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О периодической части группы Шункова, насыщенной линейными группами степени 2 над конечными полями четной характеристики [1]

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Аннотация

Понятие насыщенности, введенное в конце прошлого века, оказалось плодотворным при изучении бесконечных групп. Было получено описание различных классов бесконечных групп с различными вариантами насыщающих множеств. В частности, было установлено, что периодические группы с насыщающим множеством, состоящим из конечных простых неабелевых групп лиева типа, ранги которых ограничены в совокупности, есть в точности локально конечные группы лиева типа над подходящим локально конечным полем. Естественным шагом в дальнейших исследованиях был отказ от условия периодичности на исследуемую группу и отказ от структуры насыщающего множества, как множества, состоящего из конечных простых неабелевых групп лиева типа, ранги которых ограничены в совокупности. В настоящей работе рассматриваются смешанные группы (т.е. группы которые содержат как элементы конечного порядка, так и элементы бесконечного порядка) Шункова.

Хорошо известно, что группа Шункова не обязана обладать периодической частью (т.е. множество элементов конечного порядка в группе Шункова не обязательно является группой). В качестве насыщающего множества рассматривается множество полных линейных групп степени 2 над конечными полями четной характеристики. Отсутствие аналогов известных результатов В. Д. Мазурова о периодических группах с абелевыми централизаторами инволюций долгое время не позволяло установить структуру группы Шункова с упомянутым выше насыщающим множеством. В данной работе эту трудность удалось преодолеть. Доказывается, что группа Шункова, насыщенная полными линейными группами степени 2 над конечными полями характеристики 2, локально конечна и изоморфна полной линейной группе степени 2 над подходящим локально конечным полем характеристики 2.

Ключевые слова: Группа Шункова; группы, насыщенные заданным множеством групп.

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On the periodic part of the Shunkov group saturated with linear groups of degree 2 over finite fields of even characteristic[2]

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Abstract

The definition of saturation condition was formulated at the end of the last century. Saturation condition has become useful in study of infinite groups. A description of various classes of infinite groups with various variants of saturating sets was obtained. In particular, it was found that periodic groups with a saturating set consisting of finite simple non-Abelian groups of Lie type, under the condition that ranks of groups in saturation set are bounded in the aggregate, are precisely locally finite groups of Lie type over a suitable locally finite field. A natural step in further research was the rejection of the periodicity condition for the group under study, and the rejection of the structure of the saturating set as a set consisting of finite simple non-Abelian groups of Lie type with ranks bounded in the aggregate. In this paper, we consider mixed Shunkov groups (i.e., groups that contain both elements of finite order and elements of infinite order).

It is well known that the Shunkov group does not have to have a periodic part (i.e., the set of elements of finite order in the Shunkov group is not necessarily a group). As a saturating set, we consider the set of full linear groups of degree 2 over finite fields of even characteristic. The lack of analogues of known results V. D. Mazurova on periodic groups with Abelian centralizers of involutions for a long time did not allow us to establish the structures of the Shunkov group with the saturation set mentioned above. In this paper, this difficulty was overcome. It is proved that a Shunkov group saturated with full linear groups of degree 2 is locally finite and isomorphic to a full linear group of degree 2 over a suitable locally finite field of characteristic 2.

Keywords: Shunkov group, groups saturated with given set of groups.

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1. Introduction

Let $\mathfrak{X}$ be a set of groups. A group G is saturated with groups from the set $\mathfrak{X}$ if any finite subgroup of G contained in a subgroup of the group G, is isomorphic to a group from $\mathfrak{X}$. The set $\mathfrak{X}$ will be called *saturation set* for G [9]. Let G be a group. If all elements of finite order from G are contained in a periodic subgroup of G, then it is called the periodic part of G and denoted by $T(G)$ [3, p. 90]. Recall that a group G is called the Shunkov group if for any finite subgroup H

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of G in the factor group $N_G(H)/H$ any two conjugate elements of a prime order generate a finite group [5]. Note that the Shunkov group is not required to have a periodic part [6].

Groups with saturation conditions studied in following works: [8, 14, 15, 12, 1, 2]. In paticular in [4] it is proved that the Shunkov periodic group G saturated with the groups $GL_2(p^n)$ (neither the characteristic of the field p nor the natural n are fixed) is locally finite and isomorphic to $GL_2(P)$, where P is a suitable locally finite field. For a long time, this result could not be transferred to the entire class of Shunkov groups (without failure of periodicity) due to the lack of a description of Shunkov groups with Abelian centralizers of involutions. In this study, for the case $p = 2$, this difficulty is circumvented. The following result is proved.

Theorem. *Let the Shunkov group G be saturated with groups from the set*

$$\mathfrak{M} = \{GL_2(2^n) \mid n = 1, 2, \ldots\}.$$

Then G has a periodic part $T(G)$ that is isomorphic to $GL_2(Q)$ for a suitable locally finite field Q of characteristic 2

2. Definitions, known facts, auxiliary statements

Definition 1. Let G be a group, K be a subgroup of G, $\mathfrak{X}$ be a set of groups. By $\mathfrak{X}_G(K)$ we denote the set of all subgroups of G containing K and isomorphic to groups from $\mathfrak{X}$. If 1 is the identity subgroup of G, then $\mathfrak{X}_G(1)$ will denote the set of all subgroups of G, isomorphic to groups from $\mathfrak{X}$. If it is clear from the context which group we are talking about, then instead of $\mathfrak{X}_G(K)$ we will write $\mathfrak{X}(K)$, and instead of $\mathfrak{X}_G(1)$ we will write $\mathfrak{X}(1)$. [9]

Proposition 1. *A finite invariant set of elements of finite order in any group generates a finite normal subgroup* [3].

Proposition 2. *Shunkov periodic group G, saturated with groups from the set* Im *consisting of all groups $\{GL_2(p^n)\}$ (here p and n are not fixed), is isomorphic to $GL_2(Q)$ for suitable locally finite field* [4].

Proposition 3. *A Shunkov group with an infinite number of elements of finite order has an infinite locally finite subgroup* [10, Lemma 1].

Proposition 4. *Let G be a Shunkov group, a be an element of prime order from G, x be an involution from G. Then $\langle x, a\rangle$ is the finite group* [7, proposition 4].

Proposition 5. *Let G be a Shunkov group and H be a finite normal subgroup of G. Then the factor group $\overline{G} = G/H$ is a Shunkov group* [7, proposition 5].

Proposition 6. *If in a Shunkov group G some Sylow* 2 *-subgroup is finite, then all Sylow* 2 *-subgroups of G are finite and conjugate* [7, Proposition 9].

Proposition 7. *The Shunkov group G, in which all finite subgroups are Abelian, has an Abelian periodic part of $T(G)$* [7, proposition 7].

Proposition 8. *Let G be a Shunkov group saturated with wreathed groups. Then G has the periodic part $T(G) = (A \times B) \leftthreetimes \langle v\rangle$, where $A^v = B$, A is a locally cyclic group and $|v| = 2$* [11].

Proposition 9. *Let $G = L_2(q)$, where $q = 2^n > 2$, P is a Sylow* 2 *-subgroup of the group G. Then* :

1. *P is an elementary Abelian group, and any two different Sylow* 2 *-subgroups of group G intersect trivially.*

2. $C_G(a) = P$ *for any involution* $a \in P$.

3. $N_G(P) = P \leftthreetimes H$ *is the maximum subgroup of* G, *which is a Frobenius group with kernel* P *and a cyclic complement* H *of order* $q-1$, *acting transitively on the set* $P \setminus \{1\}$.

4. $N_G(H)$ *is a dihedral group of order* $2(q-1)$.

5. *If* K *is a subgroup of* G *and* K *has a nontrivial normal subgroup of odd order, then* $N_G(K)$ *is a dihedral group of order* $2(q-1)$ *or* $2(q+1)$ [13].

Proposition 10. *Let* $L = GL_2(2^n)$. *Then :*

1. $L = L_2(2^n) \times Z$, *where* $Z = \langle \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix} \right) \rangle$ *is the center of the group* L, $\alpha \in GF(q)$.

2. $R = \{ \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix} \right), \alpha \in GF(2^n) \}$ *is a Sylow* 2 *-subgroup of the group* L, $N_L(R) = R \leftthreetimes D$, *where* $D = \langle \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix} \right) \rangle$ *- a subgroup of diagonal matrices of the group* L, $\beta \in GF(2^n)$, $D = Z \times T$, $T = \langle \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix} \right) \rangle$, *and* $|Z| = |T| = 2^n - 1$.

3. $PGL_2(p^n) = L/Z = L_2(2^n)$ [13].

3. Proof

Suppose the theorem is false. Further, G is a counterexample to the statement of the theorem.

Lemma 1. *G is not a periodic group.*

Proof. Assume the opposite. Then $T(G) = G$ and by the proposition 2 G is isomorphic to $L_2(Q)$ for a suitable locally finite field Q of characteristic 2. Hence the lemma is proved.

Lemma 2. *G contains infinitely many elements of finite order..*

Proof. Assume the opposite. Then by the proposition 1, the group G has a finite periodic part $T(G)$, which is isomorphic, according to the saturation condition, to the group $L_2(2^m)$ for a suitable m. This contradicts the fact that G is a counterexample. Hence the lemma is proved.

Lemma 3. *All involutions from the group G are conjugate.*

Proof. Let x, y be two different involutions from G. By the proposition 4 $\langle x, y \rangle$ is a finite group. By saturation condition $\langle x, y \rangle < M < G$ and $M \in \mathfrak{M}(1)$. In this case, $M \simeq GL_2(2^l)$ for a suitable positive integer l. According to the propositions 10, 9 $x = y^g$ for some $g \in M$. Hence the lemma is proved.

Lemma 4. *Let S be a Sylow 2 -subgroup of G. Then S is an infinite elementary Abelian 2 -group.*

Proof. By Lemma 2 and Proposition 3 the group G contains an infinite locally finite subgroup L. Therefore, for any positive integer n in the group L there exists a finite subgroup K_n such that $|K_n| > n$. By saturation condition $K_n < M_n < G$ and $M_n \in \mathfrak{M}(1)$. In this case $M_n \simeq GL_2(2^l)$ for a suitable positive integer l. By the proposition 10 we have $|M_n| = (2^{2l} - 1)2^l(2^l + 1)$. Thus $n < 2^{3l}$, $(ln_2 n)/3 < l$, l can be arbitrarily large due to the arbitrariness of n and in the group G there is a finite 2-group of arbitrarily large order. Hence, using the proposition 6, S is an infinite group.

Now let s be an arbitrary element of S. By saturation condition $s \in M_n < G$ and $M_n \in \mathfrak{M}(1)$. In this case $M_n \simeq GL_2(2^l)$ for a suitable positive integer l. By the proposition 10 $|s| = 2$. Thus, S is an infinite Abelian group of period 2. Hence the lemma is proved.

Let's fix the group S of the statement of the lemma 4.

Lemma 5. *Let R be a Sylow 2-subgroup of G other than S. Then $S \cap R = 1$.*

Proof. Assume the opposite and let $1 \neq z \in S \cap R$. By the lemma, 4 z is an involution. Let's take the involution $x \in S \backslash R$ and the involution $y \in R \backslash S$. By the proposition 4 $\langle x, y\rangle$ is a finite group from $C_G(z)$ (Lemma 4). Therefore, $\langle z, x, y\rangle$ is a finite group. By saturation condition $\langle z, x, y\rangle < M < G$ and $M \in \mathfrak{M}(1)$. In this case $M \simeq GL_2(2^l)$ for a suitable positive integer l. According to the propositions 10, 9 we have $xy = yx$. Since the choice of involutions x, y is arbitrary, we conclude that the groups S, R are elementwise permutable. Therefore, SR is an elementary Abelian 2-group. By the lemma 4 $S = SR = R$. Which is a contradiction with the condition of the lemma. Hence the lemma is proved.

Lemma 6. *Let R be a Sylow 2 -subgroup of G other than S. Then $S = R^g$ for some $g \in G$.*

Proof. Let us take the involution $x \in S$ and the involution $y \in R$. By the lemma 3 $x = y^g$ and $x \in S \cap R^g$ for some $g \in G$. By the lemma 5 $S = R^g$. Hence the lemma is proved.

Lemma 7. *$N_G(S)$ has a countable periodic part $T = T(N_G(S))$.*

Proof. 1. Let $1 \neq K$ be a finite subgroup of S. By saturation condition $K < M < G$ and $M \in \mathfrak{M}(1)$. In this case $M \simeq GL_2(2^l)$ for a suitable positive integer l. Let S_M be a Sylow 2-subgroup of M containing K. By the lemma 4 $S_M < S$, $N_M(S_M) < N_G(S)$. According to the propositions 10 (clause 1.), 9 (clause 3.) $N_M(S_M)$ contains a finite subgroup $R_{S_M} = V_{S_M} \times Z_M$ of order $(2^l-1)^2$ such that $N_M(S_M) = S_M \leftthreetimes R_M$, $S_M \leftthreetimes V_{S_M}$ is a Frobenius group with kernel S_M and cyclic complement V_{S_M} of order $2^l - 1$ acting transitively on the set $S_M \setminus 1$, $Z_M = Z(M)$ is a cyclic group of order $2^l - 1$. Thus, $N_G(S)$ contains finite subgroups of an arbitrarily large odd order due to the fact that l can be arbitrarily large.

Let us consider the factor group $\overline{N} = N_G(S)/S$. Since $\pi(S) \cap \pi(\overline{N}) = \varnothing$, then $\overline{N}$ is a Shunkov group (by the proposition 5). Let us show that all finite subgroups of $\overline{N}$ are Abelian. Let $\overline{K}$ be a finite subgroup of $\overline{N}$ and K be some finite inverse image of $N_G(S)$ such that $S \cap K \neq 1$. By saturation condition $K < M < G$ and $M \in \mathfrak{M}(1)$. In this case $M \simeq GL_2(2^l)$ for a suitable positive integer l. Let S_M be a Sylow 2-subgroup of M containing K. By the lemma 4 $S_M < S, N_M(S_M) < N_G(S)$. According to the propositions 10 (clause 1.), 9 (clause 3.) $N_M(S_M)$ contains a finite subgroup $R_{S_M} = V_{S_M} \times Z_M$ of order $(2^l-1)^2$ such that $N_M(S_M) = S_M \leftthreetimes R_M$, $S_M \leftthreetimes V_{S_M}$ is a Frobenius group with kernel S_M and cyclic complement V_{S_M} of order $2^l - 1$ acting transitively on the set $S_M \setminus 1$, $Z_M = Z(M)$ is a cyclic group of order $2^l - 1$. Hence,

$$\overline{K} = KS/S < SN_M(S_M)/S = SR_M/S = S(V_{S_M} \times Z_M)/S =$$

$$= SV_{S_M}/S \times SZ_M/S = \overline{V_{S_M}} \times \overline{Z_M}$$

is an Abelian group, as required. It is clear that in this case $\overline{N}$ is saturated with finite Abelian groups and, by the proposition 7 $\overline{N}$, has the periodic part $T(\overline{N})$ which is an Abelian group.

Let us show that for any $p \in \pi(T((\overline{N}))$, $p - \mathrm{rank} T(\overline{N})$ equal to 2. Let's assume the opposite, for some p-subgroup of $p \in \pi(T(\overline{N}))$, $\overline{K} = \langle\overline{a}\rangle \times \langle\overline{b}\rangle \times \langle\overline{c}\rangle$ is an elementary Abelian p-subgroup of $T(\overline{N})$. Denote by K some finite inverse image of K in the group $T(N_G(S)$ such that $S \cap K \neq 1$. By saturation condition $K < M < G$ and $M \in \mathfrak{M}(1)$. In this case $M \simeq GL_2(2^l)$ for a suitable positive integer l. Let S_M be a Sylow 2-subgroup of M such that $N_M(S_M)$ contains K. By the lemma 4 we have $S_M < S, N_M(S_M) < N_G(S)$. Therefore, $N_M(S_M)$ contains elementary Abelian p-subgroup $\langle a\rangle \times \langle b\rangle \times \langle c\rangle$ of order p^3, which is impossible due to the propositions 10 (clause 1.), 9 (clause 3.)

Since an Abelian locally finite group is a direct product of its cyclic subgroups, $T(\overline{N})$ is a countable group. Let's take two different involutions x, y in S. By saturation condition $\langle x, y\rangle < M \in \mathfrak{M}(1)$. Let S_M be a Sylow 2-subgroup of M such that $N_M(S_M)$ contains $\langle x, y\rangle$. Clearly, $S_M < S$. By 10 (clause 1.), 9 (clause 3.) there is an element a in $N_M(S_M)$ such that $x^{sa} = y$ for any $s \in S$. Hence, taking into account the countability of $T(\overline{N})$, follows the countability of the groups S, T. Hence the lemma is proved.

Lemma 8. *In G there exists an infinite sequence of subgroups*

$$G_1, G_2, \cdots, G_n, \cdots,$$

with the following properties.

A. $G_n \simeq GL_2(2^{l_n})$ *and* l_n *divides* l_{n+1}

B. $T \cap G_n = N_{G_n}(S_n) = S_n \leftthreetimes (Z_n \times V_n)$, *where* S_n *is a Sylow* 2*-subgroup of* G_n, $Z_n = Z(G_n)$,

C. $N_{G_n}(S_N) < N_{G_{n+1}}(S_{n+1})$.

D.

$$T = \bigcup_{n=1}^{\infty} N_{G_n}(S_n) = \bigcup_{n=1}^{\infty} (S_n \leftthreetimes (Z_n \times V_n)) = S \leftthreetimes (Z \times V),$$

where

$$Z = \bigcup_{n=1}^{\infty} Z_n$$

is a locally cyclic subgroup of T generated by all elements of odd orders centralising at least one involution from S.

$$V = \bigcup_{n=1}^{\infty} V_n$$

is a locally cyclic subgroup of T which is isomorphic to the group Z and acts transitively on S.

Proof. By the lemma, 7 T is a countable group. We number the elements of the group T with elements of the natural series: $T = \{t_1 \cdots t_n \cdots\}$ and let's assume that $1 \neq t_1 \in S$. By saturation condition $\langle t_1 \rangle < M < G$ and $M \in \mathfrak{M}(1)$. Let S_M be a Sylow 2-subgroup of M containing $\langle t_1 \rangle$. By the lemma 4 we have $S_M < S, N_M(S_M) < T$. Let $M = G_1$. It is clear that $G_1 \simeq GL_2(2^{l_1})$ and $G_1 \cap T = N_{G_1}(S_1) = S_1 \leftthreetimes (Z_1 \times V_1)$, where $S_M = S$ is a Sylow 2-subgroup of G_1, $Z_1 = Z(G_1)$, V_1 is a cyclic subgroup of $N_{G_1}(S_1)$ acting regularly and transitively on S_1. (Propositions 10 (clause 1.), 9 (clause 3.)

Suppose that for $n \geqslant 1$ a subgroup G_n satisfying the conclusion of the lemma is constructed. Let t_m be an element from $T \setminus N_{G_n}(S_n)$ with the smallest possible index value of m. By the saturation condition, the finite group $\langle t_m, N_{G_n}(S_n) \rangle < M < G$ and $M \in \mathfrak{M}(1)$. Let S_M be a Sylow 2-subgroup of M containing $\langle t_m, S_n \rangle$. By the lemma 4 we have $S_M < S, N_M(S_M) < T$. Let $M = G_{n+1}$. It is clear that $G_{n+1} \simeq GL_2(2^{l_{n+1}})$ and $G_{n+1} \cap T = N_{G_{n+1}}(S_{n+1}) = S_{n+1} \leftthreetimes (Z_{n+1} \times V_{n+1})$, where $S_M = S_{n+1}$ is a Sylow 2-subgroup of G_{n+1}, $Z_{n+1} = Z(G_{n+1})$, V_{n+1} is a cyclic subgroup of $N_{G_{n+1}}(S_{m+1})$ acting regularly and transitively on S_{n+1}. (Propositions 10 (clause 1.), 9 (clause 3.) By construction, the points A,B,C,D hold. The lemma is proved.

In the notation of the lemma 8, the following statement holds.

Lemma 9. $N_G(Z_n \times V_n)$ *has a periodic part of* $T(N_G(Z_n \times V_n))$ *and*

$$T(N_G(Z_n \times V_n)) = T(N_G(Z \times V)) = (Z \times V) \leftthreetimes \langle w_n \rangle,$$

where w_n involution from G_n, such that for any $z \in Z, z^{w_n} = z$ and for any $v \in V, v^{w_n} = v^{-1}$.

Proof. Let's show that the group $N_G(Z_n \times V_n)$ is saturated with finite wreathed groups. Let K be a finite subgroup of $N_G(Z_n \times V_n)$. By the saturation condition, the finite group $\langle (Z_n \times V_n), K \rangle < M \in \mathfrak{M}(\langle (Z_n \times V_n), K \rangle)$. According to the propositions 10 (point 1.), 9 (point 3.) $N_M(Z_n \times V_n) = (Z_M \times V_M) \leftthreetimes \langle w \rangle$, where $Z_M = Z(M)$, $Z_n < Z_M$, $V_n < V_M$ and for any $x \in V_M, x^w = x^{-1}$. Let $Z_M = \langle x \rangle$, $V_M = \langle y \rangle$. Then $N_M(Z_n \times V_n) = (\langle xy \rangle \times \langle x^{-1}y \rangle) \leftthreetimes \langle w \rangle$ is an wreathed group. Due to randomness of choice of K as a finite subgroup of $N_G(Z_n \times V_n)$ the saturation of the latter with finite wreathed groups is proved. By the lemma 8 we have $(Z \times V) < N_G(Z_n \times V_n)$, therefore, by the proposition 8 $T(N_G(Z_n \times V_n)) = (Z \times V) \leftthreetimes \langle w_n \rangle$. The lemma is proved.

Lemma 10. *In G there exists an infinite sequence of subgroups*

$$M_1, M_2, \cdots, M_n, \cdots$$

with the following properties.

a. $M_n \simeq G_n$.

b. $M_n \cap G_n = (Z_n \times V_n)$.

c. $N_{M_n}(Z_n \times V_n) = (Z_n \times V_n) \leftthreetimes \langle w \rangle) < N_{M_{n+1}}(Z_{n+1} \times V_{n+1}) = (Z_{n+1} \times V_{n+1} \leftthreetimes \langle w \rangle)$, *where w is an involution from $T(N_G(Z_n \times V_n))$.*

Proof. Let's fix some involution $w_{n_0} \in T(N_G(Z_{n_0} \times V_{n_0})) = T(N_G(Z \times V)) = (Z \times V) \leftthreetimes \langle w_{n_0} \rangle$. Let $w = w_{n_0}$, $R = (Z \times V) \leftthreetimes \langle w \rangle$. By the lemma 9, all involutions in R are conjugate. Therefore, for any n in R there is an element x_n such that $w_n^{x_n} = w_n$. Let $M_n = G_n^{x_n}$. By construction sequence

$$M_1, M_2, \cdots, M_n, \cdots,$$

has the properties a.,b.,c. Hence the lemma is proved.

Lemma 11. *Subgroup sequence*

$$M_1, M_2, \cdots, M_n, \cdots,$$

forms a chain

$$M_1 < M_2 < \cdots < M_n < \cdots.$$

Proof. Denote by S_n a Sylow 2-subgroup of M_n which contains the involution w. Since the involution w lies in each of the groups M_n (by the lemma 10) and the order of the group M_n divides the order of the group M_{n+1}, then $S_n < S_{n+1}$. Hence, the group M_{n+1} contains two different Sylow 2-subgroups as subgroups S_n and $S_n^v (1 \neq v \in V_n)$ of M_n, and group $N_{M_n}(Z_n \times V_n) = (Z_n \times V_n) \leftthreetimes \langle w \rangle)$. By the propositions 10 (clause 1.), 9 (clause 3.) $M_n = \langle S_n, S_n^v, N_{M_n}(Z_n \times V_n)$. Hence, $M_n < M_{n+1}$. The lemma is proved.

Let us complete the proof of the theorem. By the proposition 2 and by the lemma 11

$$L = \bigcup_{n=1}^{\infty} M_n \simeq GL_2(Q)$$

for a suitable locally finite field Q of characteristic 2. By the lemma 1 $L < G$. Let us show that $L = T(G)$. Suppose that M contains all involutions from G. In this case, G contains the characteristic subgroup $L_1 \simeq L_2(Q)$, and all elements of finite order from $C_G(L_1)$ generate the subgroup Z and $L = L_1 \times Z$. Any element g of finite order from G is represented as $g = xz$, where $x \in L_1$, and $z \in Z$. Thus $L = T(G) \neq G$ (by the lemma 1).

Suppose that there is an involution $v \in G \setminus L$. Take the involution $k \in S < L$. By the saturation condition, finite group $\langle v, k \rangle < R \in \mathfrak{M}(\langle v, k \rangle)$. Denote by S_R a Sylow 2-subgroup of R, such that $k \in R$. By the lemmas 5, 7 $N_R(S_K) < T$. Therefore, for some positive integer n, $N_R(S_R) < N_{M_n}(S_n)$. But then $v \in R < M_n < M$. A contradiction with the choice of the involution v. Thus, all involutions from G lie in M, and in this case the statement of the theorem holds. The theorem is proved.

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