphi), \\ \dot y = \sin\varphi(\sin\alpha\cos\varphi + \beta\sin\varphi). \end{cases} \tag{56}$$

The geodesic $g(t)$, $t \in \mathbb{R}$, with $g(0) = e$ is a solution of this system with initial data $\varphi(0) = x(0) = y(0) = 0$. In this case, $|u(g(t))| \equiv 1$, i.e.

$$g(t) \in M_1 = \{(\sin\alpha\cos\varphi + \beta\sin\varphi)^2 + (\cos\alpha + y\sin\alpha - \beta x)^2 = 1\} \subset SE(2). \tag{57}$$

Therefore there exist differentiable functions $\omega = \omega(t) = 2\xi(t)$ such that

$$\sin\frac{\omega(t)}{2} = \sin\alpha\cos\varphi + \beta\sin\varphi, \quad \cos\frac{\omega(t)}{2} = \cos\alpha + y\sin\alpha - \beta x. \tag{58}$$

Given the equality $\varphi(0) = x(0) = y(0) = 0$, we can assume that $\omega(0) = 2\xi(0) = 2\alpha$.

On the ground of formula (58) the system (56) is written in a form

$$\begin{cases} \dot\varphi = \cos\frac{\omega}{2}, \\ \dot x = \sin\frac{\omega}{2}\cos\varphi, \\ \dot y = \sin\frac{\omega}{2}\sin\varphi. \end{cases} \tag{59}$$

Differentiating the first and the second equalities in (58) and using (59), we get

$$\frac{\dot\omega}{2}\cos\frac{\omega}{2} = -\left(\sin\alpha\sin\varphi - \beta\cos\varphi\right)\dot\varphi = -\cos\frac{\omega}{2}\left(\sin\alpha\sin\varphi - \beta\cos\varphi\right),$$

$$-\frac{\dot\omega}{2}\sin\frac{\omega}{2} = \dot y\sin\alpha - \beta\dot x = \sin\frac{\omega}{2}\left(\sin\alpha\sin\varphi - \beta\cos\varphi\right),$$

whence

$$\dot\omega = 2(\beta\cos\varphi - \sin\xi\sin\varphi), \quad \dot\omega(0) = 2\dot\xi(0) = 2\beta. \tag{60}$$

Differentiating the last equality, we get in view of formulae (58) and (59)

$$\ddot\omega = -2(\beta\sin\varphi + \sin\alpha\cos\varphi)\dot\varphi = -2\sin\frac{\omega}{2}\cos\frac{\omega}{2} = -\sin\omega. \tag{61}$$

We get again the mathematical pendulum equation.

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Аграчев А. А., Сачков Ю. Л. Геометрическая теория управления. — М.: ФИЗМАТЛИТ, 2005.

2. Берестовский В. Н. Субметрии пространственных форм неотрицательной кривизны // Сиб. матем. журн. — 1987. — Т. 28, No 4. — С. 44–56.

3. Берестовский В. Н. Однородные пространства с внутренней метрикой // Докл. АН СССР. — 1988. — Т. 301, No 2. — С. 268–271.

4. Берестовский В. Н. Однородные многообразия с внутренней метрикой I // Сиб. матем. журн. — 1988. — Т. 29, No 6. — С. 17–29.

5. Берестовский В. Н. Однородные многообразия с внутренней метрикой II // Сиб. матем. журн. — 1989. — Т. 30, No 2. — С. 14–28.

6. Берестовский В. Н. Геодезические левоинвариантной неголономной римановой метрики на группе движений евклидовой плоскости // Сиб. матем. журн. — 1994. — Т. 35, No 6. — С. 1223–1229.

7. Берестовский В. Н. Геодезические неголономных левоинвариантных внутренних метрик на группе Гейзенберга и изопериметриксы плоскости Минковского // Сиб. матем. журн. — 1994. — Т. 35, No. 1. — С. 3–11.

8. Берестовский В. Н. Универсальные методы поиска нормальных геодезических на группах Ли с левоинвариантной субримановой метрикой // Сиб. мат. журн. — 2014. — Т. 55, No 5. — С. 959–970.

9. Берестовский В. Н., Зубарева И. А. Геодезические и кратчайшие специальной субримановой метрики на группе Ли $SO(3)$ // Сиб. мат. журн. — 2015. — Т. 56, No 4. — С. 762–774.

10. Берестовский В. Н., Зубарева И. А. Субриманово расстояние в группах Ли $SU(2)$ и $SO(3)$ // Мат. труды. — 2015. — Т. 18, No 2. — С. 3–21.

11. Бесчастный И. Ю., Сачков Ю. Л. Геодезические в субримановой задаче на группе $SO(3)$ // Матем. сб. — 2016. — Т. 207, No 7. — С. 29–56.

12. Кон-Фоссен С. Э. О существовании кратчайших путей // Некоторые вопросы дифференциальной геометрии в целом. — М.: Физматгиз, 1959. — С. 288–303.

13. Понтрягин Л. С., Болтянский В. Г., Гамкрелидзе Р. В., Мищенко Е. Ф. Математическая теория оптимальных процессов. 2-е изд. — М.: Физматгиз, 1969.

14. Рашевский П. К. О соединимости любых двух точек вполне неголономного пространства допустимой кривой // Уч. зап. пед. ин.-та им. Либкнехта. Сер. физ.-мат. наук 2. — 1938. — С. 83–94.

15. Bellaiche A., Risler J. (Eds.) Sub-Riemannian geometry. — Progress in Math. — Vol. 144. — Birkhäuser Verlag, Basel-Boston-Berlin, 1996.

16. Berestovskii V. N. Curvatures of homogeneous sub-Riemannian manifolds // European Journal of Mathematics. — 2017. — Vol. 3. — P. 788–807.

17. Berestovskii V. N., Gorbatsevich V. V. Homogeneous spaces with inner metric and with integrable invariant distributions // Analysis and Mathematical Physics. — 2014. — Vol. 4, no. 4. — P. 263–331.

18. Berestovskii V. N., Guijarro L. A. Metric Characterization of Riemannian Submersions // Annals of Global Analysis and Geometry. — 2000. — Vol. 18. — P. 577–588.

19. Butt Y. A., Sachkov Y. L., Bhatti A. I. Extremal Trajectories and Maxwell Strata in Sub-Riemannian Problem on Group of Motions of Pseudo Euclidean Plane // Journal of Dynamical and Control Systems. — 2014. — Vol. 20, no. 3. — P. 341–364.

20. Chow W. L. Über systeme von linearen partiellen differential gleichungen erster ordnung // Math. Ann. — 1939. — Vol. 117. — P. 98–105.

21. Golé C., Karidi R. A note on Carnot geodesics in nilpotent Lie groups // J. Dyn. Control Syst. — 1995. — Vol. 4, no. 1. — P. 535–549.

22. Jurdjevich V. Geometric control theory. — Cambridge: Cambridge Univ. Press, 1997.

23. Liu W., Sussman H. J. Shortest paths for sub-Riemannian metrics on rank-two-distributions // Memoirs of the Amer. Math. Soc. — Vol. 118, no. 564. — (Amer. Mth. Soc., Providence, 1995).

24. Milnor J. Curvatures of left invariant metrics on Lie groups // Adv. Math. — 1976. — Vol. 21. — P. 293–329.

25. Moiseev R. S., Sachkov Yu. L. Maxwell strata in sub-Riemannian problem on the group of motions of a plane // ESAIM: COCV. — 2010. — Vol. 16, no. 2. — P. 380–399.

26. Montgomery R. A tour of subriemannian geometries, their geodesics and applications. — AMS, 2002.

27. Sachkov Yu. L. Conjugate and cut time in the sub-Riemannian problem on the group of motions of a plane // ESAIM: COCV. — 2010. — V. 16, no. 4. — P. 1018–1039.

REFERENCES

1. Agrachev, A.A. & Sachkov, Yu.L. 2004, *Control theory from the geometric viewpoint*, V. 87 of Encyclopedia of Mathematical Sciences, Springer-Verlag, Berlin.

2. Berestovskii, V.N. 1987, "Submetries of space forms of nonnegative curvature", *Siber. Math. J.*, vol. 28, no. 4., pp. 552–562.

3. Berestovskii, V.N. 1989, "Homogeneous spaces with intrinsic metric", *Soviet Math. Dokl.*, vol. 38, no. 1, pp. 60–63.

4. Berestovskii, V.N. 1988, "Homogeneous manifolds with intrinsic metric. I", *Siber. Math. J.*, vol. 29, no. 6, pp. 887–897.

5. Berestovskii, V.N. 1989, "Homogeneous manifolds with intrinsic metric. II", *Siber. Math. J.*, vol. 30, no. 2, pp. 180-–191.

6. Berestovskii, V.N. 1994, "Geodesics of a left-invariant nonholonomic Riemannian metric on the group of motions of the Euclidean plane", *Siberian Math. J.*, vol. 35, no. 6, pp. 1083–1088.

7. Berestovskii, V.N. 1994, "Geodesics of nonholonomic left-invariant inner metrics on the Heisenberg group and isoperimetrics of Minkowski plane", *Siber. Math. J.*, vol. 35, no. 1, pp. 1–8.

8. Berestovskii, V.N. 2014, "Universal methods of the search of normal geodesics on Lie groups with left-invariant sub-Riemannian metric", *Siberian Math. J.* vol. 55, no. 5, pp. 783–791.

9. Berestovskii, V.N. & Zubareva, I.A. 2015, "Geodesics and shortest arcs of a special sub-Riemannian metric on the Lie group $SO(3)$", *Siberian Math. J.*, vol. 56, no. 4, pp. 601–611.

10. Berestovskii, V.N. & Zubareva, I.A. 2016, "Sub-Riemannian distance in the Lie groups $SU(2)$ and $SO(3)$", *Siberian Adv. Math.*, vol. 26, no. 2, pp. 77–89.

11. Beschastnyi, I.Yu. & Sachkov, Yu.L., "Geodesics in the sub-Riemannian problem on the group $SO(3)$", *Sb. Math.*, vol. 207, no. 7, pp. 915–941.

12. Cohn-Vossen, S. 1936, "Existenz Kürzester Wege", *Compositio Math.*, vol. 3, pp. 441–452.

13. Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V. & Mishchenko E.F. 1962, *The mathematical theory of optimal processes*, Interscience Publishers John Wiley & Sons, Inc., New York–London.

14. Rashevsky, P.K. 1938, "Any two points of a totally nonholonomic space may be connected by an admissible line" (Russian), *Uch. Zap. Ped Inst. im. Liebknechta*, vol. 2, pp. 83–94.

15. Bellaiche, A. & Risler, J.(Eds.) 1996, *Sub-Riemannian geometry*, Progress in Math., Vol. 144. Birkhäuser Verlag, Basel-Boston-Berlin.

16. Berestovskii, V.N. 2017, “Curvatures of homogeneous sub-Riemannian manifolds”, *European Journal of Mathematics*, vol. 3, pp. 788–807.

17. Berestovskii, V.N. & Gorbatsevich, V.V. 2014, “Homogeneous spaces with inner metric and with integrable invariant distributions”, *Analysis and Mathematical Physics*, vol. 4, no. 4, pp. 263–331.

18. Berestovskii, V.N. & Guijarro, L. 2000, “A Metric Characterization of Riemannian Submersions”, *Annals of Global Analysis and Geometry*, vol. 18, pp. 577–588.

19. Butt, Y.A., Sachkov, Y.L. & Bhatti, A.I. 2014, “Extremal Trajectories and Maxwell Strata in Sub-Riemannian Problem on Group of Motions of Pseudo Euclidean Plane”, *Journal of Dynamical and Control Systems*, vol. 20, no. 3, pp. 341–364.

20. Chow, W.L. 1938, “Über systeme von linearen partiellen differential gleichungen erster ordnung”, *Math. Ann.* , vol. 117, pp. 98–105.

21. Golé, C. & Karidi, R. 1995, “A note on Carnot geodesics in nilpotent Lie groups”, *J. Dyn. Control Syst.*, vol. 4, no. 1, pp. 535–549.

22. Jurdjevich, V. 1997, *Geometric control theory*, Cambridge University Press, Cambridge.

23. Liu, W. & Sussman H.J. 1995, “Shortest paths for sub-Riemannian metrics on rank-two-distributions”, *Memoirs of the Amer. Math. Soc.*, vol. 118. no. 564., (Amer. Mth. Soc., Providence, 1995).

24. Milnor, J. 1976, “Curvatures of left invariant metrics on Lie groups”, *Adv. Math.*, vol. 21, pp. 293–329.

25. Moiseev, R.S. & Sachkov, Yu.L. 2010, “Maxwell strata in sub-Riemannian problem on the group of motions of a plane”, *ESAIM: COCV*, vol. 16, no. 2, pp. 380–399.

26. Montgomery, R. 2002, *A tour of subriemannian geometries, their geodesics and applications*, AMS.

27. Sachkov, Yu.L. 2010, “Conjugate and cut time in the sub-Riemannian problem on the group of motions of a plane”, *ESAIM: COCV*, vol. 16, no. 4, pp. 1018–1039.

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