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Об одном свойстве нильпотентных матриц над алгебраически замкнутым полем

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Аннотация

Предположим, что F - алгебраически замкнутое поле. Докажем, что кольцо $\prod_{n=1}^{\infty}\mathbb{M}_n(F)$ обладает специальным свойством, которое несколько параллельно (и немного лучше) свойству, установленному Šter (LAA, 2018) для колец $\prod_{n=1}^{\infty}\mathbb{M}_n(\mathbb{Z}_2)$ и $\prod_{n=1}^{\infty}\mathbb{M}_n(\mathbb{Z}_4)$, где $\mathbb{Z}_2$ - конечное простое поле из двух элементов и $\mathbb{Z}_4$ является конечным неразложимым кольцом из четырех элементов.

Ключевые слова: нильпотентные матрицы, идемпотентные матрицы, Жордановая каноническая форма, алгебраически замкнутые поля.

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On a Property of Nilpotent Matrices over an Algebraically Closed Field

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Abstract

Suppose F is an algebraically closed field. We prove that the ring $\prod_{n=1}^{\infty}\mathbb{M}_n(F)$ has a special property which is, somewhat, in sharp parallel with (and slightly better than) a property established by Šter (LAA, 2018) for the rings $\prod_{n=1}^{\infty}\mathbb{M}_n(\mathbb{Z}_2)$ and $\prod_{n=1}^{\infty}\mathbb{M}_n(\mathbb{Z}_4)$, where $\mathbb{Z}_2$ is the finite simple field of two elements and $\mathbb{Z}_4$ is the finite indecomposable ring of four elements.

Keywords: nilpotent matrices, idempotent matrices, Jordan canonical form, algebraically closed fields.

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All rings R are assumed here to be associative, containing the identity element 1 which differs from the zero element 0 of R. Recall that a ring R is *nil-clean* provided that each its element is a sum of a nilpotent and an idempotent, is *π-regular* provided that for every element $r \in R$ there is $n \in \mathbb{N}$ such that $r^n \in r^n R r^n$, and is *strongly π-regular* provided that $r^n \in r^{n+1}R$.

In his seminal paper [4], Šter showed that the ring $\prod_{n=1}^{\infty} \mathbb{M}_n(\mathbb{Z}_2)$ is nil-clean but *not* strongly π-regular, whereas the ring $\prod_{n=1}^{\infty} \mathbb{M}_n(\mathbb{Z}_4)$ is nil-clean but *not* π-regular. He utilizes an innovation of the method used in [1]. Specifically, for any $n \in \mathbb{N}$, it was proved there that, for every $n \times n$ matrix A over the finite field $\mathbb{Z}_2$, there exists an idempotent matrix E such that $(A-E)^4 = 0$, while the index of nilpotence over the finite ring $\mathbb{Z}_4$ is precisely 8. As usual, the symbol I will stand in the sequel the standard matrix identity. Thereby, $A = N + E$ for some $N^4 = 0$ and hence $(I-E)A = (I-E)N$, but it is not clear at all whether $[(I-E)A]^4 = 0$ will hold eventually.

On the other side, in [2] we have examined rings R having the property that, for each $a \in R$, there is an idempotent $e \in aR$ such that $(1-e)a$ is nilpotent. We shall be here even rather more precise by considering an existing idempotent $e \in aRa$ with $[(1-e)a]^2 = 0$.

It is well known that finite fields are, surely, *not* algebraically closed. So, the purpose of this very short note is to show that some (although little) improvement is possible by a strengthening of the technique utilized in [2] in the case of algebraically closed fields.

Before proceed by proving our chief result, we need the next two technical statements.

Lemma 1. *Let R be a unital ring, $n \geq 2$, and $A = \sum_{i=1}^{n-1} E_{i,i+1} \in \mathbb{M}_n(R)$, where the $E_{i,j}$ denote matrix units. Then there exits an idempotent $B \in A\mathbb{M}_n(R)A$ such that $((I-B)A)^2 = 0$.*

Proof. First, suppose that $n = 2$. Then

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and hence, taking $B = 0 \in A\mathbb{M}_n(R)A$, we have $((I-B)A)^2 = A^2 = 0$. Let us therefore assume that $n \geq 3$, and let

$$B = A\Big(\sum_{i=1}^{n-2} E_{i+2,i}\Big)A = \Big(\sum_{i=2}^{n-1} E_{i,i-1}\Big)A = \sum_{i=2}^{n-1} E_{i,i}.$$

Then $B \in A\mathbb{M}_n(R)A$, B is clearly an idempotent, and

$$((I-B)A)^2 = ((E_{1,1} + E_{n,n})A)^2 = E_{1,2}^2 = 0,$$

as desired. □

Lemma 2. *Let F be a field, $n \geq 1$, and $A \in \mathbb{M}_n(R)$ a matrix in Jordan canonical form. Then there exits an idempotent $B \in A\mathbb{M}_n(R)A$ such that $((I-B)A)^2 = 0$.*

Proof. Write

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_m \end{pmatrix},$$

where each A_i is a Jordan block of size $n_i \times n_i$. For each A_i we shall define a block B_i of the same size, such that $B_i \in A_i\mathbb{M}_{n_i}(F)A_i$ is idempotent.

If A_i is invertible as a matrix of $\mathbb{M}_{n_i}(F)$, then the identity element I_{n_i} of $\mathbb{M}_{n_i}(F)$ is in $A_i\mathbb{M}_{n_i}(F)A_i$, and we set $B_i = I_{n_i}$. If A_i is not invertible, then either $n_i = 1$ and $A_i = (0)$, or $n_i \geq 2$ and $A_i = \sum_{j=1}^{n_i-1} E_{j,j+1}$. In the first case, we let $B_i = (0)$, and in the second case, we take B_i as in Lemma 1. Then, clearly, in each case, $B_i \in A_i\mathbb{M}_{n_i}(F)A_i$ is idempotent, and it is easy to see that $((I_{n_i} - B_i)A_i)^2 = 0$ for each i.

It follows immediately that

$$B = \begin{pmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_m \end{pmatrix}$$

has the desired properties. □

Proposition 1. *Let F be an algebraically closed field, and let $R = \prod_{n=1}^{\infty} \mathbb{M}_n(F)$. Then for each $A \in R$ there is an idempotent $B \in ARA$ such that $((I - B)A)^2 = 0$.*

Proof. For each n let A_n denote the projection of A onto the component $\mathbb{M}_n(F)$ in R. Since F is algebraically closed, for each n we can find an invertible matrix $C_n \in \mathbb{M}_n(F)$ such that $D_n = C_nA_nC_n^{-1}$ is in Jordan canonical form. By Lemma 2, for each n we can find an idempotent matrix $G_n \in D_n\mathbb{M}_n(F)D_n$ such that $((I_n - G_n)D_n)^2 = 0$. Now, for each n let $B_n = C_n^{-1}G_nC_n$, and let $B = (B_1, B_2, \dots) \in R$. Since each G_n is idempotent, the same holds for each B_n, and hence also for B. Also, since $G_n \in D_n\mathbb{M}_n(F)D_n$ and C_n is invertible, we have for each n that

$$B_n = C_n^{-1}G_nC_n \in C_n^{-1}D_n\mathbb{M}_n(F)D_nC_n = A_nC_n^{-1}\mathbb{M}_n(F)C_nA_n = A_n\mathbb{M}_n(F)A_n,$$

and hence $B \in ARA$. Finally, since $((I_n - G_n)D_n)^2 = 0$, for each n we have

$$((I_n - B_n)A_n)^2 = ((I_n - C_n^{-1}G_nC_n)A_n)^2 = (C_n^{-1}(I_n - G_n)C_nA_n)^2$$

$$= (C_n^{-1}(I_n - G_n)D_nC_n)^2 = C_n^{-1}((I_n - G_n)D_n)^2C_n = 0,$$

from which it follows that $((I - B)A)^2 = 0$, as required. □

We end our work with the following challenging query:

Problem 1. *Extend the considered above property for any field F which is not necessarily algebraically closed.*

An intuitive idea could be the following one: It is enough to establish the claim for a given $\mathbb{M}_n(F)$ with the index of the nilpotent $(1 - e)a$ bounded independent of n. Since every matrix is the direct sum of a unit and a nilpotent (we do not need the field F to be algebraically closed for this), it is enough to do the assertion for units and for nilpotents. For a unit a, we take $e = 1$. Now suppose a is nilpotent. It is enough to do the statement for the Weyr canonical form of a – for more details the interested reader can see [3]. Thus assume a has Weyr structure $(n_1, n_2, ..., n_r)$. The idea is to get an idempotent e in aRa that is diagonal, has 0's in the first n_1 places and the last n_r, and such that $(1-e)a$ has zero blocks (relative to the partition $n_1, .., n_r$) except in the $(1, 2)$ block. Then index of the nilpotent $(1 - e)a$ is exactly 2.

We will illustrate in the case of a homogeneous structure $(3, 3, 3, 3)$ but the argument in the nonhomogeneous case is similar although a little trickier. Thus, in terms of 3×3 blocks and $I = I_3$, we will have that

$$a = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us now

$$r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix},$$

and

$$e = ara = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, one finds that

$$(1-e)a = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is nilpotent of index 2, as expected.

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REFERENCES

1. S. Breaz, G. Călugăreanu, P. Danchev and T. Micu, *Nil-clean matrix rings*, Lin. Alg. & Appl. **439** (2013), 3115–3119.

2. P.V. Danchev, *A generalization of π-regular rings*, Turk. J. Math. **43** (2019), 702–711.

3. K.C. O'Meara, J. Clark and C.I. Vinsonhaler, Advanced Topics in Linear Algebra: weaving matrix problems through the Weyr form, Oxford Univ. Press, 2011.

4. J. Šter, *On expressing matrices over $\mathbb{Z}_2$ as the sum of an idempotent and a nilpotent*, Lin. Alg. & Appl. **544** (2018), 339–349.

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