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# Обобщенная предельная теорема для периодической дзета-функции Гурвица

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**Аннотация**

С времен Бора и Йессена (1910–1935) в теории дзета-фуекций прмменяются вероятностные методы. В 1930 г. они доказали первую теорему для дзета-функции Римана $\zeta(s)$, $s=\sigma+it$, которая является прототипом современных предельных теорем, характеризующих поведение дзета-функции при помощи слабой сходимости вероятностных мер. Более точно, они получили, что при $\sigma>1$ существует предел

$$\lim_{T\to\infty}\frac{1}{T}\mathrm{J}\left\{t\in[0,T]:\log\zeta(\sigma+it)\in R\right\},$$

где $R$ – прямоугольник на комплексной плоскости со сторонами, паралельными осям, а $\mathrm{J}A$ обозначает меру Жордана множества $A\subset\mathbb{R}$. Два года спустя они распространили приведенный результат на полуплоскость $\sigma>\frac{1}{2}$.

Идеи Бора и Йессена были развиты в работах Винтнера, Борщсениуса, Йессена, Сельберга и других известных математиков. Современные версии теорем Бора-Йессена для широкого класса дзета-функций были получены в работах К. Матсумото.

В основном теория Бора-Йессена применялась для дзета-функций, имеющих эйлерово произведение по простым числам. В настоящей статье доказывается предельная теорема для дзета-функций, не имеющих эйлерова произведения и являющихся обобщением классичесской дзета-функции Гурвица. Пусть $\alpha$, $0<\alpha\leqslant 1$, фиксированный параметр, а $\mathfrak{a}=\{a_m : m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\}$ – периодическая последовательность комплексных чисел. Тогда периодическая дзета-функция Гурвица $\zeta(s,\alpha;\mathfrak{a})$ в полуплоскости $\sigma>1$ определяется рядом Дирихле

$$\zeta(s,\alpha;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m}{(m+\alpha)^s}$$

и мероморфно продолжается на всю комплексную плоскость. Пусть $\mathcal{B}(\mathbb{C})$ – борелевское $\sigma$-поле комплексной плоскости, meas$A$ – мера Лебега измеримого множества $A\subset\mathbb{R}$, а функция $\varphi(t)$ при $t\geqslant T_0$ имеет монотонную положительную производную $\varphi'(t)$, при $t\to\infty$ удовлетворяющую оценкам $(\varphi'(t))^{-1}=o(t)$ и $\varphi(2t)\max_{t\leqslant u\leqslant 2t}(\varphi'(u))^{-1}\ll t$. Тогда в статье получено, что при $\sigma>\frac{1}{2}$

$$\frac{1}{T}\mathrm{meas}\left\{t\in[0,T]:\zeta(\sigma+i\varphi(t),\alpha;\mathfrak{a})\in A\right\},\quad A\in\mathcal{B}(\mathbb{C}),$$

при $T\to\infty$ слабо сходится к некоторой в явном виде заданной вероятностной мере на $(\mathbb{C},\mathcal{B}(\mathbb{C}))$.

*Ключевые слова:* дзета-функция Гурвица, мера Хаара, периодическая дзета-функция Гурвица, предельная теорема, слабая сходимость.

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# A generalized limit theorem for the periodic Hurwitz zeta-function

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**Abstract**

Probabilistic methods are used in the theory of zeta-functions since Bohr and Jessen time (1910–1935). In 1930, they proved the first theorem for the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, which is a prototype of modern limit theorems characterizing the behavior of $\zeta(s)$ by weakly convergent probability measures. More precisely, they obtained that, for $\sigma > 1$, there exists the limit

$$\lim_{T\to\infty} \frac{1}{T} \mathrm{J}\left\{t \in [0, T] : \log \zeta(\sigma + it) \in R\right\},$$

where $R$ is a rectangle on the complex plane with edges parallel to the axes, and $\mathrm{J}A$ denotes the Jordan measure of a set $A \subset \mathbb{R}$. Two years latter, they extended the above result to the half-plane $\sigma > \frac{1}{2}$.

Ideas of Bohr and Jessen were developed by Wintner, Borchsenius, Jessen, Selberg and other famous mathematicians. Modern versions of the Bohr-Jessen theorems, for a wide class of zeta-functions, were obtained in the works of K. Matsumoto.

The theory of Bohr and Jessen is applicable, in general, for zeta-functions having Euler's product over primes. In the present paper, a limit theorem for a zeta-function without Euler's product is proved. This zeta-function is a generalization of the classical Hurwitz zeta-function. Let $\alpha$, $0 < \alpha \leqslant 1$, be a fixed parameter, and $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s},$$

and is meromorphically continued to the whole complex plane. Let $\mathcal{B}(\mathbb{C})$ denote the Borel $\sigma$-field of the set of complex numbers, meas$A$ be the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let the function $\varphi(t)$ for $t \geqslant T_0$ have the monotone positive derivative $\varphi'(t)$ such that $(\varphi'(t))^{-1} = o(t)$ and $\varphi(2t) \max_{t \leqslant u \leqslant 2t}(\varphi'(u))^{-1} \ll t$. Then it is obtained in the paper that, for $\sigma > \frac{1}{2}$,

$$\frac{1}{T} \mathrm{meas}\left\{t \in [0, T] : \zeta(\sigma + i\varphi(t), \alpha; \mathfrak{a}) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to a certain explicitly given probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$.

*Keywords:* Haar measure, Hurwitz zeta-function, limit theorem, periodic Hurwitz zeta-function, weak convergence.

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***In honor of Professor Antanas Laurinčikas on the occasion of his 70th birthday***

# 1. Introduction

The idea of application of probabilistic methods in the theory of zeta-functions is due to Bohr and Jessen. In [2], they proved a theorem for the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad s = \sigma + it, \ \sigma > 1,$$

which is a prototype of a modern limit theorems on weakly convergent probability measures. Denote by $\mathrm{J}A$ the Jordan measure of a measurable set $A \subset \mathbb{R}$, and let $R$ be a rectangle on the complex plane with edges parallel to the axis. Then they proved that, for $\sigma > 1$, there exists the limit

$$\lim_{T\to\infty} \frac{1}{T} \mathrm{J}\left\{t \in [0, T] : \log\zeta(\sigma + it) \in R\right\}.$$

Two years later, Bohr and Jessen extended [3] the above result to the half-plane $\sigma > \frac{1}{2}$. In this case, a problem arises because of possible zeros of $\zeta(s)$. Therefore, they defined the set

$$G = \left\{s \in \mathbb{C} : \sigma > \frac{1}{2}\right\} \setminus \bigcup_{s_j = \sigma_j + it_j} \left\{s = \sigma + it_j : \frac{1}{2} < \sigma < \sigma_j\right\},$$

where $s_j$ runs over all zeros of $\zeta(s)$ in the region $\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, and proved that there exists the limit

$$\lim_{T\to\infty} \frac{1}{T} \mathrm{J}\left\{t \in [0, T] : \sigma + it \in G, \ \log\zeta(\sigma + it) \in R\right\}.$$

In the sixth decade of the last century, the theory of weak convergence of probability measures was created. Therefore, it became possible to state Bohr-Jessen type theorems in the sense of weakly convergent probability measures, for results, see [6] and [8].

The present note is devoted to limit theorems for the periodic Hurwitz zeta-function. Let $\alpha$, $0 < \alpha \leqslant 1$ be a fixed parameter, and let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ was introduced in [7], and is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s}.$$

If $a_m \equiv 1$, then $\zeta(s, \alpha; \mathfrak{a})$ becomes the classical Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}, \quad \sigma > 1,$$

which has a meromorphic continuation to the whole complex plane with the unique simple pole at the point $s = 1$ with residue 1. The periodicity of the sequence $\mathfrak{a}$ implies, for $\sigma > 1$, the equality

$$\zeta(s, \alpha; \mathfrak{a}) = \frac{1}{q^s} \sum_{l=0}^{q-1} a_l \zeta\left(s, \frac{l + \alpha}{q}\right).$$

Therefore, the function $\zeta(s,\alpha;\mathfrak{a})$ also can be continued meromorphically to the whole complex plane with the unique simple pole at the point $s=1$ with residue

$$a \stackrel{def}{=} \frac{1}{q}\sum_{l=0}^{q-1} a_l.$$

If $a=0$, then the periodic Hurwitz zeta-function is entire.

In [4], [9] and [11], limit theorems on weakly convergent probability measures on the complex plane for the function $\zeta(s,\alpha;\mathfrak{a})$ were proved. Denote by $\mathcal{B}(X)$ the Borel $\sigma$-field of the space $X$. Then, for example, it was obtained in [10] that if the parameter $\alpha$ is transcendental and $\sigma > \frac{1}{2}$ is fixed, then, on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$, there exists a probability measure $P_\sigma$ such that

$$\frac{1}{T}\text{meas}\left\{t\in[0,T] : \zeta(\sigma+it,\alpha;\mathfrak{a})\in A\right\}, \quad A\in\mathcal{B}(\mathbb{C}),$$

converges weakly to $P_\sigma$ as $T\to\infty$. Moreover, the measure $P_\sigma$ is given explicitly.

The aim of this note is a generalization of the above theorem for

$$P_{T,\sigma,\alpha;\mathfrak{a}}(A) \stackrel{def}{=} \frac{1}{T-T_0}\text{meas}\left\{t\in[T_0,T] : \zeta(\sigma+i\varphi(t),\alpha;\mathfrak{a})\in A\right\}, \quad A\in\mathcal{B}(\mathbb{C}),$$

for certain functions $\varphi(t)$ and $T_0>0$. For its statement, we need some notation and definitions.

Let $\gamma$ be the unit circle on the complex plane, and

$$\Omega = \prod_{m=0}^{\infty}\gamma_m,$$

where $\gamma_m=\gamma$ for all $m\in\mathbb{N}_0$. With the product topology and pointwise multiplication, the torus $\Omega$ is a compact topological Abelian group. Therefore, on $(\Omega,\mathcal{B}(\Omega))$, the probability Haar measure $m_H$ can be defined. This gives the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. Denote by $\omega(m)$ the $m$th component, $m\in\mathbb{N}_0$, of an element $\omega\in\Omega$, and, on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$ define, for $\sigma>\frac{1}{2}$, the complex-valued random element $\zeta(\sigma,\alpha;\mathfrak{a})$

$$\zeta(\sigma,\alpha;\mathfrak{a}) = \sum_{m=0}^{\infty}\frac{a_m\omega(m)}{(m+\alpha)^\sigma}.$$

Let $P_{\zeta,\sigma}$ be the distribution of the random element $\zeta(\sigma,\alpha;\mathfrak{a})$, i.e.,

$$P_{\zeta,\sigma,\alpha;\mathfrak{a}}(A) = m_H\left\{\omega\in\Omega : \zeta(\sigma,\alpha;\mathfrak{a})\in A\right\}, \quad A\in\mathcal{B}(\mathbb{C}).$$

Now, define the class of functions. We say that $\varphi\in L(T_0)$ if $\varphi$ is a real differentiable function for $t\geqslant T_0>0$ such that $\varphi'(t)$ is monotonic positive, $\frac{1}{\varphi'(t)}=o(t)$ and $\varphi(2t)\max_{t\leqslant u\leqslant 2t}\frac{1}{\varphi'(u)}\ll t$ as $t\to\infty$. For example, the function $\varphi(t)=t^4+2t^3+t^2$ is an element of the class $L(1)$.

The main result of this note is the following theorem.

Theorem 1. *Suppose that the parameter $\alpha$ is transcendental, $\sigma>\frac{1}{2}$ is fixed and $\varphi\in L(T_0)$. Then $P_{T,\sigma,\alpha;\mathfrak{a}}$ converges weakly to the measure $P_{\zeta,\sigma,\alpha;\mathfrak{a}}$ as $T\to\infty$.*

## 2. Lemmas

We start with a limit theorem for probability measures on $(\Omega,\mathcal{B}(\Omega))$. For $A\in\mathcal{B}(\Omega)$, let

$$Q_{T,\alpha}(A) = \frac{1}{T-T_0}\text{meas}\left\{t\in[T_0,T] : ((m+\alpha)^{-i\varphi(t)} : m\in\mathbb{N}_0)\in A\right\}.$$

Lemma 1. *Suppose that $\varphi \in L(T_0)$. Then $Q_{T,\alpha}$ converges weakly to the Haar measure $m_H$ as $T \to \infty$.*

Proof. We apply the Fourier transform method. Let the sign "$\,'\,$" mean that only a finite number of integers $k_m$ are distinct from zero. Denote by $g_T(\underline{k})$, $\underline{k} = (k_m : k_m \in \mathbb{Z}, m \in \mathbb{N}_0)$ the Fourier transform of $Q_{T,\alpha}$. Then the definition of $Q_{T,\alpha}$ implies that

$$g_{T,\alpha}(\underline{k}) = \int_\Omega \left(\prod_{m=0}^{\infty}{}' \omega^{k_m}(m)\right) \mathrm{d}Q_{T,\alpha} = \frac{1}{T-T_0}\int_{T_0}^{T} \prod_{m=0}^{\infty}{}' (m+\alpha)^{-ik_m\varphi(t)} dt$$
$$= \frac{1}{T-T_0}\int_{T_0}^{T} \exp\{-i\varphi(t)\sum_{m=0}^{\infty}{}' k_m \log(m+\alpha)\}\mathrm{d}t. \tag{1}$$

Clearly,

$$g_{T,\alpha}(\underline{0}) = 1. \tag{2}$$

Since $\alpha$ is transcendental, the set $\{\log(m+\alpha) : m \in \mathbb{N}_0\}$ is linearly independent over the field of rational numbers, thus the finite sum

$$r \overset{def}{=} \sum_{m=0}^{\infty}{}' k_m \log(m+\alpha) \neq 0$$

for $\underline{k} \neq 0$. Obviously,

$$\int_{T_0}^{T} \exp\{-ir\varphi(t)\}\mathrm{d}t = \int_{T_0}^{T} \cos(r\varphi(t))\mathrm{d}t - i\int_{T_0}^{T} \sin(r\varphi(t))\mathrm{d}t. \tag{3}$$

If the function $\varphi'(t)$ is decreasing, then $(\varphi'(t))^{-1}$ is increasing. Thus, by the mean value theorem for integrals,

$$\int_{T_0}^{T} \cos(r\varphi(t))\mathrm{d}t = \frac{1}{r}\int_{T_0}^{T} \frac{r\varphi'(t)\cos(r\varphi(t))}{\varphi'(t)}\mathrm{d}t = \frac{1}{r\varphi'(T)}\int_{\xi}^{T} \varphi'(t)\cos(r\varphi(t))\mathrm{d}t$$
$$= \frac{1}{r\varphi'(T)}\int_{\xi}^{T} \mathrm{d}\sin(r\varphi(t)) = o(T), \tag{4}$$

as $T \to \infty$, where $T_0 \leqslant \xi \leqslant T$. Similarly, we find that

$$\int_{T_0}^{T} \sin(r\varphi(t))\mathrm{d}t = o(T), \quad T \to \infty. \tag{5}$$

If the function $\varphi'(t)$ is increasing, then $(\varphi'(t))^{-1}$ is decreasing, and we obtain by similar arguments that

$$\int_{T_0}^{T} \exp\{-ir\varphi(t)\}\,\mathrm{d}t = O\left(\frac{1}{r\varphi'(T_0)}\right). \tag{6}$$

Now, the estimates (4)–(6), and equalities (3) and (1) show that

$$\lim_{T\to\infty} g_{T,\alpha}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } k \neq 0. \end{cases}$$

The right-hand side of the latter equality is the Fourier transform of the Haar measure $m_H$. This and a continuity theorem for probability measures on compact groups prove the lemma. $\square$

Now, we will deal with absolutely convergent Dirichlet series. Let $\theta > \frac{1}{2}$ be a fixed number, and

$$v_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\theta}\right\}.$$

Define the functions

$$\zeta_n(s,\alpha;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m v_n(m,\alpha)}{(m+\alpha)^s}$$

and

$$\zeta_n(s,\alpha,\omega;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m) v_n(m,\alpha)}{(m+\alpha)^s}.$$

We note that the above series are absolutely convergent for $\sigma > \frac{1}{2}$ [5]. Consider the function $u_{n,\sigma,\alpha;\mathfrak{a}} : \Omega \to \mathbb{C}$ given by the formula

$$u_{n,\sigma,\alpha;\mathfrak{a}}(\omega) = \zeta_n(\sigma,\alpha,\omega;\mathfrak{a}), \quad \sigma > \frac{1}{2}.$$

Then the function $u_{n,\sigma,\alpha;\mathfrak{a}}$ is continuous. Moreover,

$$P_{T,n,\sigma,\alpha;\mathfrak{a}} = Q_{T,\alpha} u^{-1}_{n,\sigma,\alpha;\mathfrak{a}}.$$

This observation together with Theorem 5.1 of [1] gives the following assertion.

Lemma 2. *Suppose that $\varphi \in L(T_0)$. Then, for $\sigma > \frac{1}{2}$,*

$$P_{T,n,\sigma,\alpha;\mathfrak{a}}(A) \stackrel{def}{=} \frac{1}{T-T_0} \operatorname{meas}\{t \in [T_0,T] : \zeta_n(\sigma + i\varphi(t),\alpha;\mathfrak{a}) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

*converges weakly to measure $P_{n,\sigma,\alpha;\mathfrak{a}} = m_H u^{-1}_{n,\sigma,\alpha;\mathfrak{a}}$ as $T \to \infty$.*

Now we will approximate $\zeta(\sigma,\alpha;\mathfrak{a})$ by $\zeta_n(s,\alpha;\mathfrak{a})$. For this, we need a mean square estimate.

Lemma 3. *Suppose that $\varphi \in L(T_0)$ and $\sigma > \frac{1}{2}$ is fixed. Then, for $\tau \in \mathbb{R}$,*

$$\int_{T_0}^{T} |\zeta(\sigma + i\tau + i\varphi(t),\alpha;\mathfrak{a})|^2 \mathrm{d}t \ll_{\sigma,\alpha;\mathfrak{a}} T\left(1 + |\tau|\right).$$

Proof. Suppose that $T \geqslant T_0$. Then

$$\begin{aligned}\int_{T}^{2T} |\zeta(\sigma + i\tau + i\varphi(t),\alpha;\mathfrak{a})|^2 \mathrm{d}t &= \int_{T}^{2T} \frac{1}{\varphi'(t)} |\zeta(\sigma + i\tau + i\varphi(t),\alpha;\mathfrak{a})|^2 \mathrm{d}\varphi(t) \\ &\ll \max_{T \leqslant t \leqslant 2T} \frac{1}{\varphi'(t)} \int_{T}^{2T} \mathrm{d}\left(\int_{T_0}^{\tau+\varphi(t)} |\zeta(\sigma + iu,\alpha;\mathfrak{a})|^2 \mathrm{d}u\right) \\ &\ll \max_{T \leqslant t \leqslant 2T} \frac{1}{\varphi'(t)} \left(\int_{T_0}^{\tau+\varphi(t)} |\zeta(\sigma + iu,\alpha;\mathfrak{a})|^2 \mathrm{d}u\right)\Bigg|_{T}^{2T}. \qquad (7)\end{aligned}$$

For $\sigma > \frac{1}{2}$, the estimate

$$\int_{T_0}^{T} |\zeta(\sigma + iu,\alpha;\mathfrak{a})|^2 \mathrm{d}u \ll_{\sigma,\alpha,\mathfrak{a}} T$$

is true [5]. Therefore,

$$\left(\int_{T_0}^{\tau+\varphi(t)} |\zeta(\sigma+iu,\alpha;\mathfrak{a})|^2 \mathrm{d}u\right)\Bigg|_T^{2T} \ll_{\sigma,\alpha,\mathfrak{a}} |\tau| + \varphi(2T).$$

This together with hypothesis that $\varphi(2T)\max_{T\leqslant t\leqslant 2T}\frac{1}{\varphi'(t)} \ll T$ and (7) gives

$$\int_T^{2T} |\zeta(\sigma+i\tau+i\varphi(t),\alpha;\mathfrak{a})|^2 \mathrm{d}t \ll_{\sigma,\alpha,\mathfrak{a}} |\tau| + \varphi(2T)\max_{T\leqslant t\leqslant 2T}\frac{1}{\varphi'(t)}$$
$$\ll_{\sigma,\alpha,\mathfrak{a}} T + |\tau|\max_{T\leqslant t\leqslant 2T}\frac{1}{\varphi'(t)} \ll_{\sigma,\alpha,\mathfrak{a}} T\left(1+|\tau|\right).$$

Taking $2^{-k-1}T$ in place of $T$ and summing over $k\in\mathbb{N}$, gives the estimate of the lemma. □

Lemma 4. *Suppose that* $\varphi\in L(T_0)$ *and* $\sigma>\frac{1}{2}$. *Then*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T-T_0}\int_{T_0}^{T}|\zeta(\sigma+i\varphi(t),\alpha;\mathfrak{a})-\zeta_n(\sigma+i\varphi(t),\alpha;\mathfrak{a})|\mathrm{d}t=0.$$

Proof. Define the function

$$l_n(s,\alpha)=\frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)(m+\alpha)^s,$$

where $\Gamma(s)$ is the Euler gamma-function, and the number $\theta$ comes from the definition of $v_n(m,\alpha)$. Then the function $\zeta(s,\alpha;\mathfrak{a})$ has the integral representation [5]

$$\zeta_n(s,\alpha;\mathfrak{a})=\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z,\alpha;\mathfrak{a})\frac{l_n(z,\alpha)}{z}\mathrm{d}z.$$

Then, using the residue theorem and properties of the gamma-function, we obtain that

$$\frac{1}{T-T_0}\int_{T_0}^{T}|\zeta(\sigma+i\varphi(t),\alpha;\mathfrak{a})-\zeta_n(\sigma+i\varphi(t),\alpha;\mathfrak{a})|\mathrm{d}t$$
$$\ll_{\sigma,\alpha;\mathfrak{a}}\int_{-\infty}^{\infty}|l_n(\sigma_1+i\tau,\alpha)|\left(\frac{1}{T-T_0}\int_{T_0}^{T}|\zeta(\sigma_2+i\tau+i\varphi(t),\alpha;\mathfrak{a})|\mathrm{d}t\right)\mathrm{d}\tau+o(1)$$

as $T\to\infty$, where $\sigma_1<0$ and $\sigma_2>\frac{1}{2}$. Hence, in view of Lemma 3,

$$\frac{1}{T-T_0}\int_{T_0}^{T}|\zeta(\sigma+i\varphi(t),\alpha;\mathfrak{a})-\zeta_n(\sigma+i\varphi(t),\alpha;\mathfrak{a})|\mathrm{d}t$$
$$\ll_{\sigma,\alpha;\mathfrak{a}}\int_{-\infty}^{\infty}|l_n(\sigma_1+i\tau,\alpha)|\left(1+|\tau|\right)\mathrm{d}t+o(1)$$

as $T\to\infty$. Thus, by the properties of $l_n(s,\alpha)$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T-T_0}\int_{T_0}^{T}|\zeta(\sigma+i\varphi(t),\alpha;\mathfrak{a})-\zeta_n(\sigma+i\varphi(t),\alpha;\mathfrak{a})|\mathrm{d}t=0.$$

□

We recall that $P_{n,\sigma,\alpha;\mathfrak{a}}$ is the limit measure in Lemma 2.

Lemma 5. *The sequence* $\{P_{n,\sigma,\alpha;\mathfrak{a}}:n\in\mathbb{N}\}$ *is tight, i.e., for every* $\varepsilon>0$, *there exists a compact set* $K=K(\varepsilon)\subset\mathbb{C}$ *such that*

$$P_{n,\sigma,\alpha;\mathfrak{a}}(K)>1-\varepsilon$$

*for all* $n\in\mathbb{N}$.

Proof. Let $\xi$ be a random variable defined on a certain probability space with measure $\mathbb{P}$, and uniformly distributed on $[0,1]$. Define the complex-valued random element $X_{T,n,\alpha;\mathfrak{a}} = X_{T,n,\alpha;\mathfrak{a}}(\sigma)$ by

$$X_{T,n,\alpha;\mathfrak{a}} = \zeta_n(\sigma + i\varphi(\xi T), \alpha; \mathfrak{a}).$$

Then the assertion of Lemma 2 is equivalent to the relation

$$X_{T,n,\alpha;\mathfrak{a}} \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,\alpha;\mathfrak{a}}, \tag{8}$$

where $X_{n,\alpha;\mathfrak{a}}(\sigma)$ is the complex-valued random element having the distribution $P_{n,\sigma,\alpha;\mathfrak{a}}$. By Lemma 3 with $\tau = 0$, for $\sigma > \frac{1}{2}$,

$$\int_{T_0}^{T} |\zeta(\sigma + i\varphi(t), \alpha; \mathfrak{a})|^2 \mathrm{d}t \ll_{\sigma,\alpha;\mathfrak{a}} T.$$

Hence, the Cauchy inequality implies

$$\int_{T_0}^{T} |\zeta(\sigma + i\varphi(t), \alpha; \mathfrak{a})|^2 \mathrm{d}t \ll \left( (T - T_0) \int_{T_0}^{T} |\zeta(\sigma + i\varphi(t), \alpha; \mathfrak{a})|^2 \mathrm{d}t \right)^{1/2} \ll_{\sigma,\alpha;\mathfrak{a}} T.$$

Therefore, using Lemma 4, we obtain that, for $\sigma > \frac{1}{2}$,

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T - T_0} \int_{T_0}^{T} |\zeta_n(\sigma + i\varphi(t), \alpha; \mathfrak{a})| \mathrm{d}t \leqslant C_{\sigma,\alpha;\mathfrak{a}} < \infty. \tag{9}$$

Let $\varepsilon > 0$ be an arbitrary fixed number, and $M = M_{\sigma,\alpha;\mathfrak{a}}(\varepsilon) = C_{\sigma,\alpha;\mathfrak{a}}\varepsilon^{-1}$. Then, by (9),

$$\begin{aligned}\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \mathbb{P}\left(|X_{T,n,\alpha;\mathfrak{a}}| > M\right) &= \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T - T_0} \operatorname{meas}\left\{t \in [T_0, T] : |\zeta_n(\sigma + i\varphi(t), \alpha; \mathfrak{a})| > M\right\} \\ &\leqslant \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{(T - T_0)M} \int_{T_0}^{T} |\zeta_n(\sigma + i\varphi(t), \alpha; \mathfrak{a})| \mathrm{d}t \leqslant \varepsilon.\end{aligned}$$

This together with (8) shows that

$$\mathbb{P}\left(|X_{n,\alpha;\mathfrak{a}}| > M\right) \leqslant \varepsilon \tag{10}$$

for all $n \in \mathbb{N}$. The set $K = K(\varepsilon) = \{s \in \mathbb{C} : |s| \leqslant M\}$ is compact, and, by (10),

$$\mathbb{P}\left(X_{n,\alpha;\mathfrak{a}} \in K\right) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$, or equivalently,

$$P_{n,\sigma,\alpha;\mathfrak{a}}(K) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Thus, the sequence $\{P_{n,\sigma,\alpha;\mathfrak{a}} : n \in \mathbb{N}\}$ is tight. □

## 3. Proof of Theorem 1

The existence of the limit measure for $P_{T,\sigma,\alpha;\mathfrak{a}}$ as $T \to \infty$ easily follows from Lemmas 4 and 5, relation (8) and Theorem 4.2 of [1].

Proof. [Proof of Theorem 1] By the Prokhorov theorem [1, Theorem 6.1], and Lemma 5, the sequence $\{P_{n,\sigma,\alpha;\mathfrak{a}} : n \in \mathbb{N}\}$ is relatively compact, i.e., every subsequence $\{P_{n_k,\sigma,\alpha;\mathfrak{a}}\} \subset \{P_{n,\sigma,\alpha;\mathfrak{a}}\}$ contains a weakly convergent subsequence. Thus, there exists a subsequence $\{P_{n_r,\sigma,\alpha;\mathfrak{a}}\}$ such that $P_{n_r,\sigma,\alpha;\mathfrak{a}}$ converges weakly to a certain probability measure $P_{\sigma,\alpha;\mathfrak{a}}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $r \to \infty$. In other words,

$$X_{n_r,\alpha;\mathfrak{a}}(\sigma) \xrightarrow[r\to\infty]{\mathcal{D}} P_{\sigma,\alpha;\mathfrak{a}}. \tag{11}$$

Define one more complex-valued random element

$$X_{T,\alpha;\mathfrak{a}} = X_{T,\alpha;\mathfrak{a}}(\sigma) = \zeta(\sigma + i\varphi(\xi T), \alpha; \mathfrak{a}).$$

Then Lemma 4 shows that, for every $\varepsilon > 0$ and $\sigma > \frac{1}{2}$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mathrm{P}\left(|X_{T,\alpha;\mathfrak{a}}(\sigma) - X_{T,n,\alpha;\mathfrak{a}}(\sigma)| \geqslant \varepsilon\right) \\
&\qquad = \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \mathrm{meas}\left\{t \in [T_0,T] : |\zeta(\sigma + i\varphi(t), \alpha; \mathfrak{a}) - \zeta_n(\sigma + i\varphi(t), \alpha; \mathfrak{a})| \geqslant \varepsilon\right\} \\
&\qquad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{(T-T_0)\varepsilon} \int_{T_0}^{T} |\zeta(\sigma + i\varphi(t), \alpha; \mathfrak{a}) - \zeta_n(\sigma + i\varphi(t), \alpha; \mathfrak{a})| \mathrm{d}t = 0.
\end{aligned}$$

The later equality, relations (8) and (11), and Theorem 4.2 of [1] prove that

$$X_{T,\sigma,\alpha;\mathfrak{a}}(\sigma) \xrightarrow[T\to\infty]{\mathcal{D}} P_{\sigma,\alpha;\mathfrak{a}}. \tag{12}$$

in other words, $P_{T,\sigma,\alpha;\mathfrak{a}}$ converges weakly to $P_{\sigma,\alpha;\mathfrak{a}}$ as $T \to \infty$. Moreover, relation (12) shows that the measure $P_{\sigma,\alpha;\mathfrak{a}}$ does not depend of the subsequence $P_{n_r\sigma,\alpha;\mathfrak{a}}$. Therefore, we have the relation

$$X_{n,\alpha;\mathfrak{a}}(\sigma) \xrightarrow[n\to\infty]{\mathcal{D}} P_{\sigma,\alpha;\mathfrak{a}}.$$

This relation allows to identify the measure $P_{\sigma,\alpha;\mathfrak{a}}$. Namely, in [5], it was proved that, for $\sigma > \frac{1}{2}$,

$$\frac{1}{T} \mathrm{meas}\left\{t \in [0,T] : \zeta(\sigma + it, \alpha; \mathfrak{a}) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

as $T \to \infty$, also converges weakly to the limit measure $P_{\sigma,\alpha;\mathfrak{a}}$ of $P_{n,\sigma,\alpha;\mathfrak{a}}$ as $n \to \infty$, and that $P_{\sigma,\alpha;\mathfrak{a}}$ coincides with $P_{\zeta,\sigma,\alpha;\mathfrak{a}}$. Therefore, $P_{T,\sigma,\alpha;\mathfrak{a}}$ converges weakly to $P_{\zeta,\sigma,\alpha;\mathfrak{a}}$ as $T \to \infty$. The theorem is proved. ☐

## 4. Conclusions

In the paper, a generalized limit theorem for the periodic Hurwitz zeta-function

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}, \quad \mathrm{Re}s = \sigma > 1,$$

where $0 < \alpha \leqslant 1$ is a fixed transcendental parameter and $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0\}$ is a periodic sequence of complex numbers, is obtained. More precisely, it is proved that, for $\sigma > \frac{1}{2}$,

$$\frac{1}{T-T_0} \mathrm{meas}\left\{t \in [T_0,T] : \zeta(\sigma + i\varphi(t), \alpha; \mathfrak{a}) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the explicitly given probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$. Here the function $\varphi(t)$ for $t \geqslant T_0$ has a monotone positive derivative $\varphi'(t)$ satisfying the estimates $(\varphi'(t))^{-1} = o(t)$ and $\varphi(2t) \max_{t\leqslant u\leqslant 2t} \frac{1}{\varphi'(u)} \ll t$ as $t \to \infty$. The theorem obtained generalized previous author's results with $\varphi(t) = t$. Moreover, it can be extended to a collection of periodic Hurwitz zeta-functions. Also, the case of rational $\alpha$ can be considered.

## СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ


1. Billingsley P. Convergence of Probability Measures. New York: John Wiley and Sons, 1968.
2. Bohr H., Jessen B. Über die Wertverteilung der Riemanschen Zetafunktion, Erste Mitteilung // Acta Math. 1930. Vol. 54. P. 1–35.
3. Bohr H., Jessen B. ‘Über die Wertverteilung der Riemanschen Zetafunktion, Zweite Mitteilung // Acta Math. 1932. Vol. 58. P. 1–55.
4. Genienė D., Rimkevičienė A. A joint limit theorem for periodic Hurwitz zeta-functions with algebraic irrational parameters // Math. Modelling and Analysis. 2013. Vol. 18, no. 1. P. 149–159.
5. Javtokas A., Laurinčikas A. On the periodic Hurwitz zeta-function // Hardy-Ramanujan J. 2006. Vol. 29, no. 3. P. 18–36.
6. Laurinčikas A. Limit Theorems for the Riemann Zeta-Function. Dordrecht, Boston, London: Kluwer, 1996.
7. Laurinčikas A. The joint universality for periodic Hurwitz zeta-functions // Analysis. 2006. Vol. 26, no. 3, P. 419–428.
8. Matsumoto K. Probabilistic value-distribution theory of zeta-functions // Sugaku Expositions. 2004. Vol. 17. P. 51–71.
9. Misevičius G., Rimkevičienė A. Joint limit theorems for periodic Hurwitz zeta-functions. II // Annales Univ. Sci. Budapest., Sect. Comp. 2013. Vol. 41. P. 173–185.
10. Rimkevičienė A. Limit theorems for the periodic Hurwitz zeta-function // Šiauliai Math. Semin. 2010. Vol. 5(13). P. 55–69.
11. Rimkevičienė A. Joint limit theorems for the periodic Hurwitz zeta-functions // Šiauliai Math. Semin. 2011. Vol. 6(14). P. 53–68.


## REFERENCES


1. Billingsley, P. 1968, “Convergence of Probability Measures“, John Wiley and Sons, New York.
2. Bohr, H. & Jessen, B. 1930, “Über die Wertverteilung der Riemanschen Zetafunktion, Erste Mitteilung“, *Acta Math.*, vol. 54, pp. 1–35.
3. Bohr, H. & Jessen, B. 1932, “Über die Wertverteilung der Riemanschen Zetafunktion, Zweite Mitteilung“, *Acta Math.*, vol. 58, pp. 1–55.
4. Genienė, D. & Rimkevičienė, A. 2013, “A joint limit theorem for periodic Hurwitz zeta-functions with algebraic irrational parameters“, *Math. Modelling and Analysis*, vol. 18, no. 1, pp. 149–159.
5. Javtokas, A. & Laurinčikas, A. 2006, “On the periodic Hurwitz zeta-function“, *Hardy-Ramanujan J.*, vol. 29, no. 3, pp. 18–36.
6. Laurinčikas, A. 1996, “Limit Theorems for the Riemann Zeta-Function“, Kluwer, Dordrecht, Boston, London.
7. Laurinčikas, A. 2006, “The joint universality for periodic Hurwitz zeta-functions“, *Analysis*, vol. 26, no. 3, pp. 419–428.

8. Matsumoto, K. 2004, “Probabilistic value-distribution theory of zeta-functions“, *Sugaku Expositions*, vol. 17, pp. 51–71.

9. Misevičius, G. & Rimkevičienė, A. 2013, “Joint limit theorems for periodic Hurwitz zeta-functions. II“, *Annales Univ. Sci. Budapest., Sect. Comp.*, vol. 41, pp. 173–185.

10. Rimkevičienė, A. 2010, “Limit theorems for the periodic Hurwitz zeta-function“, *Šiauliai Math. Semin.*, vol. 5(13), pp. 55–69.

11. Rimkevičienė, A. 2011, “Joint limit theorems for the periodic Hurwitz zeta-functions“, *Šiauliai Math. Semin.* vol. 6(14), pp. 53–68.

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