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Гипотеза Якобиана для свободной ассоциативной алгебры (произвольной характеристики)

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Аннотация

Целью данной работы является использование PI-теории для упрощения результатов Дикса и Левина [4] об автоморфизмах свободной алгебры $F\{X\}$, а именно: если якобиан обратим, тогда каждый эндоморфизм является эпиморфизмом. Результаты переносятся на широкий класс колец.

Ключевые слова: Автоморфизмы, полиномиальные алгебры, свободные ассоциативные алгебры.

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The Jacobian Conjecture for the free associative algebra (of arbitrary characteristic)

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Abstract

The object of this note is to use PI-theory to simplify the results of Dicks and Lewin [4] on the automorphisms of the free algebra $F\{X\}$, namely that if the Jacobian is invertible, then every endomorphism is an epimorphism. We then show how the same proof applies to a somewhat wider class of rings.

Keywords: Automorphisms, polynomial algebras, free associative algebras.

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1. Introduction and main results

The object of this note is to use PI-theory to simplify the results of Dicks and Lewin [4] on the automorphisms of the free algebra $F\{X\}$, namely that if the Jacobian is invertible, then every endomorphism is an epimorphism. We then show how the same proof applies to a somewhat wider class of rings.

2. Hopfian rings

Definition 1. *An algebra R is* ***Hopfian*** *if every epimorphism (i.e., onto algebra homomorphism) $R \to R$ is an isomorphism.*

Dicks and Lewin [4, Proposition 3.1] proved that an endomorphism of the free associative algebra $F\{X\}$ is an epimorphism iff its Jacobian matrix is invertible. In this way, they reduced the Jacobian conjecture for $F\{X\}$ to the question of whether $F\{X\}$ is Hopfian, and proved it for the free algebra in two variables. In fact, this had already been resolved for any finite set of variables by Orzech and Ribes [6], with a more direct proof given in [3]. Also see [9] for a treatment of the Jacobian conjecture over a free algebra, and [1] for an overview of Yagzev's method to attack the Jacobian conjecture.

In this section we give a quick proof of the fact that the free associative algebra $F\{X\}$ is Hopfian, relying on considerations of growth, with a generalization obtained from the proof. . Recall that the **Gelfand-Kirillov dimension** $\mathrm{GKdim}(A)$ of an affine algebra $A = F\{a_1, \dots, a_\ell\}$ is

$$\mathrm{GKdim}(A) := \overline{\lim_{n\to\infty}} \log_n \tilde{d}_n, \tag{1}$$

where $A_n = \sum F a_{i_1} \cdots a_{i_n}$ and $d_n = \dim_F A_n$.

The standard reference on Gelfand-Kirillov dimension is [5] Although the d_n depend on the choice of the generating set $a_1, \dots, a_\ell$, $\mathrm{GKdim}(A)$ is independent of the choice of the generating set. We can tighten this fact a bit: Suppose that $A' = F\{a'_1, \dots, a'_\ell\}$ and $d'_n = \dim_F A'_n$. We say that the growth rate of the d_n is less than or equal to the growth rate of the d'_n if there are constants c, k such that $d'_n \leqslant cd_{kn}$. This defines an equivalence, and it is easy to see that the growth rate of A with respect to any two sets of generators is the same.

Lemma 1. *Suppose R is an affine algebra in which the growth of R/I is less than the growth of R, for each ideal I of R. Then R is Hopfian.*

In particular, if $\mathrm{GKdim}(R/I) < \mathrm{GKdim}(R)$ *for all ideals I of R, then R is Hopfian.*

Proof. For any epimorphism $\varphi : R \to R$, one has $\varphi(R) \cong R/\mathrm{ker}\varphi$, but then $\varphi(R)$ and R have the same growth rates, implying $\mathrm{ker}\varphi = 0$. □

The hypothesis of Lemma 1 holds for prime PI-algebras, cf. [2, Theorem 11.2.12], so we have:

Corollary 1. *Any prime affine PI-algebra is Hopfian.*

Remark 1. *R and R/I could have different growth rates even if* $\mathrm{GKdim}(R/I) < \mathrm{GKdim}(R)$. *For example, let R be the subalgebra of the free associative algebra generated by all subwords of u_n for any n, where $u_1 = xyx$ and $u_{n+1} = x^{10^n} u^n x^{10^n} yx^{10^n} u_n x^{10^n}$, a prime algebra, of* GKdim 2, *and I be the ideal generated by all words of degree 2 in y. Then* $\mathrm{GKdim}(R/I) = 2$, *although the growth rate of R/I is less than that of R. This example is not a PI-algebra.*

A T-**ideal** of an ideal R is an ideal invariant under all ring endomorphisms.

Lemma 2. *If $\mathcal{I}$ is a T-ideal of R, then any endomorphism φ of R clearly induces an endomorphism of $R/\mathcal{I}$.*

Proof. Define $\varphi : R/\mathcal{I} \to R/\mathcal{I}$ by $\varphi(a + \mathcal{I}) = \varphi(a) + \mathcal{I}$. This is well-defined since $\varphi(\mathcal{I}) \subseteq \mathcal{I}$ by hypothesis. □

Theorem 1 ([6]). *When X is a finite set of noncommuting indeterminates, the free associative algebra $F\{X\}$ is Hopfian.*

Proof. Let $\varphi : F\{X\} \to F\{X\}$ be an epimorphism, with some nonzero polynomial $f \in \mathrm{ker}(\varphi)$. Let $n = \deg(f)$. Let $\mathcal{I}_n$ be the T-ideal of identities of the algebra of generic $n \times n$ matrices. Then φ induces an endomorphism of $A : F\{X\}/\mathcal{I}_n$, whose kernel does not contain f, since the easy part of the Amitsur-Levitzki theorem says that the degree of any identity of $n \times n$ matrices is at least $2n > n$. Thus the epimorphism induced by φ has non-zero kernel, contradicting Lemma 1. □

The same idea of proof yields a stronger result. We say that R is T-**residually Hopfian** if the intersection of those T-ideals I of R for which R/I is Hopfian is 0. Examples include almost PI-algebras, and in particular the free algebra and all affine algebras of GKdim 2.

Theorem 2 ([6]). *Any T-**residually Hopfian** algebra is Hopfian.*

Proof. Let $\varphi : R \to R$ be an epimorphism, with some nonzero polynomial $f \in \mathrm{ker}(\varphi)$. By hypothesis there is some T-ideal $\mathcal{I}$ not containing r, but Lemma 2 implies that $R/\mathcal{I}$ is not Hopfian, a contradiction. □

Corollary 2.3 belongs to Alexei Kanel-Belov, his work was supported by the Russian Science Foundation under grant 17-11-01377. Louis Rowen was supported by ISF grant N 1623/16.

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Conclusions. In the paper we show that some ideas from PI-theory can be used for polynomial authomorphisms. Note that many specialists in PI-theory got different results in this arrear.

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