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Об одной сумме интегральных преобразований Ганкеля–Клиффорда функций Уиттекера

Джунесанг Чой, А. И. Нижников, И. А. Шилин

Джунесанг Чой — доктор наук, заслуженный профессор Донггукского университета (г. Кёнджу, Республика Корея).
e-mail: junesang@dongguk.ac.kr
Нижников Александр Иванович — кандидат физико-математических наук, профессор, заведующий кафедрой прикладной математики, информатики и ИТ, Московский педагогический государственный университет (г. Москва).
e-mail: ainizhnikov@mail.ru
Шилин Илья Анатольевич — кандидат физико-математических наук, доцент, доцент кафедры высшей математики НИУ МЭИ и кафедры алгебры, Московский педагогический государственный университет (г. Москва).
e-mail: ilyashilin@li.ru

Аннотация

В статье [11] авторами рассматривалась реализация T представления группы $SO(2,2)$ в одном пространстве однородных функций, заданных на 2×4-матрицах. Настоящее продолжение этой статьи посвящено вычислению матричных элементов тождественного оператора $T(e)$ и операторов представления $T(g)$ для подходящих элементов g группы относительно смешанного базиса, соответствующего двум различным базисам пространства представления, и вычислению некоторых несобственных интегралов, содержащих произведение функций Бесселя–Клиффорда и Уиттекера. Полученные результаты могут быть переписаны на языке интегральных преобразований Ганкеля–Клиффорда и их аналога. Первое и второе преобразования Ганкеля–Клиффорда, введенные сооответственно Хайеком и Перезом–Робайной, играют важную роль в теории дифференциальных операторов дробного порядка (см., например, [6, 8]). Близкий результат получен авторами недавно [12] для регулярной кулоновской функции.

Ключевые слова: группа $SO(2,2)$, матричные элементы представления, интегральные преобразования Ганкеля–Клиффорда, интегральное преобразование Макдональда–Клиффорда, функции Уиттекера, функции Бесселя–Клиффорда.

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On one sum of Hankel–Clifford integral transforms of Whittaker functions

J. Choi, A. I. Nizhnikov, I. A. Shilin

Junesang Choi — Doctor of Sciences, Emeritus Professor of Dangguo University (Gyeongju, Republic of Korea).
e-mail: junesang@dongguk.ac.kr
Nizhnikov Alexander Ivanovich — candidate of physical and mathematical Sciences, Professor, Head of the Department of Applied Mathematics, Computer Science and IT, Moscow State Pedagogical University (Moscow).
e-mail: ainizhnikov@mail.ru
Shilin Ilya Anatolevich — candidate of physical and mathematical Sciences, Associate Professor, Department of Higher Mathematics, NRU MEI and the Department of Algebra, Moscow State Pedagogical University (Moscow).
e-mail: ilyashilin@li.ru

Abstract

In [11], the authors considered the realization T of $SO(2,2)$-representation in a space of homogeneous functions on 2×4-matrices. In this sequel, we aim to compute matrix elements of the identical operator $T(e)$ and representation operator $T(g)$ for an appropriate g with respect to the mixed basis related to two different bases in the $SO(2,2)$-carrier space and evaluate some improper integrals involving a product of Bessel-Clifford and Whittaker functions. The obtained result can be rewritten in terms of Hankel-Clifford integral transforms and their analogue. The first and the second Hankel-Clifford transforms introduced by Hayek and Pérez–Robayna, respectively, play an important role in the theory of fractional order differential operators (see, e.g., [6, 8]). The similar result have been derived recently by the authors for the regular Coulomb function in [12].

Keywords: group $SO(2,2)$, matrix elements of representation, Hankel-Clifford integral transform, Macdonald-Clifford integral transform, Whittaker functions, Bessel-Clifford functions.

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1. Introduction and preliminaries

We recall the definitions and notations in [11]. The group $SO(2,2)$, which preserves the quadratic form $\mathcal{E}$ defined in $\mathbb{R}^4$ whose matrix with respect to the canonical basis is a diagonal matrix $e_{2,2} = \operatorname{diag}(1,1,-1,-1)$, which is called the split orthogonal group and consists of real 4×4 matrices g satisfying the equality $g\, e_{2,2}\, g^{\mathrm{t}} = e_{2,2}$. Here and throughout, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^{+}$, $\mathbb{R}^{-}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, negative real numbers, integers and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Let L be a real linear space consisting of real 2×4 matrices. We define the cone Λ in L by the subset of matrices of rank 2 satisfying the equation $x\, e_{2,2}\, x^{\mathrm{t}} = \operatorname{diag}(0,0)$. Let $\mathfrak{L}$ be the complex linear space consisting of infinitely differentiable functions defined on Λ and satisfying the equality $f(bx) = |b_{11}|^{\sigma_1}\, |b_{22}|^{\sigma_2}\, f(x)$ for a fixed pair $(\sigma_1, \sigma_2) \in \mathbb{C}^2$ and arbitrary non-degenerate matrix $b = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}$. We consider the $SO(2,2)$-representation T in $\mathfrak{L}$ defined by formula $T(g)[f(x)] = f(xg)$. In [14], with a view to investigating some special functions of matrix argument, this construction has been used.

Shilin and Choi [11] dealt with the spherical section ω_1 of Λ consisting of matrices

$$\tilde{x}(\alpha_1, \beta_1) = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & \cos\beta_1 & -\sin\beta_1 \\ \sin\alpha_1 & \cos\alpha_1 & \sin\beta_1 & \cos\beta_1 \end{pmatrix} \quad (\alpha_1,\, \beta_1 \in [0, 2\pi)). \tag{1}$$

In particular, they [11] showed that for any $x \in \Lambda$ there are a low triangular non-degenerate 2×2-matrix b and $\tilde{x} \in \omega_1$ such that $x = b\tilde{x}$. If $\mathfrak{L}_1$ is the linear space of restrictions of functions $f \in \mathfrak{L}$ on ω_1, we can realize the representations T as the same representation in $\mathfrak{L}_1$. They also showed that the function $f = \exp(\mathbf{i} p_1 \alpha_1) \exp(\mathbf{i} q_1 \beta_1)$ defined on ω_1 does not belong to $f = \exp(\mathbf{i} p_1 \alpha_1) \exp(\mathbf{i} q_1 \beta_1)$ if and only if the sum $p_1 + q_1$ is not divisible by 2, and defined the canonical basis

$$\tilde{B}_1 = \{\tilde{f}_{p_1,q_1}(\alpha_1, \beta_1) = \exp(\mathbf{i} p_1 \alpha_1) \exp(\mathbf{i} q_1 \beta_1) \mid p_1,\, q_1 \in \mathbb{Z},\, p_1 + q_1 \equiv 0 \pmod 2\},$$

which is orthonormal with respect to the scalar product

$$f * g = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(\alpha_1, \beta_1)\, \overline{g(\alpha_1, \beta_1)}\, \mathrm{d}\alpha_1\, \mathrm{d}\beta_1$$

in $\mathfrak{L}_1$. Writing Cartan decomposition $g = g_1 g_2 g_3$ for an arbitrary element g of the group $SO(2,2)$, where $g_1, g_3 \in SO(2) \times SO(2)$ and

$$g_2 \in \exp \begin{pmatrix} 0 & \operatorname{diag}(\mu, \nu) \\ \operatorname{diag}(\mu, \nu) & 0 \end{pmatrix},$$

they showed that in case $|\nu| \neq |\mu|$ the matrix elements of the linear operator $T(g)$ with respect to $\tilde{B}_1$ can be written as a product of four exponential functions, depending respectively on four parameters of the rotations g_1 and g_3, and two Gaussian hypergeometric functions depending (respectively) on $\left(\tanh \frac{\mu \pm \nu}{2}\right)^2$.

The parabolic section ω_2 of Λ has been defined as the subset consisting of matrices

$$\tilde{x}(\alpha_2, \beta_2) = \begin{pmatrix} 1 & \alpha_2 & \cos\beta_2 - \alpha_2 \sin\beta_2 & \sin\beta_2 + \alpha_2 \cos\beta_2 \\ 0 & 1 & -\sin\beta_2 & \cos\beta_2 \end{pmatrix},$$

where $\alpha_2 \in \mathbb{R}$ and $\beta_2 \in [0, 2\pi)$. If $\mathfrak{L}_2$ is the linear space of restrictions of functions $f \in \mathfrak{L}$ on ω_2, then the canonical basis in $\mathfrak{L}_2$ can be defined as follows:

$$\tilde{B}_2 = \{\tilde{f}_{p_2,q_2}(\alpha_2, \beta_2) = \exp(\mathbf{i} p_2 \alpha_2) \exp(\mathbf{i} q_2 \beta_2) \mid p_2 \in \mathbb{R},\, q_2 \in [0, 2\pi)\}.$$

They [11] established the one-to-one correspondence between the restrictions of $T(g)$ to $\mathfrak{L}_2$ and integral operators whose kernels can be described in terms of some Bessel functions.

2. Two bases in $\mathfrak{L}$ and our purpose

Let us denote the determinant $\det \begin{pmatrix} x_{1m} & x_{1n} \\ x_{2m} & x_{2n} \end{pmatrix}$ inside the matrix $x \in \Lambda$ by $\Delta_{m,n}$ and introduce the basis

$$B_1 = \{f_{p_1,q_1}(x) \mid p_1, q_1 \in \mathbb{Z}, p_1 + q_1 \equiv 0 \pmod 2\}$$

in $\mathfrak{L}$, consisting of functions

$$f_{p_1,q_1}(x) = (x_{11}^2 + x_{12}^2)^{\frac{\sigma_1-\sigma_2+q_1-p_1}{2}} \, |\Delta_{1,2}|^{\sigma_2-q_1} \, (x_{11} - \mathbf{i}x_{12})^{p_1-q_1} \, (\Delta_{1,3} + \mathbf{i}\Delta_{1,4})^{q_1}.$$

Obviously the restriction of f_{p_1,q_1} to ω_1 coincides with $\mathbf{i}^{q_1}\tilde{f}_{p_1,-q_1}$:

$$f_{p_1,q_1}|_{\omega_1} = \mathbf{i}^{q_1}\tilde{f}_{p_1,-q_1}. \tag{2}$$

In this paper, we also use the basis

$$B_2 = \{f_{p_2,q_2}(x) \mid p_2 \in \mathbb{R},\, q_2 \in \mathbb{Z}\},$$

where

$$f_{p_2,q_2}(x) = |x_{11}|^{\sigma_1-\sigma_2} \, |\Delta_{1,2}|^{\sigma_2-q_2} \, (\Delta_{1,3} + \mathbf{i}\Delta_{1,4})^{q_2} \, \exp\frac{\mathbf{i}p_2 x_{12}}{x_{11}}.$$

It is easy to see that f_{p_2,q_2} is an extension of the function $\mathbf{i}^{q_2}\tilde{f}_{p_2,q_2}$ to Λ.

Let $\mathrm{span}(f_{p_1,q_1},\, f_{\hat{p}_1,\hat{q}_1})$ be the subspace in $\mathfrak{L}_1$. It is invariant with respect to the linear operator $T(g)$ (for some fixed g) and its basis vectors f_{p_1,q_1} and $f_{\hat{p}_1,\hat{q}_1}$ which are not eigenfunctions of this operator. In this paper, we aim to establish dependence between the matrix elements of the operator $T(g)$ with respect to the *ordinary* basis B_2 and the *mixed* basis $B_2|B_1$ and matrix elements of the operator $\mathrm{id} \equiv T(e)$ with respect to $B_2|B_1$. Choosing here the group element as follows:

$$h^* = \mathrm{diag}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathrm{diag}(1,1)\right),$$

we will show that the above dependence can be rewritten as a representation of Whittaker function of the second kind in the form of integral involving Whittaker and Bessel-Clifford functions. The Bessel-Clifford functions are used, for example, for solution of wave equation [1] and are a particular case of more generalized so-called Bessel-Maitland functions (see [8]).

The above-described approach, together with other methods, was used by Shilin and Choi [10] who considered another realization of the representation of the group $SO(2,2)$ and representation operators corresponding to some diagonal and block-diagonal matrices which belong to the split orthogonal group.

3. Transitive subgroups and invariant measures

It is obvious that ω_1 is an orbit of the subgroup $H_1 \simeq SO(2)\times SO(2)$, consisting of the matrices

$$h_1(\varphi_1,\psi_1) = \begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 & 0 & 0 \\ \sin\varphi_1 & \cos\varphi_1 & 0 & 0 \\ 0 & 0 & \cos\psi_1 & -\sin\psi_1 \\ 0 & 0 & \sin\psi_1 & \cos\psi_1 \end{pmatrix}.$$

Let us consider the matrices

$$h(\varrho) = \begin{pmatrix} 2 & \varrho & 0 & \varrho \\ -\varrho & 2 & \varrho & 0 \\ 0 & \varrho & 2 & \varrho \\ \varrho & 0 & -\varrho & 2 \end{pmatrix}$$

and the points $\tilde{x}(\alpha_2,\beta_2)$ and $\tilde{x}(\hat{\alpha}_2,\hat{\beta}_2)$ belong to the subset ω_2 of Λ. Since the matrix elements $h_{ij}(\varrho)$ of the matrix $h(\varrho)$ satisfy the equalities

$$h_{i1}(\varrho) + h_{i2}(\varrho) - h_{i3}(\varrho) - h_{i4}(\varrho) = 4\,\mathrm{sign}(2.5 - i),$$

we get $\frac{1}{2}h(\varrho) \in SO(2,2)$. It is easy to see that

$$\tilde{x}(\alpha_2,\beta_2)h_1(0,-\beta_2) = \tilde{x}(\alpha_2,0) \equiv \begin{pmatrix} 1 & \alpha_2 & 1 & \alpha_2 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$
$$\frac{1}{2}\tilde{x}(\alpha_2,0)h(\hat{\alpha}_2-\alpha_2) = \tilde{x}(\hat{\alpha}_2,0)$$

and

$$\tilde{x}(\hat{\alpha}_2,0)h_1(0,\hat{\beta}_2) = \tilde{x}(\hat{\alpha}_2,\hat{\beta}_2).$$

Thus the matrix

$$h_2(\hat{\alpha}_2-\alpha_2,\hat{\beta}_2-\beta_2) = \frac{1}{2}h_1(0,-\beta_2)h(\hat{\alpha}_2-\alpha_2)h_1(0,\hat{\beta}_2)$$

transforms the point $\tilde{x}(\alpha_2,\beta_2)$ into the point $\tilde{x}(\hat{\alpha}_2,\hat{\beta}_2)$. It means that the subgroup

$$H_2 = \{h_2(\varphi_2,\psi_2) \mid \varphi_2 \in \mathbb{R},\ \psi_2 \in [0;2\pi)\}$$

acts transitively on ω_2. Also we find that $\mathrm{d}\omega_2 = \mathrm{d}\alpha_1\,\mathrm{d}\beta_2$ is an H_2-invariant measure on ω_2. It is found that f_{p_2,q_2} is an eigenfunction of the linear operator $T(h_2(\varphi_2,\psi_2))$, more exactly,

$$T(h_2(\varphi_2,\psi_2))[f_{p_2,q_2}] = \exp(\mathbf{i})f_{p_2,q_2}.$$

Similarly $\mathrm{d}\omega_1 = \mathrm{d}\alpha_1\,\mathrm{d}\beta_1$ is an H_1-invariant measure on the spherical section ω_1 and f_{p_1,q_1} is an eigenfunction of the operator $T(h_1(\varphi_1,\psi_1))$, namely

$$T(h_1(\varphi_1,\psi_1))[f_{p_1,q_1}] = \exp(\mathbf{i}p_1\varphi_1)\exp(\mathbf{i}q_1\psi_1)f_{p_1,q_1}.$$

4. Functionals F_1 and F_2 and assorted spaces

Let us introduce the following bilinear functionals defined on the direct product $\mathfrak{L}\times\mathfrak{L}^\bullet$ of two representation spaces:

$$\mathsf{F}_i:\ (u,v^\bullet) \longmapsto \iint\limits_{\omega_i} u(\alpha_i,\beta_i)\,v^\bullet(\alpha_i,\beta_i)\,\mathrm{d}\omega_i \quad (i=1,\,2),$$

where the functions on $\mathfrak{L}^\bullet$ are $(\sigma_1^\bullet,\sigma_2^\bullet)$-homogeneous.

Lemma 1. *The functional F_1 coincides with F_2 if and only if*

$$\sigma_1^\bullet - \sigma_2^\bullet = \sigma_2 - \sigma_1 - 4. \tag{3}$$

Доказательство. It was shown in [11] that for any point $x \in \Lambda$ there are a low triangular non-degenerate 2×2-matrix b_x and the point $\tilde{x}(\alpha_2,\beta_2)_x \in \omega_2$ such that $x = b_x\tilde{x}(\alpha_2,\beta_2)_x$, and $b_{11}=x_{11}$, $b_{21}=x_{21}$, $\alpha_2 = \frac{x_{12}}{x_{11}}$. In particular, for an arbitrary point (1) belonging to ω_1, we have

$$b_{\tilde{x}(\alpha_1,\beta_1)} = \begin{pmatrix} \cos\alpha_1 & 0 \\ \sin\alpha_1 & \sec\alpha_1 \end{pmatrix},$$

and, therefore, the correspondence $\tilde{x}(\alpha_1,\beta_1) \mapsto \tilde{x}(\alpha_2,\beta_2)_{\tilde{x}(\alpha_1,\beta_1)}$ is one-to-one. Since the operands $u\in\mathfrak{L}$ and $v^\bullet\in\mathfrak{L}^\bullet$ of the functional F_1 are (σ_1,σ_2)- and $(\sigma_1^\bullet,\sigma_2^\bullet)$-homogeneous, respectively, we have

$$u(\alpha_1,\beta_1)\,v^\bullet(\alpha_1,\beta_1) = (\cos\alpha_1)^{\sigma_1+\sigma_1^\bullet-\sigma_2-\sigma_2^\bullet}\,u(\alpha_2,\beta_2)\,v^\bullet(\alpha_2,\beta_2).$$

Considering that ω_1-coordinates depend on ω_2-coordinates according to the formulae

$$\alpha_1 = \arccos\left(\pm\frac{1}{\sqrt{1+\alpha_2^2}}\right), \quad \beta_1 = \arctan\frac{\alpha_2\cos\beta_2+\sin\beta_2}{\alpha_2\sin\beta_2-\cos\beta_2}, \quad \left|\frac{\partial(\alpha_1,\alpha_2)}{\partial(\alpha_2,\alpha_2)}\right| = (1+\alpha_2^2)^{-1},$$

we get

$$\mathsf{F}_1(u,v^\bullet) = \iint\limits_{\omega_1} u(\alpha_1,\beta_1)\, v^\bullet(\alpha_1,\beta_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\beta_1 = \iint\limits_{\omega_2} \frac{u(\alpha_2,\beta_2)\, v^\bullet(\alpha_2,\beta_2)\,\mathrm{d}\alpha_2\,\mathrm{d}\beta_2}{(1+\alpha_2^2)^{\sigma_1+\sigma_1^\bullet-\sigma_2-\sigma_2^\bullet+4}}.$$

It is clear that the equality $\mathsf{F}_1=\mathsf{F}_2$ is equivalent to $\sigma_1+\sigma_1^\bullet-\sigma_2-\sigma_2^\bullet+4=0$. □

Further we assume that representation spaces $\mathfrak{L}$ and $\mathfrak{L}^\bullet$ are *mutually assorted*, i.e., the pair $(\sigma_1^\bullet,\sigma_2^\bullet)$ for the representation space $\mathfrak{L}^\bullet$ is connected with the pair (σ_1,σ_2) for $\mathfrak{L}$ by the equality (3).

5. Matrix elements of the $B_1^\bullet \longrightarrow B_2^\bullet$ basis transformation

Let us express the function f_{p_2,q_2} as a linear combination of the functions belonging to the basis $B_2^\bullet$:

$$f^\bullet_{p_2,q_2}(x) = \sum_{p_1,q_1\in\mathbb{Z}} c_{p_1,q_1,p_2,q_2}\, f^\bullet_{p_1,q_1}(x). \tag{4}$$

In view of Lemma 1, we have

$$\mathsf{F}_2(f^\bullet_{p_2,q_2}, f_{\hat{p}_1,\hat{q}_1}) = \sum_{p_1,q_1\in\mathbb{Z}} c_{p_1,q_1,p_2,q_2}\,\mathsf{F}_1(f^\bullet_{p_1,q_1}, f_{\hat{p}_1,\hat{q}_1}).$$

Since

$$\mathsf{F}_1(f^\bullet_{p_1,q_1}, f_{\hat{p}_1,\hat{q}_1}) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \exp(\mathbf{i}[p_1+\hat{p}_1]\alpha_1)\,\exp(\mathbf{i}[q_1+\hat{q}_1]\beta_1)\,\mathrm{d}\alpha_1\,\mathrm{d}\beta_1,$$

we obtain

$$c_{p_1,q_1,p_2,q_2} = \frac{1}{4\pi^2}\,\mathsf{F}_2(f^\bullet_{p_2,q_2}, f_{-p_1,-q_1}).$$

We compute the matrix elements of the linear operator acting in $\mathfrak{L}^\bullet$ and transforming the basis $B_1^\bullet$ into $B_2^\bullet$, asserted by the following theorem.

Теорема 1. *Let* p_1, q_1, $q_2\in\mathbb{Z}$, $p_2\in\mathbb{R}\setminus\{0\}$, *and* $\mathrm{Re}(\sigma_1-\sigma_2)>-3$. *Then*

$$\begin{aligned} c_{p_1,q_1,p_2,q_2} &= 2^{\frac{\sigma_2-\sigma_1}{2}-3}\pi^{-1}\delta_{q_1,-q_2}\,|p_2|^{\frac{\sigma_1-\sigma_2}{2}+1} \\ &\times\left[\Gamma\left(\frac{\sigma_1-\sigma_2+(q_1-p_1)\operatorname{sign}p_2}{2}\right)\right]^{-1} W_{\frac{(q_1-p_1)\operatorname{sign}p_2}{2},\,\frac{\sigma_2-\sigma_1-3}{2}}(2|p_2|), \end{aligned} \tag{5}$$

where Γ *is the gamma function,* $W_{\mu,\nu}$ *is the Whittaker function of the second kind, and* $\delta_{s,t}$ *is the Kronecker symbol.*

Доказательство. Using iterated integrals for

$$\begin{aligned}\mathsf{F}_2(f_{p_2,q_2}, f^\bullet_{-p_1,-q_1}) = \mathbf{i}^{q_1+q_2}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi}(1+\alpha_2^2)^{\frac{\sigma_2-\sigma_1-p_1+q_1}{2}-2}(1-\mathbf{i}\alpha_2)^{p_1-q_1} \\ \times\exp\left(\mathbf{i}[p_2\alpha_2+(q_1+q_2)\beta_2]\right)\mathrm{d}\alpha_2\,\mathrm{d}\beta_2,\end{aligned}$$

we find that c_{p_1,q_1,p_2,q_2} can be expressed as an exponential Fourier transform:

$$\begin{aligned}\mathsf{F}_2(f_{p_2,q_2}, f^{\bullet}_{-p_1,-q_1}) =& 2\pi\, \delta_{q_1,-q_2} \int_{-\infty}^{\infty} (1+\mathbf{i}\alpha_2)^{\frac{\sigma_2-\sigma_1+q_1-p_1}{2}-2} \\ &\times (1-\mathbf{i}\alpha_2)^{\frac{\sigma_2-\sigma_1+p_1-q_1}{2}-2} \exp(\mathbf{i}p_2\alpha_2)\,\mathrm{d}\alpha_2.\end{aligned} \tag{6}$$

The integral in (6) can be evaluated by the following known formulae (see, e.g., [4, Entry 3.2.(12)])

$$\begin{aligned}\int_{-\infty}^{\infty} (\alpha+\mathbf{i}x)^{-2\mu}\,(\beta-\mathbf{i}x)^{-2\nu} \exp(-\mathbf{i}xy)\,\mathrm{d}x =& 2\pi(\alpha+\beta)^{-\nu-\mu}\,[\Gamma(2\nu)]^{-1} \\ &\times \exp\left(\frac{(\beta-\alpha)y}{2}\right) y^{\nu+\mu-1}\, W_{\nu-\mu,\frac{1}{2}-\nu-\mu}([\alpha+\beta]y)\end{aligned} \tag{7}$$

$$\left(\mathrm{Re}(\mu+\nu) > \frac{1}{2},\ \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\} > 0;\ y \in \mathbb{R}^+\right)$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} (\alpha+\mathbf{i}x)^{-2\mu}\,(\beta-\mathbf{i}x)^{-2\nu} \exp(-\mathbf{i}xy)\,\mathrm{d}x =& 2\pi(\alpha+\beta)^{-\mu-\nu}\,[\Gamma(2\mu)]^{-1} \\ &\times \exp\left(\frac{(\alpha-\beta)y}{2}\right) (-y)^{\nu+\mu-1}\, W_{\mu-\nu,\frac{1}{2}-\nu-\mu}(-[\alpha+\beta]y)\end{aligned} \tag{8}$$

$$\left(\mathrm{Re}(\mu+\nu) > \frac{1}{2},\ \min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta)\} > 0;\ y \in \mathbb{R}^-\right).$$

□

6. Matrix elements of the operator $T^{\bullet}(h^*)$ with respect to $B_2^{\bullet}$

For any $g \in SO(2,2)$, let $t^{\bullet}_{p_2,q_2,\hat{p}_2,\hat{q}_2}(g)$ be a matrix element of the linear operator $T^{\bullet}(g)$ with respect to the basis $B_2^{\bullet}$, that is,

$$T^{\bullet}(g)[f^{\bullet}_{p_2,q_2}] = \sum_{\hat{q}_2\in\mathbb{Z}} \int_0^{\infty} t^{\bullet}_{p_2,q_2,\hat{p}_2,\hat{q}_2}(g)\, f^{\bullet}_{\hat{p}_2,\hat{q}_2}\,\mathrm{d}\hat{p}_2. \tag{9}$$

In view of Lemma 1, we get

$$\begin{aligned}\mathsf{F}_i(T^{\bullet}(g)[f^{\bullet}_{p_2,q_2}], f_{\tilde{p}_2,\tilde{q}_2}) &= \sum_{\hat{q}_2\in\mathbb{Z}} \int_0^{\infty} t^{\bullet}_{p_2,q_2,\hat{p}_2,\hat{q}_2}(g)\, \mathsf{F}_2(f^{\bullet}_{\hat{p}_2,\hat{q}_2}, f_{\tilde{p}_2,\tilde{q}_2})\,\mathrm{d}\hat{p}_2 \\ &= 4\pi^2 \int_0^{\infty} t^{\bullet}_{p_2,q_2,\hat{p}_2,-\tilde{q}_2}(g)\, \delta(\hat{p}_2+\tilde{p}_2)\,\mathrm{d}\hat{p}_2,\end{aligned}$$

where $\delta(\hat{p}_2+\tilde{p}_2)$ is the $(-\tilde{p}_2)$-delayed Dirac delta function. We therefore have

$$t^{\bullet}_{p_2,q_2,\hat{p}_2,\hat{q}_2}(g) = \frac{1}{4\pi^2}\mathsf{F}_i(T^{\bullet}(g)[f^{\bullet}_{p_2,q_2}], f_{-\hat{p}_2,-\hat{q}_2}).$$

In Theorem 2, we show that the matrix elements $t^{\bullet}_{p_2,q_2,\hat{p}_2,\hat{q}_2}(h^*)$ can be described in terms of either Bessel–Clifford functions of the first kind

$$\mathcal{C}_{\nu}(z) = z^{-\frac{\nu}{2}}\, J_{\nu}\left(2\sqrt{z}\right) \tag{10}$$

or modified Bessel-Clifford functions of the second kind

$$\mathcal{K}_{\nu}(z) = z^{-\frac{\nu}{2}}\, K_{\nu}\left(2\sqrt{z}\right) \tag{11}$$

depending on $\mathrm{sign}(p_2\hat{p}_2)$ in both cases (see [2]). Here J_ν and K_ν are Bessel functions of the first kind and modified Bessel functions of the second kind, respectively, (see, e.g., [13, Chapter 9]).

Теорема 2. *Let* $p_2, \hat{p}_2 \in \mathbb{R} \setminus \{0\}$, $q_2, \hat{q}_2 \in [0, 2\pi)$, *and* $2 < \mathrm{Re}(\sigma_2 - \sigma_1) < 4$. *Then*

$$t^{\bullet}_{p_2,q_2,\hat{p}_2,\hat{q}_2}(h^*) = -\frac{2\,\mathbf{i}^{q_2}\,\delta_{q_2,\hat{q}_2}}{\pi}\,|p_2|^{\sigma_2-\sigma_1-3}\,\sin\frac{(\sigma_2-\sigma_1)\pi}{2}\,\mathcal{K}_{\sigma_2-\sigma_1-3}(-p_2\hat{p}_2) \qquad (12)$$

$$\left(p_2\,\hat{p}_2 \in \mathbb{R}^-\right)$$

and

$$\begin{aligned} t^{\bullet}_{p_2,q_2,\hat{p}_2,\hat{q}_2}(h^*) &= \frac{\mathbf{i}^{q_2}\,\delta_{q_2,\hat{q}_2}}{2}\,\cos\frac{(\sigma_2-\sigma_1)\pi}{2} \\ &\times \left[|\hat{p}_2|^{\sigma_1-\sigma_2+3}\mathcal{C}_{\sigma_1-\sigma_2+3}(p_2\hat{p}_2) - |p_2|^{\sigma_2-\sigma_1-3}\mathcal{C}_{\sigma_2-\sigma_1-3}(p_2\hat{p}_2)\right] \end{aligned} \qquad (13)$$

$$\left(p_2\,\hat{p}_2 \in \mathbb{R}^+\right).$$

Доказательство. Since the right shift of the subset ω_2 by the matrix h^* permutates the first and the second columns of any point belonging to ω_2, we have

$$T(h^*)[f_{p_2,q_2}(x)] = |x_{12}|^{\sigma_1-\sigma_2}\,|\Delta_{1,2}|^{\sigma_2-q_2}\,(\Delta_{2,3}+\mathbf{i}\Delta_{2,4})^{q_2}\,\exp\left(-\frac{\mathbf{i}p_2x_{11}}{x_{12}}\right).$$

In view of Lemma 1, the $T^{\bullet}(h^*)$-image of the restriction of $f^{\bullet}_{p_2,q_2}$ to ω_2 is given as follows:

$$T^{\bullet}(h^*)[\tilde{f}_{p_2,q_2}](\alpha_2,\beta_2) = (-1)^{q_2}\,|\alpha_2|^{\sigma_2-\sigma_1-4}\,\exp(\mathbf{i}q_2\beta_2)\,\exp\left(-\frac{\mathbf{i}p_2}{\alpha_2}\right).$$

We obtain

$$\begin{aligned} t^{\bullet}_{p_2,q_2,\hat{p}_2,\hat{q}_2}(h^*) &= \frac{1}{4\pi^2}\mathsf{F}_2(T^{\bullet}(h^*)[f^{\bullet}_{p_2,q_2}], f_{-\hat{p}_2,-\hat{q}_2}) \\ &= \frac{\mathbf{i}^{2q_2-\hat{q}_2}}{4\pi^2}\int_{-\pi}^{\pi}\exp(\mathbf{i}[q_2-\hat{q}_2]\beta_2)\,\mathrm{d}\beta_2\int_{-\infty}^{+\infty}|\alpha_2|^{\sigma_2-\sigma_1-4}\,\exp\left(-\mathbf{i}\left[\hat{p}_2\alpha_2+\frac{p_2}{\alpha_2}\right]\right)\,\mathrm{d}\alpha_2. \end{aligned}$$

Considering here the principle value of the last integral and using the following known formulae (see, e.g., [9, Entires 2.5.24.4 and 2.5.24.7])

$$\int_0^{\infty} x^{\alpha-1}\,\cos\left(ax+\frac{b}{x}\right)\,\mathrm{d}x = \frac{\pi}{2}\left(\frac{b}{a}\right)^{\frac{\alpha}{2}}\,\sin\left(\frac{\alpha\pi}{2}\right)\,\left[J_{-\alpha}(2\sqrt{ab}) - J_{\alpha}(2\sqrt{ab})\right]$$

and

$$\int_0^{\infty} x^{\alpha-1}\,\cos\left(ax-\frac{b}{x}\right)\,\mathrm{d}x = 2\left(\frac{b}{a}\right)^{\frac{\alpha}{2}}\,\cos\left(\frac{\alpha\pi}{2}\right)\,K_{\alpha}(2\sqrt{ab})$$

$$\left(a,\, b \in \mathbb{R}^+;\ |\mathrm{Re}(\alpha)| < 1\right),$$

with the aid of (10) and (11), we complete the proof. □

7. Matrix elements of the operator $T^{\bullet}(h^*)$ with respect to the mixed basis $f_{p_2,q_2}|f_{p_1,q_1}$

From (2) and (9), we find

$$T^{\bullet}(g)[f^{\bullet}_{p_2,q_2}] = \sum_{p_1,q_1\in\mathbb{Z}}\left(\sum_{\hat{q}_2\in\mathbb{Z}}\int_{-\infty}^{\infty} t^{\bullet}_{p_2,q_2,\hat{p}_2,\hat{q}_2}(g)\,c_{p_1,q_1,\hat{p}_2,\hat{q}_2}\,\mathrm{d}\hat{p}_2\right) f^{\bullet}_{p_1,q_1}. \qquad (14)$$

The expression in the brackets in (14) can be characterised as a matrix element of the operator $T^\bullet(h^*)$ with respect to the so-called mixed basis $f_{p_2,q_2}|f_{p_1,q_1}$ (see, e.g., [3, p. 204]).

On the other hand, these matrix elements may be obtained in the following way. It is not hard to check that the linear subspace $\operatorname{span}(\tilde{f}_{p_1,q_1}, \tilde{f}_{2q_1-p_1,q_1})$ in $\mathfrak{L}_1$ is invariant with respect to the linear operator $T^\bullet(h^*)$, namely, in view of (2),

$$T^\bullet(h^*)[f^\bullet_{p_1,q_1}|_{\omega_1}] = T(h^*)[\tilde{f}_{p_1,q_1}] = \mathbf{i}^{p_1+q_1}\,\exp(\mathbf{i}[p_1\alpha_1 - q_1\beta_1) = \mathbf{i}^{p_1}\,f^\bullet_{p_1,q_1}|_{\omega_1}. \tag{15}$$

Теорема 3. *Let $p_1, q_1 \in \mathbb{Z}$ such that $p_1 + q_1 \equiv 0 (\mathrm{mod}\, 2)$, $p_2 \in \mathbb{R}^+$, and $1 < \mathrm{Re}(\theta) < 2$. Then*

$$\begin{aligned}\int_0^\infty \Big[&\frac{1}{2}\cos(\theta\pi)\, p_2^{\theta-1}\,\hat{p}_2^{4-3\theta}\,\mathcal{C}_{3-2\theta}(p_2\hat{p}_2)\,W_{\frac{q_1-p_1}{2},\theta-\frac{3}{2}}(2\hat{p}_2)\\ &- \frac{1}{2}\cos(\theta\pi)\, p_2^{3\theta-4}\,\hat{p}_2^{1-\theta}\,\mathcal{C}_{2\theta-3}(p_2\hat{p}_2)\,W_{\frac{q_1-p_1}{2},\theta-\frac{3}{2}}(2\hat{p}_2)\\ &+ (-1)^\theta\,\frac{2\,\Gamma\left(\frac{q_1-p_1}{2}-\theta\right)}{\pi\,\Gamma\left(\frac{p_1-q_1}{2}-\theta\right)}\,\sin(\theta\pi)\,p_2^{3\theta-4}\,\hat{p}_2^{1-\theta}\,\mathcal{K}_{2\theta-3}(p_2\hat{p}_2)\,W_{\frac{p_1-q_1}{2},\theta-\frac{3}{2}}(2\hat{p}_2)\Big]\,\mathrm{d}\hat{p}_2\\ = (-1)^{\frac{p_1+q_1}{2}}&\,W_{\frac{q_1-p_1}{2},\theta-\frac{3}{2}}(2p_2).\end{aligned}$$

Доказательство. From (14) and (15) we have

$$T^\bullet(h^*)[f^\bullet_{p_2,q_2}] = \sum_{p_1,q_1\in\mathbb{Z}} c_{p_1,q_1,p_2,q_2}\,T^\bullet(g)[f^\bullet_{p_1,q_1}] = \sum_{p_1,q_1\in\mathbb{Z}} \mathbf{i}^{p_1}\,c_{p_1,q_1,p_2,q_2}\,f^\bullet_{p_1,q_1}. \tag{16}$$

Since the matrix element c_{p_1,q_1,p_2,q_2} is equal to zero in case $q_1 \neq -q_2$ and the matrix element $t^\bullet_{p_2,q_2,\hat{p}_2,\hat{q}_2}(h^*)$ is equal to zero in case $q_2 \neq \hat{q}_2$, considering (14) and (16), we have

$$\int_{-\infty}^{\infty} t^\bullet_{p_2,-q_1,\hat{p}_2,-q_1}(h^*)\,c_{p_1,q_1,\hat{p}_2,-q_1}\,\mathrm{d}\hat{p}_2 = \mathbf{i}^{p_1}\,c_{p_1,q_1,p_2,-q_1}. \tag{17}$$

Using, for (16), the results from Theorems 1 and 2, and letting $\theta = \frac{\sigma_2-\sigma_1}{2}$, we complete the proof. □

8. Concluding Remarks

Setting $p_1 = q_1 = 0$ in (14) and considering the following relation between Macdonald functions K_ν and Whittaker functions $W_{0,\nu}$ (see, e.g., Entry [15, 7.8.8]):

$$K_\nu\left(\frac{z}{2}\right) = \left(\frac{\pi}{z}\right)^{\frac{1}{2}}\,W_{0,\nu}(z), \tag{18}$$

from the result in Theorem 3, we obtain the following integral formula for the K-transform (see [4]) of the linear combination of the Bessel–Clifford functions:

$$\begin{aligned}\int_0^\infty \Big[&\frac{1}{2}\cos(\theta\pi)\,p_2^{\theta-\frac{3}{2}}\,\hat{p}_2^{4.5-3\theta}\,\mathcal{C}_{3-2\theta}(p_2\hat{p}_2) - \frac{1}{2}\cos(\theta\pi)\,p_2^{3\theta-4.5}\,\hat{p}_2^{\frac{3}{2}-\theta}\,\mathcal{C}_{2\theta-3}(p_2\hat{p}_2)\\ &- \frac{2\,(-1)^\theta}{\pi}\,\sin(\theta\pi)\,p_2^{3\theta-4.5}\,\hat{p}_2^{\frac{3}{2}-\theta}\,\mathcal{K}_{2\theta-3}(p_2\hat{p}_2)\Big]\,K_{\theta-\frac{3}{2}}(\hat{p}_2)\,\mathrm{d}p_2 = K_{\theta-\frac{3}{2}}(p_2)\\ &\qquad\left(p_2\in\mathbb{R}^+,\ 1<\mathrm{Re}(\theta)<2\right).\end{aligned} \tag{19}$$

Some similar results to those in Theorem 3 and formula (19) can be obtained from (17) in case $p_2 \in \mathbb{R}^-$.

Using the following three integral transformations:

(i) The first Hankel-Clifford integral transform (see, e.g., [7, Eq. (2.7)])

$$\mathrm{H}_\sigma^{(1)}[f](\lambda) = \lambda^\sigma \int_0^\infty \mathcal{C}_\sigma(\lambda\,\hat{\lambda})\, f(\hat{\lambda})\, \mathrm{d}\hat{\lambda} \quad \left(\lambda \in \mathbb{R}^+\right);$$

(ii) The second Hankel-Clifford integral transform (see, e.g., [5]; see also [7, Eq. (2.9)])

$$\mathrm{H}_\sigma^{(2)}[f](\lambda) = \int_0^\infty \hat{\lambda}^\sigma\, \mathcal{C}_\sigma(\lambda\,\hat{\lambda})\, f(\hat{\lambda})\, \mathrm{d}\hat{\lambda} \quad \left(\lambda \in \mathbb{R}^+\right);$$

(iii) The Macdonald-Clifford transform (see [12])

$$\mathrm{K}_\sigma[f](\lambda) = \int_0^\infty \hat{\lambda}^\sigma\, \mathcal{K}_\sigma(\lambda\,\hat{\lambda})\, f(\hat{\lambda})\, \mathrm{d}\hat{\lambda} \quad \left(\lambda \in \mathbb{R}^+\right),$$

we can rewrite the identity in Theorem 3 in the following form:

$$\begin{aligned}
&\frac{\cos(\theta\pi)}{2}\left[\mathrm{H}^{(1)}_{3-2\theta}\left[\hat{p}_2^{4-3\theta}\, W_{\frac{q_1-p_1}{2},\theta-\frac{3}{2}}(2\hat{p}_2)\right](p_2) - \mathrm{H}^{(2)}_{2\theta-3}\left[\hat{p}_2^{4-3\theta}\, W_{\frac{q_1-p_1}{2},\theta-\frac{3}{2}}(2\hat{p}_2)\right](p_2)\right]\\
&\quad + \frac{2\,(-1)^\theta\,\Gamma\left(\frac{q_1-p_1}{2}-\theta\right)}{\pi\,\Gamma\left(\frac{p_1-q_1}{2}-\theta\right)}\,\mathrm{K}_{2\theta-3}\left[\hat{p}_2^{4-3\theta}\, W_{\frac{p_1-q_1}{2},\theta-\frac{3}{2}}(2\hat{p}_2)\right](p_2)\\
&\quad = (-1)^{\frac{p_1+q_1}{2}}\, p_2^{4-3\theta}\, W_{\frac{p_1-q_1}{2},\theta-\frac{3}{2}}(2p_2).
\end{aligned}$$

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