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О многочленах Нюмена без корней на единичном круге

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Аннотация

В настоящей заметке мы получим необходимое и достаточное условие на тройку неотрицательных целых чисел $a < b < c$ при выполнении которого многочлен Нюмена $\sum_{j=0}^{a} x^j + \sum_{j=b}^{c} x^j$ имеет корень на единичном круге. Используя это условие мы докажем, что для каждого $d \geq 3$ существует такое целое положительное число $n > d$, что многочлен Нюмена $1 + x + \cdots + x^{d-2} + x^n$ длины d не имеет корней на единичном круге.

Ключевые слова: многочлен Нюмена, корень из единицы.

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On Newman polynomials without roots on the unit circle

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Abstract

In this note we give a necessary and sufficient condition on the triplet of nonnegative integers $a < b < c$ for which the Newman polynomial $\sum_{j=0}^{a} x^j + \sum_{j=b}^{c} x^j$ has a root on the unit circle. From this condition we derive that for each $d \geq 3$ there is a positive integer $n > d$ such that the Newman polynomial $1 + x + \cdots + x^{d-2} + x^n$ of length d has no roots on the unit circle.

Keywords: Newman polynomial, root of unity.

Bibliography: 11 titles.

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In honor of Professor Antanas Laurinčikas on the occasion of his 70th birthday

1. Introduction

For a polynomial $f \in \mathbb{C}[x]$, let throughout

$$m(f) := \min_{|z|=1} |f(z)|$$

be the minimal value of f on the unit circle. Motivated by some questions raised by Campbell, Ferguson, Forcade [2], and Smyth [11] Boyd in [1] studied the behavior of $m(f)$ for *Newman polynomials* f. (Recall that $f(x) = \sum_{j=0}^{d} a_j x^j$ is a Newman polynomial if $a_j \in \{0,1\}$ for each $j = 0, 1, \dots, d$.) In particular, in [1] it was shown that for each $d \geq 12$ there is a Newman polynomial f of degree d satisfying $m(f) > 1$ and that for each sufficiently large $d \in \mathbb{N}$ there is a Newman polynomial f of degree d satisfying $m(f) > d^{0.137}$. Here, 12 is the smallest possible such degree. For example,

$$m(1 + x + x^2 + x^3 + x^4 + x^7 + x^8 + x^{10} + x^{12}) = 1.36237\dots.$$

See also [9] for some results on the irreducibility of Newman polynomials.

One can raise similar questions with degree of f replaced by its length (see [1], [4] and [8]). Such questions turn out to be more difficult. More precisely, let $\mathcal{N}_d$ be the set of Newman polynomials of length d, namely,

$$\mathcal{N}_d := \{x^{k_1} + \dots + x^{k_d}, \text{ where } k_1 < \dots < k_d \text{ are nonnegative integers}\}.$$

As in [8], we put

$$\mu(d) := \max_{f \in \mathcal{N}_d} m(f).$$

Evidently, $\mu(1) = 1$ and $\mu(2) = 0$. The values of $\mu(3)$ and $\mu(4)$ have been calculated in [11] and [4], respectively. (They come from polynomials $1 + x^2 + x^3$ and $1 + x^2 + x^3 + x^4$.) It is conjectured that $\mu(d) \to \infty$ as $d \to \infty$ (see [1], [8]), but even much weaker inequality $\mu(d) \geq 1$ is still not established for each $d \geq 5$ (see [8]).

In [8] Mercer proved that $\mu(d) > 0$ for each $d \geq 3$. This is equivalent to the fact that for each $d \geq 3$ there is a Newman polynomial of length d which has no roots on the unit circle. In this note we describe which Newman polynomials of the form

$$f(x) = \sum_{j=0}^{a} x^j + \sum_{j=b}^{c} x^j = 1 + \dots + x^a + x^b + \dots + x^c,$$

where $0 \leq a < b \leq c$, have roots on the unit circle and which do not have. These polynomials are naturally obtained by taking a degree c polynomial with all coefficients 1 and then replacing in it a string of consecutive coefficients by zeros. Note that Theorem 3 immediately implies the above mentioned result $\mu(d) > 0$ for every $d \geq 3$.

For a positive integer m and a prime number p, let $\nu_p(m)$ be the power of p in the prime factorization of m, and $\nu_p(0) = 0$.

THEOREM 1. *Let a, b, c be integers satisfying $0 \leq a < b \leq c$. Then the Newman polynomial $1 + \dots + x^a + x^b + \dots + x^c$ has a root on the unit circle if and only if $c = a + b$ or at least one of the inequalities*

$$\gcd(a + 1, c - b + 1) > 1, \tag{1}$$

$$\gcd(c+1, c-b+a+2) > 1, \tag{2}$$

$$\nu_2(|c-a-b|) > \nu_2(b) \tag{3}$$

holds.

In particular, selecting $a = d-2 \geq 1$ and $b = c = n > a$, we see that $c \neq a+b$ and that $\gcd(a+1, c-b+1) = \gcd(d-1, 1) = 1$, so (1) does not hold. Inserting $c+1 = n+1$, $c-b+a+2 = d$, $|c-a-b| = d-2$ and $b = n$ into (2) and (3), we obtain the following special case of Theorem 1:

Theorem 2. *Let d and n be integers satisfying $n > d-2 \geq 1$. Then the Newman polynomial $1+x+\dots+x^{d-2}+x^n \in \mathcal{N}_d$ has a root on the unit circle if and only if at least one of the inequalities*

$$\gcd(n+1, d) > 1, \tag{4}$$

$$\nu_2(d-2) > \nu_2(n) \tag{5}$$

holds.

Let $\varphi(m)$ be the Euler totient function. From Theorem 2 we shall derive the following:

Theorem 3. *Let $d \geq 3$ be an integer. Write d in the form $d = 2^m(2l+1)$ with integers $m, l \geq 0$. If $k \in \mathbb{N}$ satisfies*

$$k \geq \frac{\log d}{\varphi(2l+1)\log 2} \tag{6}$$

then the Newman polynomial $1+x+\dots+x^{d-2}+x^n \in \mathcal{N}_d$, with degree $n = 2^{\varphi(2l+1)k}$, has no roots on the unit circle.

The proof of Theorem 1 is based on the following very simple lemma which was proved in [3]. (Subsequently, it was used in a different context in [6], [7], [10].)

Lemma 1. *Suppose z_1, z_2, z_3, z_4 are complex numbers of modulus 1 satisfying*

$$z_1 + z_2 + z_3 + z_4 = 0.$$

Then $z_1 + z_j = 0$ for some $j \in \{2, 3, 4\}$.

In the next section we give the proof of Theorem 1. In Section 3 we prove Theorem 3. Finally, in Section 4 we present one more construction of Newman polynomials without roots on the unit circle.

2. Proof of Theorem 1

For $c = a+b$, the polynomial

$$1 + \dots + x^a + x^b + \dots + x^c = (1 + \dots + x^a)(1 + x^b)$$

is a product of cyclotomic polynomials, so all of its roots are roots of unity. In all what follows we will assume that $c \neq a+b$.

If a complex number ζ of modulus 1 is a root of

$$f(x) := 1 + \dots + x^a + x^b + \dots + x^c,$$

then $\zeta \neq 1$ and

$$0 = (1-\zeta)f(\zeta) = 1 - \zeta^{a+1} + \zeta^b - \zeta^{c+1}.$$

Since the four numbers on the right hand side of this equality, namely, $1, -\zeta^{a+1}, \zeta^b, -\zeta^{c+1}$, are all of modulus 1, by Lemma 1, we must have one of the following three possibilities:

$$1 - \zeta^{a+1} = \zeta^b - \zeta^{c+1} = 0,$$

$$1 - \zeta^{c+1} = -\zeta^{a+1} + \zeta^b = 0,$$

$$1 + \zeta^b = \zeta^{a+1} + \zeta^{c+1} = 0.$$

In particular, these equalities imply that if ζ of modulus 1 is a root of f, then ζ must be a root of unity.

Now, will show that in the first (resp. second and third) case the inequality (1) (resp. (2) and (3)) holds and, conversely, if (1), (2) or (3) holds then $f(\xi) = 0$ for some root of unity ξ. (Evidently, $|\xi| = 1$.)

In the first case, we have $\zeta^{a+1} = \zeta^{c-b+1} = 1$. Set $g := \gcd(a+1, c-b+1)$. Then, there are some $u, v \in \mathbb{Z}$ such that $g = u(a+1) + v(c-b+1)$, and therefore $\zeta^g = 1$. This is impossible if $g = 1$, because $\zeta \neq 1$. Hence $g > 1$ which proves (1). On the other hand, we will show that $\xi := e^{2\pi i/g} \neq 1$ is a root of f. Indeed, then, as g divides $a+1$ and $c-b+1$, we have $\xi^{a+1} = \xi^{c-b+1} = 1$, which yields $\xi^{c+1} = \xi^b$. Consequently,

$$(1-\xi)f(\xi) = 1 - \xi^{a+1} - \xi^{c+1} + \xi^b = 1 - 1 + 0 = 0,$$

giving $f(\xi) = 0$.

In the second case, we have $\zeta^{c+1} = 1$ and $\zeta^{b-a-1} = 1$. Therefore, $\zeta^{c+1} = \zeta^{c-b+a+2} = 1$. As above, putting $g_1 := \gcd(c+1, c-b+a+2)$ we derive that $\zeta^{g_1} = 1$. Hence $g_1 > 1$ which proves (2). On the other hand, we will show that $\xi := e^{2\pi i/g_1} \neq 1$ is a root of f. Indeed, then $\xi^{c+1} = \xi^{c-b+a+2} = 1$, which yields $\xi^b = \xi^{a+1}$. Consequently,

$$(1-\xi)f(\xi) = 1 - \xi^{c+1} - \xi^{a+1} + \xi^b = 1 - 1 + 0 = 0,$$

giving $f(\xi) = 0$.

In the third case, we obtain $\zeta^b = -1$ and $\zeta^{c-a} = -1$. Hence $\zeta^{c-a-b} = 1$, and so $\zeta^{|c-a-b|} = 1$. From $\zeta^b = -1$ we find that $\zeta = e^{\pi i(2k+1)/b}$ for some $k \in \mathbb{Z}$. So, using $\zeta^{|c-a-b|} = 1$, we obtain

$$e^{\pi i(2k+1)|c-a-b|/b} = 1.$$

It follows that $(2k+1)|c-a-b|/b$ must be an even integer, which is only possible when $\nu_2(|c-a-b|) > \nu_2(b)$. This implies (3). To prove that the condition (3) is sufficient, we assume that $\nu_2(b) = t \geq 0$ and $\nu_2(|c-a-b|) = s \geq t+1$. Then $b = 2^t(2q+1)$ and $|c-a-b| = 2^s(2\ell+1)$, where $q, \ell \geq 0$ are integers. (Here, we use the fact that $c \neq a+b$.) Putting $\xi := e^{\pi i/2^t}$, we deduce that

$$\xi^b = e^{2^t(2q+1)\pi i/2^t} = e^{(2q+1)\pi i} = -1$$

and

$$\xi^{|c-a-b|} = e^{2^s(2\ell+1)\pi i/2^t} = e^{2^{s-t}(2\ell+1)\pi i} = 1.$$

Thus $1 + \xi^b = 0$ and $\xi^{c-a-b} = 1$, which yields $\xi^{c+1} = -\xi^{a+1}$. It follows that ξ is a root of $(1-x)f(x) = 1 - x^{a+1} + x^b - x^{c+1}$. Since $\xi \neq 1$, it is a root of f. This completes the proof of Theorem 1.

3. Proof of Theorem 3

To derive Theorem 3 from Theorem 2 we first observe that (6) implies

$$n = 2^{\varphi(2l+1)k} = e^{\varphi(2l+1)k\log 2} \geq e^{\log d} = d,$$

so $1+x+\cdots+x^{d-2}+x^n$ is indeed a Newman polynomial of length d. Next, from (6), $n = 2^{\varphi(2l+1)k}$, $d \geq 3$ and the trivial inequality $\nu_2(c) \leq \log_2 c$ for $c \in \mathbb{N}$, it follows that

$$\nu_2(n) = \varphi(2l+1)k \geq \log_2 d > \log_2(d-2) \geq \nu_2(d-2),$$

so (5) does not hold. To show that (4) does not hold as well, we need to prove that the numbers $n+1 = 2^{\varphi(2l+1)k}+1$ and $d = 2^m(2l+1)$ are coprime. By Euler's theorem, $2^{\varphi(2l+1)k} \equiv 1 \pmod{2l+1}$. Consequently, $n+1 = 2^{\varphi(2l+1)k}+1$ modulo $2l+1$ is 2. Combining this with the fact that $n+1$ is odd, we derive that

$$\gcd(n+1, d) = \gcd(n+1, 2l+1) = \gcd(2, 2l+1) = 1.$$

This completes the proof of Theorem 3.

4. Another series of Newman polynomials without unimodular roots

Finally, in order to give one more alternative (and very short) proof of the fact that $\mu(d) > 0$ for each $d \geq 3$ we recall the following result of Filaseta, Finch and Nicol (see the proof of Theorem 4.1 in [5]):

Lemma 2. *There is an infinite sequence of nonnegative integers $S := \{s_1 < s_2 < s_3 < \dots\}$ such that for every finite set $T \subset S$ the polynomial $1 + \sum_{t\in T} x^{4^t}$ is irreducible over the rationals.*

Fix $d \geq 3$ and take any $T \subset S$ with $d-1$ elements, for instance, $T = \{t_1 < \cdots < t_{d-1}\} \subset S$. Then

$$f(x) := 1 + \sum_{t\in T} x^{4^t} \in \mathcal{N}_d.$$

We claim that f has no roots on the unit circle. Indeed, if ζ is a root of f satisfying $|\zeta| = 1$ then so is $\overline{\zeta} = 1/\zeta$, and hence the minimal polynomial g of ζ over $\mathbb{Q}$ is reciprocal. Since $g|f$ and, by Lemma 2, f is irreducible, we must have $g = f$. It follows that f is reciprocal, namely, $f(x) = x^D f(1/x)$, with $D = 4^{t_{d-1}}$. However, the identity

$$x^D + \sum_{t\in T} x^{D-4^t} = x^D f(1/x) = f(x) = 1 + \sum_{t\in T} x^{4^t}$$

does not hold in view of

$$D - (D - 4^{t_1}) = 4^{t_1} < 3 \cdot 4^{t_{d-2}} \leq 4^{t_{d-1}} - 4^{t_{d-2}} = D - 4^{t_{d-2}}.$$

Hence f has no roots on the unit circle, as claimed.

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