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О множествах ограниченных остатков для (t,s)-последовательностей ${f I}$

Мордехай Б. Левин

Левин Мордехай Борисович — кандидат физико-математических наук, Факультет математики, Университет Бар-Илан, Рамат-Ган, Израиль. *e-mail:* mlevin@math.biu.ac.il

Аннотация

Пусть $(\mathbf{x}_n)_{n\geq 0}-s$ -мерная последовательность типа Холтона, полученная из глобального функционального поля, $b\geq 2,\ \boldsymbol{\gamma}=(\gamma_1,...,\gamma_s),\ \gamma_i\in [0,1)$ с b-адическим разложением $\gamma_i=\gamma_{i,1}b^{-1}+\gamma_{i,2}b^{-2}+...,\ i=1,...,s.$

В этой статье мы докажем, что $[0,\gamma_1) \times ... \times [0,\gamma_s)$ — множество ограниченного остатка относительно последовательности $(\mathbf{x}_n)_{n \geq 0}$ тогда и только тогда, когда

$$\max_{1 \le i \le s} \max\{j \ge 1 \mid \gamma_{i,j} \ne 0\} < \infty.$$

Мы также получим аналогичные результаты для обобщенных последовательностей Нидеррайтера, последовательностей Хинга — Нидеррайтера и последовательностей Нидеррайтера — Хинга.

Kлючевые слова: множества ограниченных остатков, (t,s)-последовательности, последовательности Холтона .

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On a bounded remainder set for (t, s) sequences I

Mordechay B. Levin

Levin Mordechay Borisovich — candidat of physical and mathematical Sciences, Department of Mathematics, Bar-Ilan University, Ramat-Gan, 5290002, Israel.

 $e\hbox{-}mail\hbox{:}\ mlevin@math.biu.ac.il$

Let $\mathbf{x}_0, \mathbf{x}_1, \dots$ be a sequence of points in $[0,1)^s$. A subset S of $[0,1)^s$ is called a bounded remainder set if there exist two real numbers a and C such that, for every integer N,

$$|\operatorname{card}\{n < N \mid \mathbf{x}_n \in S\} - aN| < C.$$

Let $(\mathbf{x}_n)_{n\geq 0}$ be an s-dimensional Halton-type sequence obtained from a global function field, $b\geq 2, \ \gamma=(\gamma_1,...,\gamma_s), \ \gamma_i\in [0,1),$ with b-adic expansion $\gamma_i=\gamma_{i,1}b^{-1}+\gamma_{i,2}b^{-2}+...,$ i=1,...,s. In this paper, we prove that $[0,\gamma_1)\times...\times[0,\gamma_s)$ is the bounded remainder set with respect to the sequence $(\mathbf{x}_n)_{n\geq 0}$ if and only if

$$\max_{1 \le i \le s} \max\{j \ge 1 \mid \gamma_{i,j} \ne 0\} < \infty.$$

We also obtain the similar results for a generalized Niederreiter sequences, Xing-Niederreiter sequences and Niederreiter-Xing sequences.

Keywords: bounded remainder set, (t, s) sequence, Halton type sequences.

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1. Introduction

1.1. Bounded remainder sets. Let $\mathbf{x}_0, \mathbf{x}_1, \dots$ be a sequence of points in $[0,1)^s, S \subseteq [0,1)^s$,

$$\Delta(S, (\mathbf{x}_n)_{n=0}^{N-1}) = \sum_{n=0}^{N-1} (\mathbf{1}_S(\mathbf{x}_n) - \lambda(S)),$$

where $\mathbf{1}_{S}(\mathbf{x}) = 1$, if $\mathbf{x} \in S$, and $\mathbf{1}_{S}(\mathbf{x}) = 0$, if $\mathbf{x} \notin S$. Here $\lambda(S)$ denotes the s-dimensional Lebesgue-measure of S. We define the star discrepancy of an N-point set $(\mathbf{x}_n)_{n=0}^{N-1}$ as

$$D^*((\mathbf{x}_n)_{n=0}^{N-1}) = \sup_{0 < y_1, \dots, y_s \le 1} |\Delta([\mathbf{0}, \mathbf{y}), (\mathbf{x}_n)_{n=0}^{N-1})/N|,$$

where $[\mathbf{0}, \mathbf{y}) = [0, y_1) \times \cdots \times [0, y_s)$. The sequence $(\mathbf{x}_n)_{n \geq 0}$ is said to be uniformly distributed in $[0, 1)^s$ if $D_N \to 0$. In 1954, Roth proved that $\limsup_{N \to \infty} N(\ln N)^{-\frac{s}{2}} D^*((\mathbf{x}_n)_{n=0}^{N-1}) > 0$. According to the well-known conjecture (see, e.g., [1, p.283]), this estimate can be improved to

$$\lim \sup_{N \to \infty} N(\ln N)^{-s} D^*((\mathbf{x}_n)_{n=0}^{N-1}) > 0.$$
 (1)

See [2] and [7] for results on this conjecture.

A sequence $(\mathbf{x}_n^{(s)})_{n\geq 0}$ is of low discrepancy (abbreviated l.d.s.) if $D((\mathbf{x}_n^{(s)})_{n=0}^{N-1}) = O(N^{-1}(\ln N)^s)$ for $N\to\infty$. A sequence of point sets $((\mathbf{x}_{n,N}^{(s)})_{n=0}^{N-1})_{N=1}^{\infty}$ is of low discrepancy (abbreviated l.d.p.s.) if $D((\mathbf{x}_{n,N}^{(s)})_{n=0}^{N-1}) = O(N^{-1}(\ln N)^{s-1})$, for $N\to\infty$. For examples of such a sequences, see, e.g., [1], [3], and [11].

Definition 1. Let $\mathbf{x}_0, \mathbf{x}_1, \dots$ be a sequence of points in $[0,1)^s$. A subset S of $[0,1)^s$ is called a bounded remainder set for $(\mathbf{x}_n)_{n\geq 0}$ if the discrepancy function $\Delta(S, (\mathbf{x}_n)_{n=0}^{N-1})$ is bounded in \mathbb{N} .

Let α be an irrational number, let I be an interval in [0,1) with length |I|, let $\{n\alpha\}$ be the fractional part of $n\alpha$, n=1,2,.... Hecke, Ostrowski and Kesten proved that $\Delta(S,(\{n\alpha\})_{n=1}^N)$ is bounded if and only if $|I| = \{k\alpha\}$ for some integer k (see references in [4]).

The sets of bounded remainder for the classical s-dimensional Kronecker sequence studied by Lev and Grepstad [4]. The case of Halton's sequence was studied by Hellekalek [5].

Let b be a prime power, $\gamma = (\gamma_1, ..., \gamma_s), \gamma_i \in (0, 1)$ with b-adic expansion

$$\gamma_i = \gamma_{i,1}b^{-1} + \gamma_{i,2}b^{-2} + \dots, \quad i = 1, \dots, s,$$

and let $(\mathbf{x}_n)_{n\geq 0}$ be a uniformly distributed digital Kronecker sequence. In [7], we proved the following theorem:

Theorem A. The set $[0, \gamma_1) \times ... \times [0, \gamma_s)$ is of bounded remainder with respect to $(\mathbf{x}_n)_{n\geq 0}$ if and only if

$$\max_{1 \le i \le s} \max\{j \ge 1 \mid \gamma_{i,j} \ne 0\} < \infty. \tag{2}$$

In this paper, we prove similar results for digital sequences described in [3, Sec. 8]. Note that according to Larcher's conjecture [6, p.215], the assertion of Theorem A is true for all digital (t, s)-sequences in base b.

2. Definitions and auxiliary results.

2.1 (**T**, s) sequences. A subinterval E of $[0,1)^s$ of the form

$$E = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1) b^{-d_i}),$$

with $a_i, d_i \in \mathbb{Z}$, $d_i \ge 0$, $0 \le a_i < b^{d_i}$, for $1 \le i \le s$ is called an elementary interval in base $b \ge 2$.

Definition 2. Let $0 \le t \le m$ be integers. A (t, m, s)-net in base b is a point set $\mathbf{x}_0, ..., \mathbf{x}_{b^m-1}$ in $[0, 1)^s$ such that $\#\{n \in [0, b^m - 1] | x_n \in E\} = b^t$ for every elementary interval E in base b with $vol(E) = b^{t-m}$.

Definition 3. Let $t \ge 0$ be an integer. A sequence $\mathbf{x}_0, \mathbf{x}_1, \ldots$ of points in $[0,1)^s$ is a (t,s)-sequence in base b if, for all integers $k \ge 0$ and $m \ge t$, the point set consisting of \mathbf{x}_n with $kb^m \le n < (k+1)b^m$ is a (t,m,s)-net in base b.

By [Ni, p. 56,60], (t, m, s) nets and (t, s) sequences are of low discrepancy. See reviews on (t, m, s) nets and (t, s) sequences in [3] and [11].

Definition 4. ([3, Definition 4.30]) For a given dimension $s \ge 1$, an integer base $b \ge 2$, and a function $\mathbf{T} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\mathbf{T}(m) \le m$ for all $m \in \mathbb{N}_0$, a sequence $(\mathbf{x}_0, \mathbf{x}_1, ...)$ of points in $[0, 1)^s$ is called a (\mathbf{T}, s) -sequence in base b if for all integers $m \ge 0$ and $k \ge 0$, the point set consisting of the points $x_{kb^m}, ..., x_{kb^m+b^m-1}$ forms a $(\mathbf{T}(m), m, s)$ -net in base b.

Definition 5. ([3, Definition 4.47]) Let $m, s \geq 1$ be integers. Let $C^{(1,m)}, ..., C^{(s,m)}$ be $m \times m$ matrices over \mathbb{F}_b . Now we construct b^m points in $[0,1)^s$. For $n=0,1,...,b^m-1$, let $n=\sum_{j=0}^{m-1}a_j(n)b^j$ be the b-adic expansion of n. Choose a bijection $\phi: \mathbb{Z}_b:=\{0,1,...,b-1\} \mapsto \mathbb{F}_b$ with $\phi(0)=\bar{0}$, the neutral element of addition in \mathbb{F}_b . We identify n with the row vector

$$\mathbf{n} = (\bar{a}_0(n), ..., \bar{a}_{m-1}(n)) \in \mathbb{F}_b^m \text{ with } \bar{a}_r(n) = \phi(a_r(n)), \ r \in [0, m).$$
(3)

We map the vectors

$$y_n^{(i)} = (y_{n,1}^{(i)}, ..., y_{n,m}^{(i)}) := \mathbf{n}C^{(i,m)\top}, \quad y_{n,j}^{(i)} = \sum_{r=0}^{\infty} \bar{a}_r(n)c_{j,r}^{(i)} \in \mathbb{F}_b,$$

$$(4)$$

to the real numbers

$$x_n^{(i)} = \sum_{j=1}^m x_{n,j}^{(i)} / b^j, \quad x_{n,j}^{(i)} = \phi^{-1}(y_{n,j}^{(i)})$$
 (5)

to obtain the point

$$\mathbf{x}_n := (x_n^{(1)}, \dots, x_n^{(s)}) \in [0, 1)^s. \tag{6}$$

The point set $\{\mathbf{x}_0,...,\mathbf{x}_{b^m-1}\}$ is called a digital net (over \mathbb{F}_b) (with generating matrices $(C^{(1,m)},...,C^{(s,m)})$).

For $m = \infty$, we obtain a sequence $\mathbf{x}_0, \mathbf{x}_1, \dots$ of points in $[0,1)^s$ which is called a digital sequence (over \mathbb{F}_b) (with generating matrices $(C^{(1,\infty)}, \dots, C^{(s,\infty)})$).

We abbreviate $C^{(i,m)}$ as $C^{(i)}$ for $m \in \mathbb{N}$ and for $m = \infty$.

2.2 Duality theory (see [3, Section 7]).

Let \mathcal{N} be an arbitrary \mathbb{F}_b -linear subspace of \mathbb{F}_b^{sm} . Let H be a matrix over \mathbb{F}_b consisting of sm columns such that the row-space of H is equal to \mathcal{N} . Then we define the dual space $\mathcal{N}^{\perp} \subseteq \mathbb{F}_b^{sm}$ of \mathcal{N} to be the null space of H (see [3, p. 244]). In other words, \mathcal{N}^{\perp} is the orthogonal complement of \mathcal{N} relative to the standard inner product in \mathbb{F}_b^{sm} ,

$$\mathcal{N}^{\perp} = \{ A \in \mathbb{F}_b^{sm} \mid B \cdot A = 0 \text{ for all } B \in \mathcal{N} \}.$$

Let $C^{(1)},...,C^{(s)} \in \mathbb{F}_b^{\infty \times \infty}$ be generating matrices of a digital sequence $(\mathbf{x}_n(C))_{n\geq 0}$ over \mathbb{F}_b . For any $m \in \mathbb{N}$, we denote the $m \times m$ left-upper sub-matrix of $C^{(i)}$ by $[C^{(i)}]_m$. The matrices $[C^{(1)}]_m,...,[C^{(s)}]_m$ are then the generating matrices of a digital net. We define the overall generating matrix of this digital net by

$$[C]_m = ([C^{(1)}]_m^\top | [C^{(2)}]_m^\top | \dots | [C^{(s)}]_m^\top) \in \mathbb{F}_b^{m \times sm}$$
(7)

for any $m \in \mathbb{N}$.

Let \mathcal{C}_m denote the row space of the matrix $[C]_m$ i.e.,

$$C_m = \left\{ \left(\sum_{r=0}^{m-1} c_{j,r}^{(i)} \bar{a}_r(n) \right)_{1 \le j \le m, 1 \le i \le s} \mid 0 \le n < b^m \right\}.$$

The dual space is then given by

$$\mathcal{C}_m^{\perp} = \{ A \in \mathbb{F}_b^{sm} \mid B \cdot A^{\top} = \mathbf{0} \text{ for all } B \in \mathcal{C}_m \}.$$

Lemma A. ([3, Theorem 4.86]) Let b be a prime power. A strict digital (\mathbf{T}, s) -sequence over \mathbb{F}_b is uniformly distributed modulo one, if and only if $\liminf_{m\to\infty} (m-\mathbf{T}(m)) = \infty$.

2.3 Admissible sequences.

For $x = \sum_{j \geq 1} x_j b^{-j}$, and $y = \sum_{j \geq 1} y_j b^{-j}$ where $x_j, y_j \in \mathbb{Z}_b := \{0, 1,, b - 1\}$, we define the (b-adic) digital shifted point v by $v = x \oplus y := \sum_{j \geq 1} v_j b^{-j}$, where $v_j \equiv x_j + y_j \pmod{b}$ and $v_j \in \mathbb{Z}_b$. Let $\mathbf{x} = (x^{(1)}, ..., x^{(s)}) \in [0, 1)^s$, $\mathbf{y} = (y^{(1)}, ..., y^{(s)}) \in [0, 1)^s$. We define the (b-adic) digital shifted point \mathbf{v} by $\mathbf{v} = \mathbf{x} \oplus \mathbf{y} = (x^{(1)} \oplus y^{(1)}, ..., x^{(s)} \oplus y^{(s)})$. For $n_1, n_2 \in [0, b^m)$, we define $n_1 \oplus n_2 : = (n_1/b^m \oplus n_2)b^m)b^m$.

For $x = \sum_{j \geq 1} x_i b^{-i}$, where $x_i \in \mathbb{Z}_b$, $x_i = 0$ (i = 1, ..., k) and $x_{k+1} \neq 0$, we define the absolute valuation $\|.\|_b$ of x by $\|x\|_b = b^{-k-1}$. Let $\|n\|_b = b^k$ for $n \in [b^k, b^{k+1})$.

Definition 6. A point set $(\mathbf{x}_n)_{0 \le n \le b^m}$ in $[0,1)^s$ is d-admissible in base b if

$$\min_{0 \le k < n < b^m} \|\mathbf{x}_n \ominus \mathbf{x}_k\|_b > b^{-m-d} \quad \text{where} \quad \|\mathbf{x}\|_b := \prod_{i=1}^s \left\| x_j^{(i)} \right\|_b.$$

A sequence $(\mathbf{x}_n)_{n\geq 0}$ in $[0,1)^s$ is d-admissible in base b if $\inf_{n>k\geq 0}\|n\ominus k\|_b\times\|\mathbf{x}_n\ominus\mathbf{x}_k\|_b\geq b^{-d}$.

By [8], generalized Niederreiter's sequences, Xing-Niederreiter's sequences and Halton-type (t, s) sequences have d-admissible properties. In [8], we proved for all d-admissible digital (t, s) sequences $(\mathbf{x}_n)_{n>0}$

$$\max_{1 \le N \le b^m} ND^*((\mathbf{x}_n \oplus \mathbf{w})_{0 \le n < N}) \ge Km^s$$

with some **w** and K > 0. This result supports conjecture (1).

Definition 7. A sequence $(\mathbf{x}_n)_{n\geq 0}$ in $[0,1)^s$ is weakly admissible in base b if

$$\varkappa_m := \min_{0 \le k < n < b^m} \|\mathbf{x}_n \ominus \mathbf{x}_k\|_b > 0 \quad \forall m \ge 1 \text{ where } \|\mathbf{x}\|_b := \prod_{i=1}^s \|x^{(i)}\|_b.$$

Let
$$m \ge 1$$
, $\tau_m = [\log_q(\kappa_m)] + m$, $\mathbf{w} = (w^{(1)}, ..., w^{(s)})$, $w^{(i)} = (w_1^{(i)}, ..., w_{\tau_m}^{(i)})$,

$$g_{\mathbf{w}} = \{ A \ge 1 \mid x_{b^m A, j}^{(i)} = w_j^{(i)}, \ j \in [1, \tau_m], \ i \in [1, s] \} \text{ and } g_{\mathbf{w}} \ne \emptyset \ \forall w_j^{(i)} \in \mathbb{Z}_b.$$
 (8)

Theorem B. (see [9, Proposition]) Let $(\mathbf{x}_n)_{n\geq 0}$ be a uniformly distributed weakly admissible digital (T,s)-sequence in base b, satisfying (8) for all $m\geq m_0$. Then the set $[0,\gamma_1)\times...\times[0,\gamma_s)$ is of bounded remainder with respect to $(\mathbf{x}_n)_{n\geq 0}$ if and only if (2) is true.

2.4 Notation and terminology for algebraic function fields. For the theory of algebraic function fields, we follow the notation and terminology in the books [14] and [13].

Let b be an arbitrary prime power, \mathbb{F}_b a finite field with b elements, $\mathbb{F}_b(x)$ the rational function field over \mathbb{F}_b , and $\mathbb{F}_b[x]$ the polynomial ring over \mathbb{F}_b . For $\alpha = f/g$, $f, g \in \mathbb{F}_b[x]$, let

$$\nu_{\infty}(\alpha) = \deg(g) - \deg(f)$$

be the degree valuation of $\mathbb{F}_b(x)$. We define the field of Laurent series as

$$\mathbb{F}_b((x)) := \Big\{ \sum_{i=m}^{\infty} a_i x^i \mid m \in \mathbb{Z}, \ a_i \in \mathbb{F}_b \Big\}.$$

A finite extension field F of $\mathbb{F}_b(x)$ is called an algebraic function field over \mathbb{F}_b . Let \mathbb{F}_b be algebraically closed in F. We express this fact by simply saying that F/\mathbb{F}_b is an algebraic function field. The genus of F/\mathbb{F}_b is denoted by g.

A place \mathcal{P} of F is, by definition, the maximal ideal of some valuation ring of F. We denote by $O_{\mathcal{P}}$ the valuation ring corresponding to \mathcal{P} and we denote by \mathbb{P}_F the set of places of F. For a place \mathcal{P} of F, we write $\nu_{\mathcal{P}}$ for the normalized discrete valuation of F corresponding to \mathcal{P} , and any element $t \in F$ with $\nu_{\mathcal{P}}(t) = 1$ is called a local parameter (prime element) at \mathcal{P} .

The field $F_{\mathcal{P}} := O_{\mathcal{P}}/\mathcal{P}$ is called the residue field of F with respect to \mathcal{P} . The degree of a place \mathcal{P} is defined as $\deg(\mathcal{P}) = [F_{\mathcal{P}} : \mathbb{F}_b]$. We denote by $\operatorname{Div}(F)$ the set of divisors of F/\mathbb{F}_b .

The completion of F with respect to $\nu_{\mathcal{P}}$ will be denoted by $F^{(\mathcal{P})}$. Let t be a local parameter of \mathcal{P} . Then $F^{(\mathcal{P})}$ is isomorphic to $F_{\mathcal{P}}((t))$ (see [13, Theorem 2.5.20]), and an arbitrary element $\alpha \in F^{(P)}$ can be uniquely expanded as (see [13, p. 293])

$$\alpha = \sum_{i=\nu_{\mathcal{P}}(\alpha)}^{\infty} S_i t^i$$
 where $S_i = S_i(t,\alpha) \in F_{\mathcal{P}} \subseteq F^{(P)}$.

The derivative $\frac{d\alpha}{dt}$, or differentiation with respect to t, is defined by (see [13, Definition 9.3.1])

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \sum_{i=\nu_{\mathcal{P}}(\alpha)}^{\infty} iS_i t^{i-1}.$$
 (9)

For an algebraic function field F/\mathbb{F}_b , we define its set of differentials (or Hasse differentials, H-differentials) as

$$\Delta_F = \{ y \, \mathrm{d}z \mid y \in F, \ z \text{ is a separating element for } F/\mathbb{F}_b \}$$

(see [14, Definition 4.1.7]).

Lemma B. ([14, Proposition 4.1.8] or [13, Theorem 9.3.13]) Let $z \in F$ be separating. Then every differential $\gamma \in \Delta_F$ can be written uniquely as $\gamma = y \, \mathrm{d}z$ for some $y \in F$.

We define the order of $\alpha d\beta$ at \mathcal{P} by

$$\nu_{\mathcal{P}}(\alpha \, \mathrm{d}\beta) := \nu_{\mathcal{P}}(\alpha \, \mathrm{d}\beta/\mathrm{d}t),\tag{10}$$

where t is any local parameter for \mathcal{P} (see [13, Definition 9.3.8]).

Let Ω_F be the set of all Weil differentials of F/\mathbb{F}_b . There exists an F-linear isomorphism of the differential module Δ_F onto Ω_F (see [14, Theorem 4.3.2] or [13, Theorem 9.3.15]).

For $0 \neq \omega \in \Omega_F$, there exists a uniquely determined divisor $\operatorname{div}(\omega) \in \operatorname{Div}(F)$. Such a divisor $\operatorname{div}(\omega)$ is called a canonical divisor of F/\mathbb{F}_b . (see [14, Definition 1.5.11]). For a canonical divisor \dot{W} , we have (see [14, Corollary 1.5.16])

$$\deg(\dot{W}) = 2g - 2 \quad \text{and} \quad \ell(\dot{W}) = g. \tag{11}$$

Let α d β be a nonzero H-differential in F and let ω be the corresponding Weil differential. Then (see [13, Theorem 9.3.17], [14, ref. 4.35])

$$\nu_{\mathcal{P}}(\operatorname{div}(\omega)) = \nu_{\mathcal{P}}(\alpha \, d\beta), \quad \text{for all} \quad \mathcal{P} \in \mathbb{P}_F.$$
 (12)

Let $\alpha d\beta$ be an H-differential, t a local parameter of \mathcal{P} , and

$$\alpha d\beta = \sum_{i=\nu_{\mathcal{P}}(\alpha)}^{\infty} S_i t^i dt \in F^{(\mathcal{P})}.$$

Then the residue of $\alpha d\beta$ (see [13, Definition 9.3.10) is defined by

$$\operatorname{Res}_{\mathcal{P}}(\alpha \, \mathrm{d}\beta) := \operatorname{Tr}_{F_{\mathcal{P}}/\mathbb{F}_b}(S_{-1}) \in \mathbb{F}_b.$$

Let

$$\operatorname{Res}_{\mathcal{P},t}(\alpha) := \operatorname{Res}_{\mathcal{P}}(\alpha dt).$$

For a divisor \mathcal{D} of F/\mathbb{F}_b , let $\mathcal{L}(\mathcal{D})$ denote the Riemann-Roch space

$$\mathcal{L}(\mathcal{D}) = \{ y \in F \setminus 0 \mid \operatorname{div}(y) + \mathcal{D} \ge 0 \} \cup \{ 0 \}. \tag{13}$$

Then $\mathcal{L}(\mathcal{D})$ is a finite-dimensional vector space over \mathbb{F} , and we denote its dimension by $\ell(\mathcal{D})$. By [14, Corollary 1.4.12],

$$\ell(\mathcal{D}) = \{0\} \quad \text{for} \quad \deg(\mathcal{D}) < 0. \tag{14}$$

Theorem C (Riemann-Roch Theorem). [14, Theorem 1.5.15, and 14, Theorem 1.5.17] Let W be a canonical divisor of F/\mathbb{F}_b . Then for each divisor $A \in \operatorname{div}(F)$, $\ell(A) = \deg(A) + 1 - g + \ell(W - A)$, and

$$\ell(A) = \deg(A) + 1 - g \quad \text{for} \quad \deg(A) \ge 2g - 1.$$

3. Statements of results.

3.1 Generalized Niederreiter sequence. In this subsection, we introduce a generalization of the Niederreiter sequence due to Tezuka (see [3, Section 8.1.2]). By [3, Section 8.1], the Sobol's sequence, the Faure's sequence and the original Niederreiter sequence are particular cases of a generalized Niederreiter sequence.

Let b be a prime power and let $p_1, ..., p_s \in \mathbb{F}_b[x]$ be pairwise coprime polynomials over \mathbb{F}_b . Let $e_i = \deg(p_i) \geq 1$ for $1 \leq i \leq s$. For each $j \geq 1$ and $1 \leq i \leq s$, the set of polynomials $\{y_{i,j,k}(x) : 0 \leq k < e_i\}$ needs to be linearly independent (mod $p_i(x)$) over \mathbb{F}_b . For integers $1 \leq i \leq s, j \geq 1$ and $0 \leq k < e_i$, consider the expansions

$$\frac{y_{i,j,k}(x)}{p_i(x)^j} = \sum_{r>0} a^{(i)}(j,k,r)x^{-r-1}$$

over the field of formal Laurent series $\mathbb{F}_b((x^{-1}))$. Then we define the matrix $C^{(i)}=(c_{j,r}^{(i)})_{j\geq 1,r\geq 0}$ by

$$c_{i,r}^{(i)} = a^{(i)}(Q+1, k, r) \in \mathbb{F}_b$$
 for $1 \le i \le s, \ j \ge 1, \ r \ge 0$,

where $j - 1 = Qe_i + k$ with integers Q = Q(i, j) and k = k(i, j) satisfying $0 \le k < e_i$.

A digital sequence $(\mathbf{x}_n)_{n\geq 0}$ over \mathbb{F}_b generated by the matrices $C^{(1)},...,C^{(s)}$ is called a generalized Niederreiter sequence (see [3, p.266]).

Theorem D. (see [3, p.266] and [7, Theorem 1]) The generalized Niederreiter sequence $(\mathbf{x}_n)_{n\geq 0}$ with generating matrices, defined as above, is a digital d-admissible (t, s)-sequence over \mathbb{F}_b with $d = e_0$, $t = e_0 - s$ and $e_0 = e_1 + ... + e_s$.

In this paper, we will consider the case where $(x, p_i) = 1$ for $1 \le i \le s$. We will consider the general case in [10].

Theorem 1. With the notations as above, the set $[0, \gamma_1) \times ... \times [0, \gamma_s)$ is of bounded remainder with respect to $(\mathbf{x}_n)_{n\geq 0}$ if and only if (2) is true.

3.2 Xing-Niederreiter sequence (see [3, Section 8.4]). Let F/\mathbb{F}_b be an algebraic function field with full constant field \mathbb{F}_b and genus g. Assume that F/\mathbb{F}_b has at least one rational place P_{∞} , and let G be a positive divisor of F/\mathbb{F}_b with $\deg(G) = 2g$ and $P_{\infty} \notin \operatorname{supp}(G)$. Let $P_1, ..., P_s$ be s distinct places of F/\mathbb{F}_b with $P_i \neq P_{\infty}$ for $1 \leq i \leq s$. Put $e_i = \deg(P_i)$ for $1 \leq i \leq s$.

By [3, p.279], we have that there exists a basis $w_0, w_1, ..., w_q$ of $\mathcal{L}(G)$ over \mathbb{F}_b such that

$$\nu_{P_{\infty}}(w_u) = n_u$$
 for $0 < u < q$,

where $0 = n_0 < n_1 < \dots < n_g \le 2g$. For each $1 \le i \le s$, we consider the chain

$$\mathcal{L}(G) \subset \mathcal{L}(G+P_i) \subset \mathcal{L}(G+2P_i) \subset \dots$$

of vector spaces over \mathbb{F}_b . By starting from the basis $w_0, w_1, ..., w_g$ of $\mathcal{L}(G)$ and successively adding basis vectors at each step of the chain, we obtain for each $n \in \mathbb{N}$ a basis

$$\{w_0, w_1, ..., w_q, k_1^{(i)}, k_2^{(i)}, ..., k_{ne_i}^{(i)}\}$$

of $\mathcal{L}(G+nP_i)$. We note that we then have

$$k_j^{(i)} \in \mathcal{L}(G + ([(j-1)/e_i + 1)]P_i) \text{ for } 1 \le i \le s \text{ and } j \ge 1.$$
 (15)

Lemma C. ([3, Lemma 8.10]) The system $\{w_0, w_1, ..., w_g\} \cup \{k_j^{(i)}\}_{1 \leq i \leq s, j \geq 1}$ of elements of F is linearly independent over \mathbb{F}_b .

Let z be an arbitrary local parameter at P_{∞} . For $r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we put

$$z_r = \begin{cases} z^r & \text{if } r \notin \{n_0, n_1, ..., n_g\}, \\ w_u & \text{if } r = n_u \text{ for some } u \in \{0, 1, ..., g\}. \end{cases}$$
 (16)

Note that in this case $\nu_{P_{\infty}}(z_r) = r$ for all $r \in \mathbb{N}_0$. For $1 \le i \le s$ and $j \in \mathbb{N}$, we have $k_j^{(i)} \in \mathcal{L}(G + nP_i)$ for some $n \in \mathbb{N}$ and also $P_{\infty} \notin \text{supp}(G + nP_i)$, hence $\nu_{P_{\infty}}(k_j^{(i)}) \ge 0$. Thus we have the local expansions

$$k_j^{(i)} = \sum_{r=0}^{\infty} a_{j,r}^{(i)} z_r \quad \text{for } 1 \le i \le s \quad \text{and } j \in \mathbb{N},$$

$$\tag{17}$$

where all coefficients $a_{j,r}^{(i)} \in \mathbb{F}_b$. Let $H_1 = \mathbb{N}_0 \setminus H_2 = \{h(0), h(1), ...\}, H_2 = \{n_0, n_1, ..., n_g\}$.

For $1 \leq i \leq s$ and $j \in \mathbb{N}$, we now define the sequences

$$c_{j,r}^{(i)} = a_{j,h(r)}^{(i)}, \quad \mathbf{c}_{j}^{(i)} = (c_{j,0}^{(i)}, c_{j,1}^{(i)}, \dots) := (a_{j,n}^{(i)})_{n \in \mathbb{N}_{0} \setminus \{n_{0}, \dots, n_{g}\}} = (a_{j,h(r)}^{(i)})_{r \geq 0}$$

$$= (\widehat{a_{j,n}^{(i)}}, a_{j,n_{0}+1}^{(i)}, \dots, \widehat{a_{j,n_{1}}^{(i)}}, a_{j,n_{1}+1}^{(i)}, \dots, \widehat{a_{j,n_{g}}^{(i)}}, a_{j,n_{g}+1}^{(i)}, \dots) \in \mathbb{F}_{b}^{\mathbb{N}},$$

$$(18)$$

where the hat indicates that the corresponding term is deleted.

We define the matrices $C^{(1)}, ..., C^{(s)} \in \mathbb{F}_b^{\mathbb{N} \times \mathbb{N}}$ by

$$C^{(i)} = (\mathbf{c}_1^{(i)}, \mathbf{c}_2^{(i)}, \mathbf{c}_3^{(i)}, \dots)^{\top} \quad \text{for} \quad 1 \le i \le s,$$
 (19)

i.e., the vector $\mathbf{c}_{i}^{(i)}$ is the jth row vector of $C^{(i)}$ for $1 \leq i \leq s$.

Theorem E (see [3, Theorem 8.11] and [7, Theorem 1]). With the above notations, we have that the matrices $C^{(1)}, ..., C^{(s)}$ given by (19) are generating matrices of the Xing-Niederreiter d-admissible digital (t, s)-sequence $(\mathbf{x}_n)_{n>0}$ with $d = e_1 + ... + e_s$, $t = g + e_1 + ... + e_s - s$.

In order to obtain the bounded remainder set property, we will take a specific local parameter z. Let $P_0 \in \mathbb{P}_F$, $P_0 \not\subset \{P_1, ..., P_s, P_\infty\}$, $P_0 \not\in \text{supp}(G)$ and $\deg(P_0) = e_0$. By the Riemann-Roch theorem, there exists a local parameter z at P_∞ , with

$$z \in \mathcal{L}((2g+1)P_0 - P_\infty) \setminus \mathcal{L}((2g+1)P_0 - 2P_\infty). \tag{20}$$

Theorem 2. With the notations as above, the set $[0, \gamma_1) \times ... \times [0, \gamma_s)$ is of bounded remainder with respect to $(\mathbf{x}_n)_{n\geq 0}$ if and only if (2) is true.

3.3 Generalized Halton-type sequences from global function fields.

Let $q \geq 2$ be an integer

$$n = \sum_{j \ge 1} e_{q,j}(n)q^{j-1}, \ e_{q,j}(n) \in \{0, 1, \dots, q-1\}, \ \text{and} \ \varphi_q(n) = \sum_{j \ge 1} e_{q,j}(n)q^{-j}.$$

Van der Corput proved that $(\varphi_q(n))_{n\geq 0}$ is a 1-dimensional l.d.s. Let

$$\hat{H}_s(n) = (\varphi_{\hat{q}_1}(n), \dots, \varphi_{\hat{q}_s}(n)), \quad n = 0, 1, 2, \dots,$$

where $\hat{q}_1, \ldots, \hat{q}_s \geq 2$ are pairwise coprime integers. Halton proved that $(\hat{H}_s(n))_{n\geq 0}$ is an s-dimensional l.d.s. (see [11]).

Let $Q = (q_1, q_2,)$ and $Q_j = q_1q_2....q_j$, where $q_j \ge 2$ (j = 1, 2, ...) is a sequence of integers. Every nonnegative integer n then has a unique Q-adic representation of the form

$$n = \sum_{j=1}^{\infty} n_j q_1 \cdots q_{j-1} = n_1 + n_2 q_1 + n_3 q_1 q_2 + \cdots,$$

where $n_j \in \{0, 1, ..., q_j - 1\}$. We call this the Cantor expansion of n with respect to the base Q. Consider Cantor's expansion of $x \in [0, 1)$:

$$x = \sum_{j=1}^{\infty} x_j/Q_j$$
, $x_j \in \{0, 1, \dots, q_j - 1\}$, $x_j \neq q_j - 1$ for infinitely many j .

The Q-adic representation of x is then unique. We define the radical inverse function

$$\varphi_Q\left(\sum_{j=1}^{\infty} n_j q_1 \cdots q_{j-1}\right) = \sum_{j=1}^{\infty} \frac{n_j}{q_1 \cdots q_j}.$$

Let $p_{i,j} \geq 2$ be integers $(s \geq i \geq 1, j \geq 1)$, $g.c.d.(p_{i,k}, p_{j,l}) = 1$ for $i \neq j$, $\tilde{P}_{i,0} = 1$, $\tilde{P}_{i,j} = \prod_{1 \leq k \leq j} p_{i,k}, i \in [1, s], j \geq 1$, $\mathcal{P}_i = (p_{i,1}, p_{i,2}, ...), \mathcal{P} = (\mathcal{P}_1, ..., \mathcal{P}_s)$. In [5], Hellecaleq proposed the following generalisation of the Halton sequence:

$$H_{\mathcal{P}} = (\varphi_{\mathcal{P}_1}(n), \dots, \varphi_{\mathcal{P}_s}(n))_{n=0}^{\infty}.$$
 (21)

In [Te], Tezuka introduced a polynomial arithmetic analogue of the Halton sequence: Let p(x) be an arbitrary nonconstant polynomial over \mathbb{F}_b , $e = \deg(p)$,

$$n = a_0(n) + a_1(n)b + \dots + a_m(n)b^m$$
.

We fix a bijection $\phi: \mathbb{Z}_b \to \mathbb{F}_b$ with $\phi(0) = \bar{0}$. Denote $v_n(x) = \bar{a}_0(n) + \bar{a}_1(n)x + \cdots + \bar{a}_m(n)x^m$, where $\bar{a}_r(n) = \phi(a_r(n))$, r = 0, 1, ..., m. Then $v_n(x)$ can be represented in terms of p(x) in the following way:

$$v_n(x) = r_0(x) + r_1(x)p(x) + \dots + r_k(p(x))^k$$
, with $k = [m/e]$.

We define the radical inverse function $\varphi_{p(x)}: \mathbb{F}_b[x] \to \mathbb{F}_b(x)$ as follows

$$\varphi_{p(x)}(v_n(x)) = r_0(x)/p(x) + r_1(x)/p^2(x) + \dots + r_k/(p(x))^{k+1}$$

Let $p_1(x), ..., p_s(x)$ be pairwise coprime. Then Tezuka's sequence is defined as follows

$$\mathbf{x}_n = (\sigma_1(\varphi_{p_1(x)}(n)), \dots, \sigma_s(\varphi_{p_s(x)}(n)))_{n=0}^{\infty},$$

where each σ_i is a mapping from F to the real field defined by $\sigma_i(\sum_{j\geq w} \dot{a}_j x^{-j}) = \sum_{j\geq w} \phi^{-1}(\dot{a}_j)b^{-j}$. By [Te], $(\mathbf{x}_n)_{n\geq 0}$ is a (t,s) sequence in base b.

In 2010, Levin [7] and in 2013, Niederreiter and Yeo [12] generalized Tezuka's construction to the case of arbitrary algebraic function fields F. The construction of [12] is follows:

Let F/\mathbb{F}_b be an algebraic function field with full constant field \mathbb{F}_b and genus g. We assume that F/\mathbb{F}_b has at least one rational place, that is, a place of degree 1. Given a dimension $s \geq 1$, we choose s+1 distinct places $P_1,...,P_s,P_\infty$ of F with $\deg(P_\infty)=1$. The degrees of the places $P_1,...,P_s$ are arbitrary and we put $e_i=\deg(P_i)$ for $1\leq i\leq s$. Denote by O_F the holomorphy ring given by $O_F=\bigcap_{P\neq P_\infty}O_P$, where the intersection is extended over all places $P\neq P_\infty$ of F, and O_P is the valuation ring of P. We arrange the elements of O_F into a sequence by using the fact that $O_F=\bigcup_{m\geq 0}\mathcal{L}(mP_\infty)$. The terms of this sequence are denoted by $f_0,f_1,...$ and they are obtained as

follows. Consider the chain $\mathcal{L}(0) \subseteq L(P_{\infty}) \subseteq L(2P_{\infty}) \subseteq \cdots$ of vector spaces over \mathbb{F}_b . At each step of this chain, the dimension either remains the same or increases by 1. From a certain point on, the dimension always increases by 1 according to the Riemann-Roch theorem. Thus we can construct a sequence v_0, v_1, \ldots of elements of O_F such that $\{v_0, v_1, \ldots, v_{\ell(mP_{s+1})-1}\}$ is a \mathbb{F}_b -basis of $\mathcal{L}(mP_{s+1})$. We fix a bijection $\phi : \mathbb{Z}_b \to \mathbb{F}_b$ with $\phi(0) = \bar{0}$. Then we define

$$f_n = \sum_{r=0}^{\infty} \bar{a}_r(n) v_r \in O_F$$
 with $\bar{a}_r(n) = \phi(a_r(n))$ for $n = \sum_{r=0}^{\infty} a_r(n) b^r$.

Note that the sum above is finite since for each $n \in \mathbb{N}$. We have $a_r(n) = 0$ for all sufficiently large r. By the Riemann-Roch theorem, we have

$$\{\tilde{f} \mid \tilde{f} \in \mathcal{L}((m+g-1)P_{s+1})\} = \{f_n \mid n \in [0, b^m)\} \text{ for } m \ge g.$$

For each i=1,...,s, let \wp_i be the maximal ideal of O_F corresponding to P_i . Then the residue class field $F_{P_i}:=O_F/\wp_i$ has order b^{e_i} (see [14, Proposition 3.2.9]). We fix a bijection $\sigma_{P_i}:F_{P_i}\to Z_{b^{e_i}}$. For each i=1,...,s, we can obtain a local parameter $t_i\in O_F$ at \wp_i , by applying the Riemann-Roch theorem and choosing $t_i\in \mathcal{L}(kP_\infty-P_i)\setminus \mathcal{L}(kP_\infty-2P_i)$ for a suitably large integer k. We have a local expansion of f_n at \wp_i of the form

$$f_n = \sum_{j>0} f_{n,j}^{(i)} t_i^j$$
 with all $f_{n,j}^{(i)} \in F_{P_i}, \ n = 0, 1, \dots$

We define the map $\xi: O_F \to [0,1]^s$ by

$$\xi(f_n) = \Big(\sum_{j=0}^{\infty} \sigma_{P_1}(f_{n,j}^{(1)})(b^{e_1})^{-j-1}, \dots, \sum_{j=0}^{\infty} \sigma_{P_s}(f_{n,j}^{(s)})(b^{e_s})^{-j-1}\Big).$$

Now we define the sequence $\mathbf{x}_0, \mathbf{x}_1, \dots$ of points in $[0,1]^s$ by $\mathbf{x}_n = \xi(f_n)$ for $n = 0, 1, \dots$. From [12, Theorem 1], we get the following theorem:

Theorem F. With the notation as above, we have that $(\mathbf{x}_n)_{n\geq 0}$ is a (t,s)-sequence over \mathbb{F}_b with $t=g+e_1+...+e_s-s$.

The construction of Levin [7] is similar, but more complicated than in [12]. However in [7], we can use arbitrary pairwise coprime divisors $D_1, ..., D_s$ instead of places $P_1, ..., P_s$.

In this paper, we introduce the Hellecalek-like generalisation (21) of the above construction: Let $\mathbb{P}_F := \{P | P \text{ be a place of } F/\mathbb{F}_b\}, P_0, P_\infty \in \mathbb{P}_F, \deg(P_\infty) = 1, \deg(P_0)$

 $= e_0, P_0 \neq P_{\infty}, P_{i,j} \in \mathbb{P}_F \text{ for } 1 \leq j, 1 \leq i \leq s, P_{i_1,j_1} \neq P_{i_2,j_2} \text{ for } i_1 \neq i_2, P_{i,j} \neq P_0, P_{i,j} \neq P_{\infty} \text{ for all } i, j, \dot{n}_{i,j} = \deg(P_{i,j}), n_{i,j} = \deg(\mathcal{P}_{i,j}), \mathcal{P}_{0,j} = P_0^j,$

$$\mathcal{P}_{i,0} = 1, \ \mathcal{P}_{i,j} = \prod_{1 \le k \le j} P_{i,k}, \ n_{i,j} = \deg(\mathcal{P}_{i,j}) = n_{i,j-1} + \dot{n}_{i,j}, n_{i,0} = 0.$$
 (22)

Let $i \in [0, s]$. We will construct a basis $(w_j^{(i)})_{j \geq 0}$ of O_F in the following way. Let

$$L_{i,j} = \mathcal{L}((n_{i,j} + 2g - 1)P_{\infty}) = \mathcal{L}(A_{i,j}), \ A_{i,j} = (n_{i,j} + 2g - 1)P_{\infty},$$

$$\mathfrak{L}_{i,j} = \mathcal{L}((n_{i,j} + 2g - 1)P_{\infty} - \mathcal{P}_{i,j}) = \mathcal{L}(B_{i,j}), \ B_{i,j} = (n_{i,j} + 2g - 1)P_{\infty} - \mathcal{P}_{i,j},$$

$$\mathcal{L}_{i,j} = \mathcal{L}((n_{i,j} + 2g - 1)P_{\infty} - \mathcal{P}_{i,j-1}), \ \dot{B}_{i,j} = (n_{i,j} + 2g - 1)P_{\infty} - \mathcal{P}_{i,j-1}.$$
(23)

Using the Riemann-Roch theorem, we obtain

$$\deg(A_{i,j}) = n_{i,j} + 2g - 1, \quad \dim(L_{i,j}) = n_{i,j} + g, \quad \deg(B_{i,j}) = 2g - 1,$$

$$\dim(\mathfrak{L}_{i,j}) = g, \quad \deg(\dot{B}_{i,j}) = \dot{n}_{i,j} + 2g - 1, \quad \dim(\mathcal{L}_{i,j}) = \dot{n}_{i,j} + g.$$
(24)

Let $(u_{j,\mu}^{(i)})_{\mu=1}^g$ be a \mathbb{F}_b linear basis of $\mathfrak{L}_{i,j}$. By (23) and (24), we get that the basis $(u_{j,\mu}^{(i)})_{\mu=1}^g$ can be extended to a basis $(v_{j,1}^{(i)}, \dots, v_{j,n_{i,j}}^{(i)}, u_{j,1}^{(i)}, \dots, u_{j,g}^{(i)})$ of $\mathscr{L}_{i,j}$.

Bearing in mind that $(u_{j,\mu}^{(i)})_{\mu=1}^g$ is a \mathbb{F}_b linear basis of $\mathfrak{L}_{i,j}$, we obtain that $v_{j,\mu}^{(i)} \notin \mathfrak{L}_{i,j}$ for $\mu \in [1, \dot{n}_{i,j}]$. So

$$v_{j,\mu}^{(i)} \in \mathbb{L}_{i,j} := \mathcal{L}_{i,j} \setminus \mathfrak{L}_{i,j} \quad \text{for} \quad \mu \in [1, \dot{n}_{i,j}].$$
 (25)

Let

$$V_{i,j} := \{ v_{k,\mu}^{(i)} \mid 1 \le \mu \le \dot{n}_{i,k}, \ 1 \le k \le j \} \cup \{ u_{j,\mu}^{(i)} \mid \mu = 1, ..., g \}.$$
 (26)

We claim that vectors from $V_{i,j}$ are \mathbb{F}_b linear independent. Suppose the opposite. Assume that there exists $b_{k,u}^{(i)} \in \mathbb{F}_b$ such that

$$\dot{u} + \ddot{u} = 0$$
, where $\dot{u} = \sum_{k=1}^{j} w_k$, $w_k = \sum_{\mu=1}^{\dot{n}_{i,k}} b_{k,\mu}^{(i)} v_{k,\mu}^{(i)}$, $\ddot{u} = \sum_{\mu=1}^{g} b_{0,\mu}^{(i)} u_{j,\mu}^{(i)}$. (27)

Let $w_l \neq 0$ for some $l \in [1, j]$ and let $w_k = 0$ for all $k \in [1, l)$. Using (23) - (25), we get

$$w_l \in \mathbb{L}_{i,l} = \mathcal{L}((n_{i,l} + 2g - 1)P_{\infty} - \mathcal{P}_{i,l-1}) \setminus \mathcal{L}((n_{i,l} + 2g - 1)P_{\infty} - \mathcal{P}_{i,l}).$$

Applying definition (13) of the Riemann-Roch space, we obtain

$$w_l \in \mathcal{L}((n_{i,j}+2g-1)P_{\infty}-\mathcal{P}_{i,l-1}) \setminus \mathcal{L}((n_{i,j}+2g-1)P_{\infty}-\mathcal{P}_{i,l}).$$

But from (27), (22) and (25), we have

$$-w_l = \dot{u} + \ddot{u} - \sum_{k=1}^l w_k = \sum_{k=l+1}^j w_k + \ddot{u} \in \mathcal{L}((n_{i,j} + 2g - 1)P_{\infty} - \mathcal{P}_{i,l}).$$

We have a contradiction. Hence vectors from $V_{i,j}$ are \mathbb{F}_b linear independent. By (23) - (26), we have $V_{i,j} \subset L_{i,j}$ and

$$\operatorname{card}(V_{i,j}) = \sum_{k=1}^{j} \dot{n}_{i,k} + g = n_{i,j} + g = \dim(L_{i,j}).$$

Hence vectors from $V_{i,j}$ are the \mathbb{F}_b linear basis of $L_{i,j}$.

Now we will find a basis of $L_{i,j-2g}$. We claim that $u_{j,\mu}^{(i)} \notin L_{i,j-2g}$ for $\mu \in [1,g]$. Suppose the opposite. By (23) and (24), we get

$$u_{j,\mu}^{(i)} \in L_{i,j-2g} \cap \mathfrak{L}_{i,j} = \mathcal{L}((n_{i,j-2g} + 2g - 1)P_{\infty}) \cap \mathcal{L}((n_{i,j} + 2g - 1)P_{\infty} - \mathcal{P}_{i,j})$$

= $\mathcal{L}((n_{i,j-2g} + 2g - 1)P_{\infty} - \mathcal{P}_{i,j}) = \mathcal{L}(T).$

By (22), $\deg(T) = n_{i,j-2g} + 2g - 1 - n_{i,j} < 0$. Hence $\mathcal{L}(T) = \{0\}$. We have a contradiction. Bearing in mind that $V_{i,j}$ is \mathbb{F}_b linear basis of $L_{i,j}$, we obtain that a basis of $L_{i,j-2g}$ can be chosen from the

set
$$(v_{1,1}^{(i)},...,v_{1,\dot{n}_{i,1}}^{(i)},...,v_{j,1}^{(i)},...,v_{j,\dot{n}_{i,j}}^{(i)})$$

= $V_{i,j}\setminus\{u_{j,\mu}^{(i)}\mid\mu=1,...,g\}$. From (23) - (25), we get

$$v_{k,\mu}^{(i)} \in \mathcal{L}_{i,k} \subseteq L_{i,j-2g}$$
 for $\mu \in [1, \dot{n}_{i,k}]$ and $1 \le k \le j-2g$

Hence vectors

$$v_{1,1}^{(i)},...,v_{1,\dot{n}_{i,1}}^{(i)},...,v_{j-2g,1}^{(i)},...,v_{j-2g,\dot{n}_{i,j-2g}}^{(i)},\tilde{v}_{j,1}^{(i)},...,\tilde{v}_{j,g}^{(i)}\quad\text{with}\quad \tilde{v}_{j,\mu}^{(i)}=v_{k,\rho}^{(i)},$$

 $1 \le \mu \le g$, for some $\rho \in [1, \dot{n}_{i,k}]$ and $k \in (j-2g, j]$ are an \mathbb{F}_b linear basis of $L_{i,j-2g}$ $(0 \le i \le s)$.

Therefore $(v_{k,\mu}^{(i)})_{1 \leq \mu \leq \hat{n}_{i,k,k \geq 1}}$ is the \mathbb{F}_b linear basis of $O_F = \bigcup_{j \geq 1} L_{i,j}$. We put in order the basis $(v_{k,\mu}^{(i)})_{1 \leq \mu \leq \hat{n}_{i,k},k \geq 1}$ as follows

$$w_{n_{i,j-1}+\mu-1}^{(i)} = v_{j,\mu}^{(i)}, \quad \text{with} \quad n_{i,0} = 0, \ 1 \le \mu \le \dot{n}_{i,j}, \ 0 \le i \le s.$$
 (28)

So we proved the following lemma:

Lemma 1. For all $i \in [0, s]$ there exists a sequence $(w_j^{(i)})_{j \geq 0}$ such that $(w_j^{(i)})_{j \geq 0}$ is a \mathbb{F}_b linear basis of O_F and for all $j \geq 1$ a \mathbb{F}_b linear basis of $L_{i,j}$ can be chosen from the set $\{w_0^{(i)}, ..., w_{n_{i,i+2a-1}}^{(i)}\}$.

Bearing in mind that $(w_j^{(i)})_{j\geq 0}$ is the \mathbb{F}_b linear basis of O_F , we obtain for all $i\in [1,s]$ and $r\geq 0$ that there exists $c_{j,r}^{(i)}\in \mathbb{F}_b$ and integers $l_r^{(i)}$ such that

$$w_r^{(0)} = \sum_{j=1}^{l_r^{(i)}} c_{j,r}^{(i)} w_{j-1}^{(i)}, \quad c_{j,j-1}^{(0)} = 1, \text{ and } c_{j,r}^{(0)} = 0 \text{ for } j - 1 \neq r.$$
(29)

Let $n = \sum_{r \geq 0} a_r(n) b^r$. We fix a bijection $\phi : \mathbb{Z}_b \to \mathbb{F}_b$ with $\phi(0) = \bar{0}$. Then we define

$$f_n = \sum_{r=0}^{\infty} \bar{a}_r(n) w_r^{(0)} \in O_F \text{ with } \bar{a}_r(n) = \phi(a_r(n)) \text{ for } n = 0, 1, \dots$$
 (30)

By (29), we have for $i \in [0, s]$

$$f_n = \sum_{r=0}^{\infty} \bar{a}_r(n) \sum_{j=1}^{l_{i,r}} c_{j,r}^{(i)} w_{j-1}^{(i)} = \sum_{j=1}^{\infty} w_{j-1}^{(i)} \sum_{r=0}^{\infty} \bar{a}_r(n) c_{j,r}^{(i)} = \sum_{j=1}^{\infty} y_{n,j}^{(i)} w_{j-1}^{(i)}$$
(31)

where $y_{n,j}^{(i)} = \sum_{r \geq 0} \bar{a}_r(n) c_{j,r}^{(i)} \in \mathbb{F}_b$, $y_{n,j}^{(0)} = \bar{a}_{j-1}(n)$. We map the vectors

$$y_n^{(i)} = (y_{n,1}^{(i)}, y_{n,2}^{(i)}, \dots)$$
(32)

to the real numbers

$$x_n^{(i)} = \sum_{j \ge 1} \phi^{-1}(y_{n,j}^{(i)})/b^j$$

to obtain the point

$$\mathbf{x}_n := (x_n^{(1)}, \dots, x_n^{(s)}) \in [0, 1)^s. \tag{33}$$

Theorem 3. With the notations as above, the set $[0, \gamma_1) \times ... \times [0, \gamma_s)$ is of bounded remainder with respect to $(\mathbf{x}_n)_{n\geq 0}$ if and only if (2) is true.

Remark. It is easy to verify that Hellekalek's sequence and our generalized Halton-type sequence $(\mathbf{x}_n)_{n\geq 0}$ are l.d.s if

$$\limsup_{m \to \infty} m^{-s} \sum_{i=1}^{s} \sum_{j=1}^{m} \log(p_{i,j}) < \infty \text{ and } \limsup_{m \to \infty} m^{-s} \sum_{i=1}^{s} \sum_{j=1}^{m} \deg(P_{i,j}) < \infty.$$

3.4 Niederreiter-Xing sequence (see [3, Section 8.3]). Let F/\mathbb{F}_b be an algebraic function field with full constant field \mathbb{F}_b and genus g. Assume that F/\mathbb{F}_b has at least s+1 rational places. Let $P_1, ..., P_{s+1}$ be s+1 distinct rational places of F. Let $G_m = m(P_1 + ... + P_s) - (m-g+1)P_{s+1}$, and let t_i be a local parameter at P_i , $1 \le i \le s+1$. For any $f \in \mathcal{L}(G_m)$ we have $\nu_{P_i}(f) \ge -m$, and so the local expansion of f at P_i has the form

$$f = \sum_{j=-m}^{\infty} f_{i,j} t_i^j$$
, with $f_{i,j} \in \mathbb{F}_b$, $j \ge -m$, $1 \le i \le s$.

For $1 \leq i \leq s$, we define the \mathbb{F}_b -linear map $\psi_{m,i} : \mathcal{L}(G_m) \to \mathbb{F}_b^m$ by

$$\psi_{m,i}(f) = (f_{i,-1}, ..., f_{i,-m}) \in \mathbb{F}_b^m, \text{ for } f \in \mathcal{L}(G_m).$$

Let

$$\mathcal{M}_m = \mathcal{M}_m(P_1, ..., P_s; G_m) := \{ (\psi_{m,1}(f), ..., \psi_{m,s}(f)) \in \mathbb{F}_b^{ms} \mid f \in \mathcal{L}(G_m) \}.$$

Let $C^{(1)},...,C^{(s)} \in \mathbb{F}_b^{\infty \times \infty}$ be the generating matrices of a digital sequence $\mathbf{x}_n(C)_{n\geq 0}$, and let $(\mathcal{C}_m)_{m\geq 1}$ be the associated sequence of row spaces of overall generating matrices $[C]_m$, m=1,2,... (see (7)).

Theorem G. (see [3, Theorem 7.26 and Theorem 8.9]) There exist matrices $C^{(1)}, ..., C^{(s)}$ such that $(\mathbf{x}_n(C))_{n\geq 0}$ is a digital (t,s)-sequence with t=g and $\mathcal{C}_m^{\perp}=\mathcal{M}_m(P_1,...,P_s;G_m)$ for $m\geq g+1$, $s\geq 2$.

In [8, p.24], we proposed the following way to get $\mathbf{x}_n(C)_{n\geq 0}$:

We consider the H-differential dt_{s+1} . Let ω be the corresponding Weil differential, $\operatorname{div}(\omega)$ the divisor of ω , and $W := \operatorname{div}(dt_{s+1}) = \operatorname{div}(\omega)$. By (9)-(11), we have $\operatorname{deg}(W) = 2g - 2$. We consider a sequence $\dot{v}_0, \dot{v}_1, \ldots$ of elements of F such that $\{\dot{v}_0, \dot{v}_1, \ldots, \dot{v}_{\ell((m-g+1)P_{s+1}+W)-1}\}$ is an \mathbb{F}_b linear basis of $L_m := \mathcal{L}((m-g+1)P_{s+1}+W)$ and

$$\dot{v}_r \in L_{r+1} \setminus L_r$$
, $\nu_{P_{s+1}}(\dot{v}_r) = -r + g - 2$, $r \ge g$, and $\dot{v}_{r,r+2-g} = 1$, $\dot{v}_{r,j} = 0$ (34)

for $2 \le j < r + 2 - g$, where

$$\dot{v}_r := \sum_{j \le r-g+2} \dot{v}_{r,j} t_{s+1}^{-j} \quad \text{for} \quad \dot{v}_{r,j} \in \mathbb{F}_b \text{ and } r \ge g.$$

According to Lemma B, we have that there exists $\tau_i \in F$ $(1 \leq i \leq s)$ such that $dt_{s+1} = \tau_i dt_i$, for $1 \leq i \leq s$.

Bearing in mind (10), (12) and (34), we get

$$\nu_{P_i}(\dot{v}_i \tau_i) = \nu_{P_i}(\dot{v}_i \tau_i dt_i) = \nu_{P_i}(\dot{v}_i dt_{s+1}) \ge \nu_{P_i}(\operatorname{div}(dt_{s+1}) - W) = 0, \quad j \ge 0.$$

We consider the following local expansions

$$\dot{v}_r \tau_i := \sum_{j=1}^{\infty} \dot{c}_{j,r}^{(i)} t_i^{j-1}, \text{ where all } \dot{c}_{j,r}^{(i)} \in \mathbb{F}_b, \ 1 \le i \le s, \ j \ge 1.$$
 (35)

Now let $\dot{C}^{(i)} = (\dot{c}_{j,r}^{(i)})_{j-1,r\geq 0}, \ 1\leq i\leq s$, and let $(\dot{C}_m^{\perp})_{m\geq 1}$ be the associated sequence of row spaces of overall generating matrices $[\dot{C}]_m, \ m=1,2,...$ (see (7)).

Theorem H (see [8, Theorem 5]). With the above notations, $(\mathbf{x}_n(\dot{C}))_{n\geq 0}$ is a digital d-admissible (t,s) sequence with d=g+s, t=g, and $\dot{\mathcal{C}}_m^{\perp}=\mathcal{M}_m(P_1,...,P_s;G_m)$ for all $m\geq g+1$.

We note that condition (34) is required in the proof of Theorem H only in order to get the discrepancy lower bound. While the equality $\dot{\mathcal{C}}_m^{\perp} = \mathcal{M}_m(P_1,...,P_s;G_m)$ is true for arbitrary sequence $\dot{v}_0,\dot{v}_1,...$ of elements of \mathbb{F}_b such that for all $m \geq 1$

$$\{\dot{v}_0, \dot{v}_1, ..., \dot{v}_{\ell((m-g+1)P_{s+1}+W)-1}\}\$$
 is a \mathbb{F}_b linear basis of L_m . (36)

In order to obtain the bounded remainder property, in this paper, we will construct from $(\dot{v}_n)_{n\geq 0}$ a special basis $(\ddot{v}_n)_{n\geq 0}$ as follows:

Let $P_0 \in \mathbb{P}_F$, $P_0 \neq P_i$ (i = 1, ..., s + 1), and let t_0 be a local parameter of P_0 . For simplicity, we suppose that $\deg(P_0) = 1$. Let

$$L_m = \mathcal{L}((m-g+1)P_{s+1} + W), \quad \mathcal{L}_m = \mathcal{L}((m+2)P_{s+1} + W - mP_0),$$

$$\mathfrak{L}_m = \mathcal{L}((m+2)P_{s+1} + W - (m+1)P_0).$$
(37)

It is easy to verify that

$$\deg(\mathcal{L}_m) = 2g, \quad \dim(\mathcal{L}_m) = g+1, \quad \deg(\mathfrak{L}_m) = 2g-1, \quad \dim(\mathfrak{L}_m) = g,$$

for $m \ge 0$, $\deg(L_m) = m + g - 1$, $\dim(L_m) = m$, for $m \ge g$. (38)

Using the Riemann-Roch theorem, we have that there exists

$$w_m \in \mathcal{L}_m \setminus \mathfrak{L}_m$$
, and $w_m \in L_{m+q+1}$, $m = 0, 1, \dots$ (39)

According to Lemma B, we have that there exists $\tau_0 \in F$, such that $dt_{s+1} = \tau_0 dt_0$.

Let $u \in L_m = \mathcal{L}((m-g+1)P_{s+1} + W)$ with $m \ge 0$. Bearing in mind (10), (12), (37)-(39) and the Riemann-Roch theorem, we get

$$\nu_{P_0}(u\tau_0) = \nu_{P_0}(u\tau_0 dt_0) = \nu_{P_0}(udt_{s+1}) = \nu_{P_0}(\operatorname{div}(u) + W) \ge 0 \tag{40}$$

and

$$\nu_{P_0}(w_m \tau_0) = \nu_{P_0}(\operatorname{div}(w_m) + W) = m \quad \text{for} \quad m = 0, 1, \dots$$
 (41)

We consider the sequence $(\dot{v}_j)_{j\geq 0}$ (34). By (36), $(\dot{v}_j)_{j=0}^{m-1}$ is an \mathbb{F}_b linear basis of L_m . Let

$$V_j = \{\dot{v}_j + \sum_{k=0}^{j-1} b_k \dot{v}_k \mid b_k \in \mathbb{F}_b, \ k \in [0, j)\}, \ \alpha(j) = \max_{v \in V_j} \nu_{P_0}(v\tau_0).$$
 (42)

It is easy to verify that $\alpha(j) \neq \alpha(j)$ for $i \neq j$. We construct a sequence $(\ddot{v}_i)_{i>0}$ as follows:

$$\ddot{v}_0 = \dot{v}_0, \quad \ddot{v}_j \in \{ v \in V_j \mid \nu_{P_0}(v\tau_0) = \alpha(j) \}, \qquad j = 1, 2, \dots$$
(43)

It is easy to see that $(\ddot{v}_j)_{j\geq 0}$ satisfy the condition (36). Bearing in mind (40)-(42) and that $\ddot{v}_j \in L_m$ for j < m, we get

$$\nu_{P_0}(\ddot{v}_j\tau_0) \neq \nu_{P_0}(\ddot{v}_k\tau_0) \text{ for } j \neq k, \text{ and } \nu_{P_0}(\ddot{v}_j\tau_0) = \alpha(j) \geq 0, \quad j \geq 0.$$
 (44)

Hence, for all $f \in L_m$, we have

$$\nu_{P_0}(f\tau_0) \in \{\alpha(0), \alpha(1), ...\} =: \dot{H}.$$

Taking into account (41) and (44), we obtain

$$\dot{H} = \{ n \mid n \ge 0 \} = \mathbb{N}_0. \tag{45}$$

Suppose that $\alpha(j) > j + g$. By (36) - (38), $\ddot{v}_j \in L_{j+1} = \mathcal{L}((j-g+2)P_{s+1} + W)$. Hence $\ddot{v}_j \in \mathcal{L}(X)$, with $X = (j-g+2)P_{s+1} + W - (j+g+1)P_0$.

Bearing in mind that $\deg(P_0) = \deg(P_{s+1}) = 1$ and $\deg(W) = 2g - 2$, we get $\deg(X) = -1$. Therefore $\mathcal{L}(X) = \{0\}$ and we have a contradiction. Hence

$$\alpha(j) \le j + g. \tag{46}$$

By (45), we have that for every integer $k \geq 0$ there exists $r \geq 0$ with $\alpha(r) = k$. Therefore the map $\alpha : \mathbb{N}_0 \to \mathbb{N}_0$ is an isomorphism. Hence there exist integers $\beta(k) \geq 0$ such that

$$\beta(k) = \alpha^{-1}(k), \ \alpha(\beta(k)) = k \text{ and } \beta(\alpha(k)) = k \text{ for } k = 0, 1, \dots$$
 (47)

From (46), we get for $j = \beta(k)$

$$k = \alpha(\beta(k)) = \alpha(j) \le j + g = \beta(k) + g. \tag{48}$$

Let

$$B_j = \{ r \ge 0 \mid \alpha(r) < j \}. \tag{49}$$

Taking $r = \beta(k)$, we get $\alpha(r) = k$ and

$$B_j = \{\beta(0), \beta(1), ..., \beta(j-1)\} \text{ for } j \ge 1.$$
 (50)

Suppose $j \notin B_{j+g+1}$ for some j, then $j = \beta(j+g+l)$ for some $l \ge 1$. Using (48) with k = j+g+l, we obtain

$$j + l = (j + q + l) - q < \beta(j + q + l) = j.$$

We have a contradiction. Hence

$$j \in B_{j+q+1}$$
 for all $j \ge 0$.

We consider the local expansion (35), applied to i = 0:

$$\dot{v}_r \tau_0 := \sum_{j=1}^{\infty} \dot{c}_{j,r}^{(0)} t_0^{j-1}, \text{ where } \dot{c}_{j,r}^{(0)} \in \mathbb{F}_b, \ j \ge 1, \ \dot{C}^{(0)} = (\dot{c}_{j,r}^{(0)})_{j-1,r \ge 0}.$$
 (51)

Let $(x_n^{(0)}(\dot{C}^{(0)}))_{n\geq 0}$ be the digital sequence generated by the matrix $\dot{C}^{(0)}$.

Now we consider the matrix $\tilde{C}^{(i)} = (\dot{c}_{j,r}^{(i)})_{j-1,r\geq 0}$, obtained from equation (35) and (51), where we take \ddot{v}_r instead of \dot{v}_r (i=0,1,...,s). Using Theorem H, we obtain that $(x_n^{(0)}(\tilde{C}^{(0)}),\mathbf{x}_n(\tilde{C}))_{n\geq 0}$ is the digital (t,s+1)-sequence with t=g. Therefore we have proved the following lemma:

Lemma 2. There exists a sequence $(\ddot{v}_j)_{j\geq 0}$ such that $(x_n^{(0)}(\tilde{C}^{(0)}), \mathbf{x}_n(\tilde{C}))_{n\geq 0}$ is the digital (t, s+1)-sequence with t=g and $\{0, 1, ..., m-1\} \subset B_{m+g}$.

In §4.4, we will prove

Theorem 4. With the notations as above, the set $[0, \gamma_1) \times ... \times [0, \gamma_s)$ is of bounded remainder with respect to $(\mathbf{x}_n(\tilde{C}))_{n\geq 0}$ if and only if (2) is true.

4. Proof

Consider the following condition

$$\liminf_{m \to \infty} (m - \mathbf{T}(m)) = \infty.$$
(52)

We will prove (52) for the generalized Halton sequence in §4.3. For other considered sequences, assertion (52) follows from Theorem D, Theorem E and Theorem H.

The sufficient part of all considered theorems follows from Definition 2 and (52). Therefore we need only consider the case of necessity.

4.1 Generalized Niederreiter sequence. Proof of Theorem 1.

From Theorem D, we have that $(\mathbf{x}_n)_{n\geq 0}$ is the uniformly distributed digital weakly admissible (t,s)-sequence in base b. By Theorem B, in order to prove Theorem 1, we need only to check condition (8). By (8, p.26, ref 4.6), we get

$$y_{n,j}^{(i)} = \underset{P_{\infty}, x^{-1}}{\text{Res}} \left(\frac{y_{i,l,k(i,j)}(x)}{p_i(x)^l} \sum_{r=0}^{m-1} \bar{a}_r(n) x^{r+2} \right) = \underset{P_{\infty}, x^{-1}}{\text{Res}} \left(\frac{y_{i,l,k(i,j)}(x)}{p_i(x)^l} n(x) \right)$$
(53)

with
$$l = Q(i, j) + 1$$
, $n(x) = \sum_{j=0}^{m-1} \bar{a}_j(n) x^{j+2}$ and $\bar{a}_j(n) = \underset{P_{\infty}, x^{-1}}{\text{Res}} (n(x) x^{-j-1})$.

We take $\dot{y}_{i,j,k}(x) = x^m y_{i,j,k}(x)$ instead of $y_{i,j,k}(x)$. Now using Theorem D, we obtain from (53), (4) - (6) that $(\dot{\mathbf{x}}_n)_{0 \leq n < b^{\dot{m}}}$ is a (t, \dot{m}, s) net for $\dot{m} = s\tau_m + t$ with $\dot{x}_{n,j}^{(i)} = \phi^{-1}(\dot{y}_{n,j}^{(i)}) = x_{b^m n,j}^{(i)}$. Bearing in mind that $\dot{\mathbf{x}}_n = \mathbf{x}_{b^m n}$, we obtain (8). Hence Theorem 1 is proved.

4.2 Xing-Niederreiter sequence. Proof of Theorem 2.

By Theorem B and Theorem E, in order to prove Theorem 2, we need only to check condition (8).

From (3) - (6), we get that in order to obtain (8), it suffices to prove that

$$\#\{n \in [0, b^M) \mid y_{n,j}^{(i)} = u_j^{(i)}, \ j \in [1, \tau_m] \text{ for } i \in [1, s], \text{ and } a_{j-1}(n) = u_j^{(0)}$$

$$j \in [1, m]\} > 0, \quad \text{with} \quad M = s\tau_m + (m + 2g)(2g + 1)e_0 + m_0,$$

$$(54)$$

 $m_0 = 2g + 2 + e_1 + \dots + e_s$, for all $u_j^{(i)} \in \mathbb{F}_b$ Let

$$\delta(\mathfrak{T}) = \begin{cases} 1, & \text{if } \mathfrak{T} \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $k_{j+1}^{(0)} = z_{h(j)} = z^{h(j)}$ for $j \in H_1$ with $H_1 = \mathbb{N}_0 \setminus H_2 = \{h(0), h(1), ...\}, H_2 = \{n_0, n_1, ..., n_g\}$. From (17), we have

$$a_{j,r}^{(0)} = \delta(j-1 = r \in H_1), \quad j \ge 1.$$

Let $c_{j,r}^{(0)} := a_{j,h(r)}^{(0)}$. By (4), (5) and (20), we get

$$c_{j,r}^{(0)} = \delta(j - 1 = h(r)), \qquad y_{n,j}^{(0)} = \sum_{r \ge 0} \bar{a}_r(n) c_{j,r}^{(0)} = \bar{a}_{h(j-1)}(n), \tag{55}$$

$$x_n^{(0)} = \sum_{j>1} a_{h(j-1)}(n)/b^j$$
 and $k_j^{(0)} = z^{h(j-1)} \in \mathcal{L}(h(j-1)(2g+1)P_0), j \ge 1.$

So, we obtain a digital s+1-dimensional sequence $(x_n^{(0)}, \mathbf{x}_n)_{n\geq 0}$. Let $n=\sum_{r=0}^{M-1}a_r(n)b^r$ and let

$$\dot{n} = \sum_{r \in H_1} a_r(n)b^r, \ \ddot{n} = \sum_{r \in H_2} a_r(n)b^r, \ \dot{U} = \{\dot{n}|n \in [0, b^M)\}, \\ \ddot{U} = \{\ddot{n}|n \in [0, b^M)\}.$$

By (4), (18) and (55), we get

$$\begin{aligned} y_{n,j}^{(i)} &= \sum_{r \geq 0} \bar{a}_r(n) c_{j,r}^{(i)} = \sum_{r \in H_1} \bar{a}_r(n) c_{j,r}^{(i)} + \sum_{r \in H_2} \bar{a}_r(n) c_{j,r}^{(i)} = y_{n,j}^{(i)} + y_{n,j}^{(i)}, \\ i \in [1,s], \quad y_{n,j}^{(0)} &= y_{n,j}^{(0)} = \bar{a}_{h(j-1)}(n), \quad y_{n,j}^{(0)} = 0, \quad j \geq 1. \end{aligned}$$

We fix $\tilde{n} \in \ddot{U}$. Let

$$A_{\mathbf{u},\tilde{n}} = \{ n \in [0, b^{M}) \mid y_{n,j}^{(i)} = u_{j}^{(i)}, \ j \in [1, \tau_{m}], \ i \in [1, s],$$

$$y_{n,j}^{(0)} = u_{j}^{(0)}, \ j \in [1, m], \quad \ddot{n} = \tilde{n} \}.$$

$$(56)$$

It is easy to verify that statement (54) follows from the next assertion

$$\#A_{\mathbf{u},\tilde{n}} > 0 \quad \forall \ u_j^{(i)} \in \mathbb{F}_b, \quad \tilde{n} \in \ddot{U}.$$
 (57)

Taking into account that $y_{n,j}^{(i)} = y_{n,j}^{(i)} + y_{n,j}^{(i)}$, we get

$$A_{\mathbf{u},\tilde{n}} = \{ \dot{n} \in \dot{U} \mid y_{\dot{n},j}^{(i)} = \dot{u}_{j}^{(i)}, \ j \in [1, \tau_{m}], \ i \in [1, s], \ y_{\dot{n},j}^{(0)} = u_{j}^{(0)}, \ j \in [1, m] \},$$

where $\dot{u}_j^{(i)}=\dot{u}_j^{(i)}-y_{\tilde{n},j}^{(i)}$. According to (4), (18) and (55), in order to prove (57), it suffices to show that the vectors

$$\pi_M(\mathbf{c}_i^{(i)}) = (c_{i \ 0}^{(i)}, ..., c_{i \ M-1}^{(i)}) \in F_h^M, \text{ with } 1 \le j \le d_i, \ 0 \le i \le s,$$
 (58)

 $d_i = \tau_m, 1 \leq i \leq s$ and $d_0 = m$, are linearly independent over \mathbb{F}_b .

To prove this statement, we closely follow [3, p.282]. Suppose that we have

$$\sum_{j=1}^{m} f_j^{(0)} \pi_M(\mathbf{c}_j^{(0)}) + \sum_{i=1}^{s} \sum_{j=1}^{\tau_m} f_j^{(i)} \pi_M(\mathbf{c}_j^{(i)}) = \mathbf{0} \in F_b^M$$

for some $f_j^{(i)} \in \mathbb{F}_b$ with $\sum_{j=1}^m |\phi^{-1}(f_j^{(0)})| + \sum_{i=1}^s \sum_{j=1}^{\tau_m} |\phi^{-1}(f_j^{(i)})| > 0$. We put $f_r^{(0)} = 0$ for r > m. Hence

$$\sum_{i=1}^{m} f_{j}^{(0)} c_{j,r}^{(0)} + \sum_{i=1}^{s} \sum_{j=1}^{\tau_{m}} f_{j}^{(i)} c_{j,r}^{(i)} = 0 \quad \text{for} \quad r \in [0, M).$$

By (18) and (55), we obtain $c_{j,r}^{(i)} = a_{j,h(r)}^{(i)}$ for $1 \le i \le s$ and $c_{j,r}^{(0)} = \delta(j-1)$ = h(r)). Therefore

$$0 = \sum_{j=1}^{m} f_j^{(0)} \delta(j-1) = h(r) + \sum_{i=1}^{s} \sum_{j=1}^{\tau_m} f_j^{(i)} a_{j,h(r)}^{(i)} = f_{h(r)+1}^{(0)} + \sum_{i=1}^{s} \sum_{j=1}^{\tau_m} f_j^{(i)} a_{j,h(r)}^{(i)}$$
(59)

for $r \in [0, M)$.

Now consider the element $\alpha \in \mathbb{F}_b$ given by $\alpha = \alpha_1 + \alpha_2$, where

$$\alpha_1 = \sum_{r=0}^{m-1} f_{h(r)+1}^{(0)} z_{h(r)}, \quad \alpha_2 = \sum_{i=1}^{s} \sum_{j=1}^{\tau_m} f_j^{(i)} k_j^{(i)} - \sum_{i=1}^{s} \sum_{j=1}^{\tau_m} f_j^{(i)} \sum_{u=0}^{g} a_{j,n_u}^{(i)} w_u.$$
 (60)

Using (17), we get

$$\alpha_2 = \sum_{i=1}^s \sum_{j=1}^{\tau_m} f_j^{(i)} \Big(\sum_{r=0}^\infty a_{j,r}^{(i)} z_r - \sum_{u=0}^g a_{j,n_u}^{(i)} z_{n_u} \Big) = \sum_{r \in H_1} \Big(\sum_{i=1}^s \sum_{j=1}^{\tau_m} f_j^{(i)} a_{j,r}^{(i)} \Big) z_r.$$

From (18), (59) and (60), we obtain

$$\alpha = \sum_{r \geq 0} \left(f_{h(r)+1}^{(0)} + \sum_{i=1}^{s} \sum_{j=1}^{\tau_m} f_j^{(i)} a_{j,h(r)}^{(i)} \right) z_{h(r)} = \sum_{r \geq M} \left(f_{h(r)+1}^{(0)} + \sum_{i=1}^{s} \sum_{j=1}^{\tau_m} f_j^{(i)} a_{j,h(r)}^{(i)} \right) z_{h(r)}.$$

Hence

$$\nu_{P_{\infty}}(\alpha) \ge M. \tag{61}$$

Furthermore, (15), (16), (20), (55) and (60) yield

$$\alpha_1 \in \mathcal{L}((m+2g)(2g+1)P_0), \quad \alpha_2 \in \mathcal{L}\left(G + \sum_{i=1}^s ([\tau_m/e_i] + 1)P_i\right).$$
 (62)

Combining (61) and (62), we obtain

$$\alpha \in \mathcal{L}\Big(G + \sum_{i=1}^{s} ([\tau_m/e_i] + 1)P_i + (m+2g)(2g+1)P_0 - MP_\infty\Big).$$

But from (54), we have

$$\deg \left(G + \sum_{i=1}^{s} ([\tau_m/e_i] + 1)P_i + (m+2g)(2g+1)P_0 - MP_{\infty}\right)$$

$$= 2g + \sum_{i=1}^{s} ([\tau_m/e_i] + 1)e_i + (m+2g)(2g+1)e_0 - M$$

$$\leq 2g + s\tau_m + e_1 + \dots + e_s + (m+2g)(2g+1)e_0 - M < 0.$$

Hence

$$\mathcal{L}\left(G + \sum_{i=1}^{s} ([\tau_m/e_i] + 1)P_i + (m+2g)(2g+1)P_0 - MP_{\infty}\right) = \{0\}$$

by (14) and therefore we have $\alpha = 0$.

By (15), we have $\nu_{P_0}(k_j^{(i)}) \geq 0$ and $\nu_{P_0}(w_u) \geq 0$ for all i, j, u. According to (60), we get $\nu_{P_0}(\alpha_2) \geq 0$. Suppose that $\alpha_1 \neq 0$. Taking into account that $z_0 = z_{n_0} = w_0 \neq z_{h(r)}$ for $r \geq 0$, we obtain from (60) that $\nu_{P_0}(\alpha_1) < 0$. We have a contradiction. Hence $\alpha_1 = 0$ and $\alpha_2 = 0$. From Lemma C, we conclude that $f_j^{(i)} = 0$ for all i, j. Hence the system (58) is linearly independent over \mathbb{F}_b .

Thus (54) is true and $(\mathbf{x}_n)_{n\geq 0}$ satisfies the condition (8). By Theorem E, $(\mathbf{x}_n)_{n\geq 0}$ is the d-admissible uniformly distributed digital (t,s)-sequence in base b. Applying Theorem B, we get the assertion of Theorem 2.

4.3 Generalized Halton-type sequence. Proof of Theorem 3.

Lemma 3. The sequence $(\mathbf{x}_n)_{n\geq 0}$ is uniformly distributed in $[0,1)^s$.

Proof. By Lemma A, in order to prove Lemma 3, it suffices to show that $m-T(m) \to \infty$ for $m \to \infty$. Let $R_k = \max_{1 \le i \le s} n_{i,k}, k = 1, 2, \dots$. We define $j_{i,k}$ from the following condition $n_{i,j_{i,k}} \ge R_k > n_{i,j_{i,k}-1}$. Let $\tilde{R}_k = \sum_{i=1}^s n_{i,j_{i,k}}$.

We consider the definition of (t, m, s) net. Suppose that for all

$$E = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1)b^{-d_i}), \text{ with } a_i = \sum_{j=1}^{d_i} a_{i,j} b^{j-1}, \ a_{i,j} \in \mathbb{Z}_b, \ d_i \ge 0,$$

 $1 \leq i \leq s$, $d_1 + \cdots + d_s = R_k$, we have

$$\#\{n \in [0, b^m) \mid x_n \in E\} = \#\{n \in [0, b^m) \mid y_{n,j}^{(i)} = u_j^{(i)}, \ j \in [1, d_i], i \in [1, s]\}$$

$$= b^{m - R_k}, \quad \text{where} \quad m \ge \tilde{R}_k + (3g + 3)e_0, \ u_i^{(i)} = \phi^{-1}(a_{i,j}) \in \mathbb{F}_b,$$

$$(63)$$

 $j \in [1, d_i], i \in [1, s].$

By Definition 2, we get that $(\mathbf{x}_n)_{n\geq 0}$ is a (T,s)-sequence in base b with $m-R(k)\geq T(m)$ for $m \geq R_k + (3g+3)e_0$. Bearing in mind that $R(k) \to \infty$ for $k \to \infty$, we obtain the assertion of

Taking into account that $d_i \leq R_k \leq n_{i,j_{i,k}}$ for $1 \leq i \leq s$, we get that in order to prove (63), it suffices to verify that

$$\#\{n \in [0, b^m) \mid y_{n,j}^{(i)} = u_j^{(i)}, \ j \in [1, n_{i, j_{i,k}}], \ i \in [1, s]\} = b^{m - \tilde{R}_k}$$

$$(64)$$

for all $u_j^{(i)} \in \mathbb{F}_b$, with $j \in [1, n_{i, j_{i, k}}], i \in [1, s]$. Let $\mathcal{M} = (m_0 e_0 + 2g - 1) P_{\infty}$ with $m_0 = [m/e_0] - 2g - 1$.

By Lemma 1, we obtain that there exist sets H_1 and H_2 such that $H_1 \cup H_2 = \{0, 1, ..., m-1\}$, $H_1 \cap H_2 = \emptyset$, $(w_r^{(0)})_{r \in H_1}$ is the \mathbb{F}_b linear basis of $\mathcal{L}(\mathcal{M})$ and $\#H_2 = m - m_0 e_0 - g =: g_1$, with $g_1 - e_0(2g+1) - g \in [0, e_0)$. Let $n = \sum_{r=0}^{m-1} a_r(n)b^r$ and let

$$\dot{n} = \sum_{r \in H_1} a_r(n) b^r, \ \ddot{n} = \sum_{r \in H_2} a_r(n) b^r, \ \dot{U} = \{ \dot{n} | n \in [0, b^m) \}, \ \ddot{U} = \{ \ddot{n} | n \in [0, b^m).$$

So

$$f_n = \sum_{r=0}^{m-1} \bar{a}_r(n) w_r^{(0)} \in \mathcal{L}(\mathcal{M}) \quad \Longleftrightarrow \quad n = \dot{n}, \text{ for } n \in [0, b^m).$$

We fix $\tilde{n} \in \ddot{U}$. Let

$$A_{\mathbf{u},\tilde{n}} = \{ n \in [0, b^m) \mid y_{n,j}^{(i)} = u_j^{(i)}, \ j \in [1, n_{i,j_{i,k}}], \ i \in [1, s], \ \ddot{n} = \tilde{n} \}.$$

It is easy to see that statement (64) follows from the next assertion

$$#A_{\mathbf{u},\tilde{n}} = b^{m-\tilde{R}_k - g_1} \quad \forall \ u_j^{(i)} \in \mathbb{F}_b, \quad \tilde{n} \in \ddot{U}.$$

$$(65)$$

Taking into account that $y_{n,j}^{(i)} = y_{n,j}^{(i)} + y_{n,j}^{(i)}$, we get

$$A_{\mathbf{u},\tilde{n}} = \{ \dot{n} \in \dot{U} \mid y_{\dot{n},j}^{(i)} = \dot{u}_{j}^{(i)}, \ j \in [1, n_{i,j_{i,k}}], \ i \in [1, s] \},$$

where $\dot{u}_{j}^{(i)} = \dot{u}_{j}^{(i)} - y_{\tilde{n},j}^{(i)}$. Let

$$\hat{\psi}(f_n) := (y_{n,1}^{(1)}, ..., y_{n,n_{1,j_{1,k}}}^{(1)}, ..., y_{n,1}^{(s)}, ..., y_{n,n_{s,j_{s,k}}}^{(s)}) \in \mathbb{F}_b^{\tilde{R}_k}.$$

We consider the map $\check{\psi}: \mathcal{L}(\mathcal{M}) \to \mathbb{F}_b^{\tilde{R}_k}$ defined by

$$\check{\psi}(\dot{f}) := \hat{\psi}(f_n) \text{ where } \mathcal{L}(\mathcal{M}) \ni \dot{f} = f_n \text{ with some } n \in \dot{U}.$$

Note that $\check{\psi}$ is a linear transformation between vector spaces over \mathbb{F}_b . It is clear that in order to prove (65), it suffices to verify that $\check{\psi}$ is surjective. To prove this, it is enough to show that

$$\dim \left(\mathcal{L}(\mathcal{M}) / \ker(\check{\psi}) \right) = \tilde{R}_k. \tag{66}$$

Using (23), (25) and (28), we get that $w_l^{(i)} \equiv 0 \pmod{\mathcal{P}_{i,j_{i,k}}}$ for $l \geq n_{i,j_{i,k}}$. By (23), (25), (28), and (31), we derive that $y_{n,j}^{(i)} = 0$ for all $j \in [1, n_{i,j_{i,k}}]$ if and only if $\dot{f} = f_n \equiv 0 \pmod{\mathcal{P}_{i,j_{i,k}}}$ for $i \in [1, s]$.

From the definition of $\check{\psi}$ it is clear that

$$\ker(\check{\psi}) = \mathcal{L}(H), \text{ with } H = \mathcal{M} - \sum_{i=1}^{s} \mathcal{P}_{i,n_{i,j_{i,k}}}.$$

Using Riemann-Roch's theorem, we obtain that $\dim(\mathcal{M}) = m_0 e_0 + g = m - g_1$, where $\deg(\mathcal{M}) = m_0 e_0 + 2g - 1$ and

$$\deg(H) = m_0 e_0 + 2g - 1 - \sum_{i=1}^{s} n_{i,j_{i,k}} = m + g - 1 - g_1 - \tilde{R}_k.$$

Hence $\dim(\ker(\psi)) = m - \tilde{R}_k - g_1 \ge (3g+3)e_0 - g_1 \ge (3g+3)e_0 - (2g+2)e_0 - g \ge 1$, $\dim(\mathcal{M}) = m - g_1$ and (66) follows. So $\check{\psi}$ is indeed surjective. Therefore (65) and Lemma 3 are proved.

Lemma 4. The sequence $(\mathbf{x}_n)_{n\geq 0}$ satisfies condition (8).

Proof. Let

$$M = ([M_1/e_0] + 3g + 1)e_0, \quad M_1 = \sum_{i=0}^{s} n_{i,j_{i,m}} \text{ where } n_{j_{i,m}} \ge \tau_m > n_{j_{i,m}-1}$$
 (67)

for $i \in [1, s]$, $n_{0, j_{0,m}} = ([m/e_0] + 1)e_0 \ j_{0,m} = [m/e_0] + 1$.

Bearing in mind that $y_{n,j}^{(0)} = \bar{a}_{j-1}(n)$, (j = 1, 2, ...), we get from (32) - (33), that in order to obtain (8), it suffices to prove that

$$\#\{n \in [0, b^M) \mid y_{n,j}^{(i)} = u_j^{(i)}, \quad j \in [1, n_{i,j_{i,m}}] \quad \text{for } i \in [0, s]\} > 0$$

$$(68)$$

for all $u_j^{(i)} \in \mathbb{F}_b$. Let $\mathcal{M} = (([M_1/e_0] + 1)e_0 + 2g - 1)P_{\infty}$. By (22), $\deg(P_{\infty}) = 1$. Hence $\deg(\mathcal{M}) = ([M_1/e_0] + 1)e_0 + 2g - 1$. Using Riemann-Roch's theorem, we obtain that

$$\dim(\mathcal{M}) = ([M_1/e_0] + 1)e_0 + g = M_1 + g_1 + g \quad \text{with} \quad g_1 := ([M_1/e_0] + 1)e_0 - M_1. \tag{69}$$

By Lemma 1, we get that an \mathbb{F}_b linear basis of $\mathcal{L}(\mathcal{M})$ can be chosen from the set $\{w_0^{(0)},...,w_{M-1}^{(0)}\}$ with $M = ([M_1/e_0] + 3g + 1)e_0 = n_{0,[M_1/e_0]+3g+1}$.

Let $n = \sum_{r=0}^{M-1} a_r(n)b^r$ and let $f_n = \sum_{r=0}^{M-1} \bar{a}_r(n)w_r^{(0)}$. We get that for all $\dot{f} \in \mathcal{L}(\mathcal{M})$ there exists $n \in [0, b^M)$ such that $\dot{f} = f_n$.

From (31), we have

$$f_n = \sum_{j=1}^{\infty} y_{n,j}^{(i)} w_{j-1}^{(i)}, \qquad 0 \le i \le s.$$
 (70)

Let

$$\psi(f_n) := (y_{n,1}^{(0)}, ..., y_{n,n_{0,j_{0,m}}}^{(0)}, ..., y_{n,1}^{(s)}, ..., y_{n,n_{s,j_{s,m}}}^{(s)}) \in \mathbb{F}_b^{M_1}.$$
 (71)

Consider the map $\dot{\psi}: \mathcal{L}(\mathcal{M}) \to \mathbb{F}_b^{M_1}$ defined by

$$\dot{\psi}(\dot{f}) := \psi(f_n)$$
 where $\dot{f} = f_n$ with some $n \in [0, b^M)$.

We see that in order to obtain (68), it suffices to verify that $\dot{\psi}$ is surjective.

To prove this, it suffices to show that

$$\dim \left(\mathcal{L}(\mathcal{M}) / \ker(\dot{\psi}) \right) = M_1. \tag{72}$$

Using (23), (25) and (28), we get that $w_k^{(i)} \equiv 0 \pmod{\mathcal{P}_{i,j_{i,m}}}$ for $k > n_{i,m}$. From (70), (23), (25) and (28), we derive that $y_{n,j}^{(i)} = 0$ for all $j \in [1, n_{i,j_{i,m}}]$ if and only if $f_n \equiv 0 \pmod{\mathcal{P}_{i,j_{i,m}}}$ for $i \in [0, s]$.

From the definition of ψ it is clear that

$$\ker(\dot{\psi}) = \mathcal{L}(H), \text{ with } H = \mathcal{M} - \sum_{i=0}^{s} \mathcal{P}_{i,j_{i,m}}.$$

Using (67), (69), (22) and Riemann-Roch's theorem, we obtain that

$$\deg(H) = M_1 + g_1 + 2g - 1 - \sum_{i=0}^{s} n_{i,j_{i,m}} = g_1 + 2g - 1$$

and dim(ker(ψ)) = $g_1 + g$. By (69), dim(\mathcal{M}) = $M_1 + g_1 + g$. Hence dim ($\mathcal{L}(\mathcal{M})/\text{ker}(\dot{\psi})$) = M_1 . Therefore (72) is true. So ψ is indeed surjective and (68) follows. Therefore Lemma 4 is proved.

Lemma 5. The sequence $(\mathbf{x}_n)_{n\geq 0}$ is weakly admissible.

Proof. Suppose that $x_n^{(i)} = x_k^{(i)}$ for some i, n, k. From (71) and (32)-(33), we get that $y_{n,j}^{(i)} = y_{k,j}^{(i)}$ for $j \ge 1$.

Using (70), we have

$$f_n = \sum_{i \ge 1} y_{n,j}^{(i)}(n) w_{j-1}^{(i)}.$$

Hence $f_n = f_k$. Taking into account that $(w_r^{(0)})_{r \geq 0}$ is an \mathbb{F}_b linear basis of O_F , we obtain from (30), that n = k. By Definition 7, Lemma 5 is proved.

Applying Theorem B, we get the assertion of Theorem 3.

4.4 Niederreiter-Xing sequence. Proof of Theorem 4.

Similarly to the proof of Lemma 5, we get that $(\mathbf{x}_n)_{n\geq 0}$ is weakly admissible. By Lemma 2, $(\mathbf{x}_n)_{n\geq 0}$ is the digital uniformly distributed sequence.

According to (4), (5), (8) and Theorem B, in order to prove Theorem 4, it is enough to verify that

$$\#\{n \in [0, b^M) | y_{n,j}^{(i)} = u_j^{(i)}, \ j \in [1, \tau_m], i \in [1, s], \ a_j(n) = 0 \text{ for } j \in [0, m)\} > 0$$
 (73)

for all $u_i^{(i)} \in \mathbb{F}_b$, where $M = s\tau_m + m + 2g + 2$.

Bearing in mind that by Lemma 2 $(x_n^{(0)}, \mathbf{x}_n)_{n\geq 0}$ is a (g, s+1) sequence, we obtain

$$\#\{n \in [0, b^M)|y_{n,j}^{(i)} = u_j^{(i)}, j \in [1, \tau_m], i \in [1, s], \ y_{n,j}^{(0)} = 0, j \in [1, m + g + 2]\} > 0$$

for all $u_j^{(i)} \in \mathbb{F}_b$.

Therefore, in order to prove (73), it suffices to verify that

if
$$y_{n,j}^{(0)} = 0$$
 for $j \in [1, m+g+2]$ then $a_j(n) = 0$ for $j \in [0, m)$. (74)

Now we will prove (74):

From (35) and (43), we have $\ddot{v}_r \dot{\tau}_0 = \sum_{j \geq 1} \dot{c}_{j,r}^{(0)} t_0^{j-1}$ with $\nu_{P_0}(\ddot{v}_r \tau_0)$ = $\alpha(r)$. Hence $\dot{c}_{j,r}^{(0)} = 0$ for $j \leq \alpha(r)$ and $\dot{c}_{j,r}^{(0)} \neq 0$ for $j = \alpha(r) + 1$.

Using (4), (47) and (49) we obtain $\dot{c}_{i,\beta(i-1)}^{(0)} \neq 0$ and

$$y_{n,j}^{(0)} = \sum_{r \ge 0} \bar{a}_r(n) \dot{c}_{j,r}^{(0)} = \sum_{\alpha(r) < j} \bar{a}_r(n) \dot{c}_{j,r}^{(0)} = \sum_{r \in B_j} \bar{a}_r(n) \dot{c}_{j,r}^{(0)}, \quad j \ge 1.$$

We apply induction and consider the case j=1. By (50), we see that $\bar{a}_{\beta(0)}(n)=0$ if $y_{n,1}^{(0)}=0$. Suppose that $\bar{a}_{\beta(0)}(n) = \cdots = \bar{a}_{\beta(l-1)}(n) = 0$ if $y_{n,1}^{(0)} = \cdots = y_{n,l}^{(0)} = 0$ for some $l \geq 1$. Now let $y_{n,1}^{(0)} = \dots = y_{n,l}^{(0)} = y_{n,l+1}^{(0)} = 0$. We see

$$0 = y_{n,l+1}^{(0)} = \sum_{r \in B_{l+1}} \bar{a}_r(n) \dot{c}_{l+1,r}^{(0)} = \sum_{r \in B_{l+1} \setminus B_l} \bar{a}_r(n) \dot{c}_{l+1,r}^{(0)} = \bar{a}_{\beta(l)}(n) \dot{c}_{l+1,\beta(l)}^{(0)}.$$

Bearing in mind that $\dot{c}_{l+1,\beta(l)}^{(0)} \neq 0$, we get $\bar{a}_{\beta(l)}(n) = 0$.

Therefore if $y_{n,j}^{(0)} = 0$ for all $1 \le j \le m + g + 1$, then $a_{\beta(j-1)}(n) = 0$ for all $1 \le j \le m + g + 1$. Using Lemma 2, we get $a_r(n) = 0$ for all $0 \le r \le m-1$.

Hence (74) is true and Theorem 4 follows. ■

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Критерий периодичности непрерывных дробей ключевых элементов гиперэллиптических полей 1

В. П. Платонов, Г. В. Федоров

Платонов Владимир Петрович — Федеральное государственное учреждение «Федеральный научный центр Научно-исследовательский институт системных исследований Российской академии наук» (ФГУ ФНЦ НИИСИ РАН); Математический институт им. В. А. Стеклова РАН (МИАН), г. Москва.

e-mail: platonov@niisi.ras.ru

Федоров Глеб Владимирович — кандидат физико-математических наук, Федеральное государственное учреждение «Федеральный научный центр Научно-исследовательский институт системных исследований Российской академии наук» (ФГУ ФНЦ НИИСИ РАН); Московский государственный университет имени М. В. Ломоносова (МГУ имени М. В. Ломоносова), г. Москва.

 $e ext{-}mail: fedorov@mech.math.msu.su$

Аннотация

Периодичность и квазипериодичность функциональных непрерывных дробей в гиперэллиптическом поле $L=\mathbb{Q}(x)(\sqrt{f})$ имеет более сложную природу, чем периодичность числовых непрерывных дробей элементов квадратичных полей. Известно, что периодичность непрерывной дроби элемента \sqrt{f}/h^{g+1} , построенной по нормированию, связанному с многочленом h первой степени, эквивалентна наличию нетривиальных S-единиц в поле L рода g и эквивалентна наличию нетривиального кручения в группе классов дивизоров. В данной статье найден точный промежуток значений $s \in \mathbb{Z}$ таких, что элементы \sqrt{f}/h^s имеют периодическое разложение в непрерывную дробь, где $f\in\mathbb{Q}[x]$ — свободный от квадратов многочлен четной степени. Для многочленов f нечетной степени проблема периодичности непрерывных дробей элементов вида \sqrt{f}/h^s рассмотрена в статье [5], причем там доказано, что длина квазипериода не превосходит степени фундаментальной S-единицы поля L. Проблема периодичности непрерывных дробей элементов вида \sqrt{f}/h^s для многочленов f четной степени является более сложной. Это подчеркивается найденным нами примером многочлена f степени 4, для которого соответствующие непрерывные дроби имеют аномально большую длину периода. Ранее в статье [5] также были найдены примеры непрерывных дробей элементов гиперэллиптического поля L с длиной квазипериода значительно превосходившей степень фундаментальной S-единицы поля L.

Ключевые слова: непрерывные дроби, фундаментальные единицы, *S*-единицы, кручение в якобианах, гиперэллиптические поля, дивизоры, группа классов дивизоров.

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