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О некоторых характерах представлений групп

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Аннотация

Мы изучаем поля реализации и целочисленность характеров дискретных и конечных подгрупп $SL_2(\mathbf{C})$ и связанные с ним решетки, а также целочисленность характеров конечных групп G.

Теория характеров конечных и бесконечных групп играет центральную роль в теории групп, теории представлений конечных групп и ассоциативных алгебр. Классические результаты связаны с некоторыми арифметическими задачами: описание целочисленых представлений существенно для конечных групп над кольцами целых чисел в числовых полях, локальных полях или, в более общем случае, для дедекиндовых колец.

Существенная часть этой статьи посвящена следующему вопросу, восходящему к В. Бернсайду: каждое ли представление над числовым полем может быть сделано целочисленным.

Всякое ли линейное представление $\rho : G \to GL_n(K)$ конечной группы G над числовым полем $K/\mathbf{Q}$ сопряжено в $GL_n(K)$ с представлением $\rho : G \to GL_n(O_K)$ над кольцом целых чисел O_K поля K? Чтобы изучить этот вопрос, используется связь целочисленых представлений и решеток.

Этот вопрос тесно связан с глобально неприводимыми представлениями; концепция, предложенная Дж. Томпсоном и Б. Гроссом, была изучена Фам Хыу Тиепом и обобщена Ф. Ван Ойстаеном и А. Е. Залесским, однако остается много открытых вопросов.

Нас интересуют арифметические аспекты целочисленной реализуемости представлений конечных групп, и, в частности, рассматриваются условия реализуемости в терминах символов Гильберта и алгебр кватернионов.

Ключевые слова: Гиперболические решетки, группы, порожденные отражениями, характеры дискретных и конечных групп, индекс Шура, дедекиндовы кольца, глобально неприводимые представления, простые алгебры над числовыми полями, кватернионы, решетки в простых алгебрах, символ Гильберта, роды, поля расщепления.

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On some characters of group representations

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Abstract

We study realization fields and integrality of characters of discrete and finite subgroups of $SL_2(\mathbf{C})$ and related lattices with a focus on on the integrality of characters of finite groups G. Theory of characters of finite and infinite groups plays the central role in the group theory and the theory of representations of finite groups and associative algebras. The classical results are related to some arithmetic problems: the description of integral representations are essential for finite groups over rings of integers in number fields, local fields, or, more generally, for Dedekind rings. A substantial part of this paper is devoted to the following question, coming back to W. Burnside: whether every representation over a number field can be made integral. Given a linear representation $\rho : G \to GL_n(K)$ of finite group G over a number field $K/\mathbf{Q}$, is it conjugate in $GL_n(K)$ to a representation $\rho : G \to GL_n(O_K)$ over the ring of integers O_K? To study this question, it is possible to translate integrality into the setting of lattices.

This question is closely related to globally irreducible representations; the concept introduced by J. G. Thompson and B. Gross, was developed by Pham Huu Tiep and generalized by F. Van Oystaeyen and A.E. Zalesskii, and there are still many open questions. We are interested in the arithmetic aspects of the integral realizability of representations of finite groups, splitting fields, and, in particular, consider the conditions of realizability in the terms of Hilbert symbols and quaternion algebras.

Keywords: Hyperbolic lattices, groups generated by reflections, characters of discrete and finite groups, Schur index, Dedekind ring, globally irreducible representations, simple algebras over number fields, quaternions, lattices in simple algebras, Hilbert symbol, genera, splitting fields.

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Dedicated to the 80th anniversary of the birth of Professor Michel Deza

1. Introduction

In this paper we are interested to study the integrality of characters of discrete subgroups of $SL_2(\mathbf{C})$ and related lattices.

Hyperbolic lattices in dimension three, that is, discrete cofinite subgroups of $SL_2(\mathbf{C})$, show a preference for having integrally valued character functions, see [29]. Probably, the first known lattice with non-integral character seems to be the one presented by Vinberg at the very end of his fundamental paper [29] where it plays the role of an example for reflection groups. We can present a version of this example and then discuss a series of lattices which contains, most probably, infinitely many with no integer valued character. This is a lattice in three-dimensional hyperbolic space generated by reflections. Let P be the solid in H^3 described combinatorially as a prism with two opposite triangular and three planar quadrangular faces.

Proposition 1. (Vinberg [29]). The group Γ generated by reflections on the faces of P is a cofinite but not cocompact lattice in hyperbolic space H^3. It is not arithmetic.

Consider the following presentation of a subgroup Γ_1 of Γ:
Generators: $\sigma_1, \sigma_2, \tau_1, \tau_2$,
Relations:
(1) $\sigma_1^2 = \sigma_2^2 = (\sigma_1\tau_1)^2 = (\sigma_2\tau_1)^2 = (\sigma_2\tau_2)^2 = (\tau_2^{-1}\tau_1)^2 = I$,
(2) $(\sigma_1\tau_2)^3 = I$,
(3) $\tau_1^6 = \tau_2^6 = I$.

Proposition 2. Γ_1 has trace field equal to $\mathbf{Q}(\sqrt{-3})$, the field of cube roots of unity. Its character values (squared) are unbounded at the nonarchimedian valuation at the prime 2 and integral at all other non-archimedian places. It is cofinite with exactly one cusp.

The character of Γ_1 is determined by the following representation
$G \to GL_n(\mathbf{C})$

$$\sigma_1 = \begin{pmatrix} -i & \frac{3}{\sqrt{2}}i \\ 0 & i \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$\tau_1 = \begin{pmatrix} \frac{\sqrt{3}-3i}{2} & \sqrt{2}i \\ -\sqrt{2}i & \frac{\sqrt{3}-3i}{2} \end{pmatrix}, \tau_2 = \begin{pmatrix} \frac{\sqrt{3}+i}{2} & 0 \\ 0 & \frac{\sqrt{3}-i}{2} \end{pmatrix}$$

In [10] H. Helling considered explicit hyperbolic manifolds obtained by Dehn surgeries of type $(4n, n)$ on the figure of eight knot 4_1. These share the properties of an earlier paper [15] and the above propositions of having associated lattices $SL_2(\mathbf{C})$ with non-integrally valued character functions. See also [15] and [27]. This gives a series of examples of lattices $SL_2(\mathbf{C})$ having non-integral characters.

2. Integrality of characters for finite groups

Starting from this section in this paper we focus on the integrality of characters of finite groups G. Though the traces of $g \in G$ are always algebraic integers, the representations $G \to GL_n(K)$ are not always realizable in the rings of integers of algebraic number fields K.

Let us consider the following

Assumption 1. Let G be a finite group, K a number field with the ring of integers O_K and $\rho : G \to GL_n(K)$ an irreducible representation of G. We denote by M_K the associated irreducible KG-module.

Definition. The representation $\rho : G \to GL_n(K)$ is integral, if and only if $\rho(g) \in GL_n(O_K)$ for all $g \in G$. We say that $\rho(G)$ can be made integral, if and only if there exists an integral representation $G \to GL_n(O_K)$ which is equivalent to ρ. We call M_K integral if $\rho(G)$ can be made integral.

In other words, $\rho(G)$ can be made integral if and only if we can apply a base change such that all matrices have integral entries.

W. Burnside asked the question whether every representation over a number field can be made integral. To study this question, it is possible to translate integrality into the setting of lattices.

Question. *(W. Burnside, I. Schur, later W. Feit, J.-P. Serre). Given a linear representation* $\rho : G \to GL_n(K)$ *of finite group* G *over a number field* $K/\mathbf{Q}$*, is it conjugate to a representation* $\rho : G \to GL_n(O_K)$ *over the ring of integers* O_K*?*

There is an algorithm which efficiently answers this question, it decides whether this representation can be made integral, and, if this is the case, a conjugate integral representation can be computed. Integral realizability of ρ ver the ring of integers O_K depends strongly on the class number cl_K of K. The following proposition is well-known, see e.g. [4].

Proposition 3A. *Assume that one of the conditions hold:*
(i) We have $K = \mathbf{Q}$.
(ii) We have $cl_K = 1$.
(iii) We have the greatest common divisor $GCD(cl_K; n) = 1$.
(iv) We have $cl_K/cl_K^2 = 1$.
Then the representation $\rho : G \to GL_n(K)$ *can be made integral.*

In the papers by D. K. Faddeev (1965, 1995), see [6] and [7], some new ideas on generalized integral representations over Dedekind rings were discussed.

The following theorem is contained in [2].

Theorem 1 *(Cliff, Ritter, Weiss, [2]). Let* G *be a finite solvable group. Then every absolutely irreducible character* χ *of* G *can be realized over* $\mathbf{Z}[\zeta_{\mathbf{m}}]$*, where* m *is the exponent of* G.

Example. The metacyclic group $G = \langle x; y | x^9 = y^{19} = 1; y^x = y^7 \rangle$ admits an absolutely irreducible representation $G \to GL_3(K)$ which cannot be made integral, where K is the unique subfield of $\mathbf{Q}(\zeta_{\mathbf{57}})$ of degree 12.

Theorem 2 *(Serre, [28]).*
Let $G = Q_8$, $K = \mathbf{Q}(\sqrt{-\mathbf{d}})$, *and* $d > 0$. *Then*
1) G *is realizable over* K, $\rho : G \to GL_2(K)$, *if and only if* $d = a^2 + b^2 + c^2$ *for some integers* a, b, c.
2) G *is realizable over* O_K, $\rho : G \to GL_2(O_K)$, *if and only if* $d = a^2 + b^2$ *for some integers* a, b *or* $d = a^2 + 2b^2$ *for some integers* a, b.

The starting point of studying absolutely irreducible representations of finite groups with the property of irreducibility modulo all primes was the concept of of global irreducibility. The notion of globally irreducible representations for the ring of rational integers appeared in papers by B. H. Gross, see [8], [9] in order to explain new series of Euclidean lattices discovered by N. Elkies and T. Shioda using Mordell-Weil lattices of elliptic curves.

The concept of global irreducibility for arithmetic rings has been introduced by F. Van Oystaeyen and A.E. Zalesskii: a finite group $G \subset GL_n(F)$ over an algebraic number field F is globally irreducible if for every non-archimedean valuation v of F a Brauer reduction reduction of G (mod v) is absolutely irreducible. The following theorem is proven in [25].

Theorem 3 (F. Van Oystaeyen and A.E. Zalesskii, see [25]).
O_F-span $O_F G$ of a group $G \subset GL_n(O_F)$ is equal to $M_n(O_F)$ if and only if $G \subset GL_n(O_F)$ is globally irreducible.

The natural problem is to describe the possible n and arithmetic rings O_F such that there is a globally irreducible $G \subset GL_n(O_F)$. In our particular situation it is interesting, what happens for $n = 2$? This question was considered in [20], [22]. The answer is given in the theorem below, see [22], Theorem, p. 9.

Theorem 4 ([22]).
1) Let $G = Q_{4m}$ be the group of generalized quaternions, and let $H = G = Q_8$ be the group of quaternions. Then there is a quadratic subfield $K_1 \subset K$ and an $O_{K_1}H$-module I which is an ideal in an extended field $L_1 = K_1(i)$, such that: $G = Q_{4m}$ is realizable over O_K if and only if H is realizable over O_{K_1}, and all Hilbert symbols $\left(\frac{-d, N_{L_1/\mathbf{Q}}(I)}{p}\right) = 1$ for all $p|d$.
2) If $G = Q_{4m}$ is not realizable over O_K, the minimal realization field such that H is realizable over its ring of integers is a biquadratic extension $\mathbf{Q}(\sqrt{\mathbf{d_1}}, \sqrt{\mathbf{d_2}})$, where $d = d_1 d_2$ and d_1, d_2 are integers not equal to $\pm$ 1 or to $\pm$ d.
3) The explicit computation of I in $L_1 = K_1(i)$ is relevant to a representation of the integer $d = a^2 + b^2 + c^2$. N_{L_1/K_1} of either of these ideals is a principal ideal in O_{K_1} if:
(1) *$b = c$; then $d = a^2 + 2b^2$ ($(a, b) = 1$) – equivalently, d has no prime factors $p \equiv 5(\mathrm{mod}8)$ and $p \equiv 7(\mathrm{mod}8)$*, or
(2) *$c = 0$; then $d = a^2 + b^2$ ($(a, b) = 1$) or equivalently, d has no prime factors $p \equiv 3(\mathrm{mod}4)$.*

Let G be a finite group and χ its complex irreducible character. A number field $K/\mathbf{Q}$ is a splitting field of χ, if there exists a representation of G over K affording χ. A splitting field K is of the minimal degree, if there is no splitting field of χ with degree smaller than K. We say that a splitting field K of χ is integral, if any representation of G over K affording χ can be made integral. Otherwise, the splitting field K is nonintegral.

Let χ be an irreducible complex character of a finite group. All minimal splitting fields of χ have the same relative degree over the character field $\mathbf{Q}(\chi)$, which is called the Schur index of χ over $\mathbf{Q}$, [18] . Let us use for this degree the following notation: $m_{\mathbf{Q}(\chi)}(\chi)$.

For each place v of $\mathbf{Q}(\chi)$, there is an associated local Schur index of χ at v, denoted by $m_{\mathbf{Q}(\chi)_\mathbf{v}}(\chi)$, and the least common multiple

$$m_{\mathbf{Q}(\chi)}(\chi) = LCM_v\{m_{\mathbf{Q}(\chi)_\mathbf{v}}(\chi).\}$$

The field $\mathbf{Q}(\chi) \subset \mathbf{K}$ is a splitting field of χ if and only if $m_{\mathbf{Q}(\chi)_\mathbf{v}}(\chi)$ divides $[K_w : \mathbf{Q}(\chi)_\mathbf{v}]$ for all places v of $\mathbf{Q}(\chi)$ and all divisors w of v.

If $m_\mathbf{Q}(\chi) > 1$, then there are infinitely many minimal splitting fields of χ, and if $m_\mathbf{Q}(\chi) = 1$, then the field of characters $\mathbf{Q}(\chi)$ is the unique minimal splitting field of χ.

Do there exist integral and nonintegral minimal splitting fields of a given character? If so, how many are there?

Let us consider the case of trivial Schur index. In this case $\mathbf{Q}(\chi)$ is the only minimal splitting field of χ. The example above shows that it can be nonintegral. On the other hand, for a character χ with $\mathbf{Q}(\chi) = \mathbf{Q}$ the minimal splitting field of χ is integral. Thus in general both cases will occur. We will now concentrate on the case $m_\mathbf{Q}(\chi) > 1$, more precisely on the case $m_\mathbf{Q}(\chi) > 1, \mathbf{Q}(\chi) = \mathbf{Q}$ and $deg(\chi) = 2$.

Let G be a finite group, K a number field with the ring of integers $O = O_K$. We will now concentrate on a special situation, originally treated by Serre in [Ser08], for which the existence of integral and nonintegral minimal splitting fields is closely connected to the theory of quaternion algebras and Hilbert symbols. Below we consider (the Hilbert symbol (a, b) over $\mathbf{Q}$ and for a place v of $\mathbf{Q}$ we denote by $(a; b)_v$ the corresponding local Hilbert symbol over $\mathbf{Q_v}$. By $Br_2(\mathbf{Q})$ we denote the subgroup of the Brauer group of $\mathbf{Q}$ generated by quaternion algebras.

We denote by Cl_K the group of ideal classes of K. For a finitely generated O_K-module (a lattice) M we denote by $cl(M)$ its Steinitz class. The simple component of $\mathbf{QG}$, corresponding to the irreducible character χ, is a non-split quaternion algebra over $\mathbf{Q}$, which we denote by D. The proof of the proposition 3 below is contained in the paper by J.-P. Serre [28].

Definition (see [28].) Let K be an imaginary quadratic number field with discriminant $-d, d > 0$. We define the map

$$e_K : Cl_K/Cl_K^2 \to Br_2(\mathbf{Q}); [\mathbf{a}] \to (\mathbf{N}(\mathbf{a}), -\mathbf{d}).$$

Proposition 3 (see [28]). *Let K be an imaginary quadratic number field with discriminant $-d$, which splits D and which we consequently view as a subfield of D. Then the following conditions hold true:*
(i) The map e_K is well-defined and injective.
(ii) Let R be a maximal order of D containing O. Then the O-module R is G-invariant. In particular R is an OG-lattice.
(iii) If R and R_0 are two maximal orders of D containing O, then $cl(R) = cl(R_0)$ in Cl_K/Cl_K^2.
(iv) Let R be a maximal order of D containing O. Then we have $e_K(cl(R)) = (D) \cdot (d_D; -d)$, where d_D is the product of all primes ramified in D including -1 if ∞ is ramified, (D) is the class of D in $Br_2(\mathbf{Q})$.

Proposition 4 ([22], proposition 5).

(1) *An algebraic number field K is a splitting field for the group G of quaternions if and only if K is totally imaginary and for all localizations K_v for all prime divisors v of 2 the local degree $[K_v : \mathbf{Q_2}]$ is even.*

(2) *If K is a splitting field for the group G of quaternions, then $[K : \mathbf{Q}]$ is even.*

(3) *K is a splitting field for the group G of quaternions and $K/\mathbf{Q}$ is abelian, then K has a quadratic subfield $\mathbf{Q}(\sqrt{\mathbf{d}})$.*

For the convenience of the reader we include the proof of proposition 4.

Proof. By the theorem of Hasse-Brauer-Noether, K is a splitting field for $\langle G\rangle_{\mathbf{Q}}$ if and only if the localization K_v is a splitting field locally for $\langle G\rangle_{\mathbf{Q_p}} = \mathbf{Q_p}G$ for all prime divisors v of p. Since the quaternion algebra has invariants $1/2$ at 2 and ∞ in the Brauer group, and 0 at all other primes p, K is a splitting field for G if and only if K is totally imaginary and for all localizations K_v for all prime divisors v of 2 the local degree $[K_v : \mathbf{Q_2}]$ is even [5], Satz 2, ch. VII, sect. 5.

Since $[K : \mathbf{Q}]$ is the sum of $[K_v : \mathbf{Q_2}]$, it must be even, and this implies (2).

If $K/\mathbf{Q}$ is abelian, its degree is even, and its Galois group has a subgroup of index 2, therefore, the fixed subfield of this subgroup is a quadratic extension of $\mathbf{Q}$.

This completes the proof of proposition 4.

Let us consider the following

Assumption 2. Let G be a finite group and let χ be an irreducible character of a finite group with $m_{\mathbf{Q}}(\chi) > 1, \mathbf{Q}(\chi) = \mathbf{Q}$ and $deg(\chi) = 2$.

Consider the simple component D of $\mathbf{QG}$, corresponding to the irreducible character χ, which was used in proposition 3 above. Below we consider classes of sublattices $L(R)$ of a maximal order R of D. Recall that a quaternion algebra is just a 4-dimensional $\mathbf{Q}$-algebra with center $\mathbf{Q}$. We have the following equivalence:

(1) A quadratic field K is a splitting field of χ,

(2) all places v of $\mathbf{Q}$ with $m_{\mathbf{Q_v}}(\chi) = 2$ do not split in K over $\mathbf{Q}$,

(3) the field K can be embedded as a maximal subfield of D,

(4) For all places v of $\mathbf{Q}$ at which D is ramified, the field K_w splits D_v for all places w of K lying above v.

Let K be an imaginary quadratic field which splits D. Then we can view D as a KG-module, which we denote by M_K, and we have seen that a maximal order R of D containing O is an OG-lattice of M_K. To determine integrality, it is now sufficient to consider the set $cl(L(R))$ of classes of sublattices of R or, since e_K is injective, the set $e_K(cl(L(R)))$. Let K be a minimal splitting field of the character χ. Let us denote by S_K the set of prime ideals of $O = O_K$ such that a Brauer reduction of M_K is reducible, let $S' = S'_K$ be the set of rational primes lying over ideals in S_K. Let S be the intersection of all $S' = S'_K$ over all minimal splitting fields K; following [28], we denote by $e(D, K) = e_K(cl(R))$ for a maximal order R of D containing O. Remind that $-d$ is the discriminant of K.

Lemma 1.
$$e_K(cl(L(M_K))) \subset e(D, K) \cdot \{\Pi_{p \in S_0}(p, -d) | S_0 \subset S\}$$

Proof. It follows from [26], theorem 2.5, and the observation that the class of a sublattice of R can only change by a square or the class of $[I] \in Cl_K/Cl_K^2$, where I is a prime ideal whose I-reduction is reducible, that $cl(L(M_K)) \subset cl(R) \cdot \{\Pi_{I|p \in S_0}[I] | S_0 \subset S\}$. By applying the map e_K to the equation obtained, we get $e_K(cl(L(M_K))) \subset e(D, K) \cdot \{\Pi_{p \in S_0}(p^{f(I)}, -d) | S_0 \subset S\}$, where $f(I)$ is the inertia index of $f(I)$ in $K/\mathbf{Q}$. Assume that $I \in S_K$, but not in S and I is a prime ideal of K above p. Then there exists a minimal splitting field L and a prime ideal q of L lying above p such that the reduction of M_K modulo I is reducible, while the reduction of M_K modulo q is irreducible. This is only possible if the residue field of I is strictly larger than the residue field of I. Thus the norm $N(I) = p^2$ and therefore $(p^{f(I)}, -d) = 1$.

This completes the proof of lemma 1.

Lemma 2 *([3], Theorem 5.3.2, see also [24], Theorem 2.8, compare also [16], sect 81, p.144, Theorem 112). Let $(a_i)_{i \in I}$ be a finite set of elements of $\mathbf{Q}^*$, and let $(\epsilon_{i,v})_{i \in I, v \in P}$ be a set of numbers equal to ± 1. There exist an infinite number of $x \in \mathbf{Q}^*$ such that $(a_i, x) = \epsilon_{i,v}$ for all $i \in I$ and all $v \in P$ if and only if the following three conditions are satisfied:*

(1) Almost all of the $\epsilon_{i,v}$ are equal to 1, say, $\epsilon_{i,v} = 1$ for $v \notin P_0$ and a finite subset $P_0 \subset P$.

(2) For all $i \in I$ we have $\Pi_{v \in P}(\epsilon_{i,v}) = 1$.

(3) For all $v \in P$ there exists $x_v \in \mathbf{Q}_\mathbf{v}^*$ such that $(a_i, x_v)_v = \epsilon_{i,v}$ for all $i \in I$.

Note that infiniteness of the number of x follows from Dirichlet's theorem on primes in arithmetic progressions which is involved in the proof.

Lemma 3. *There is an infinite number of splitting fields $K = \mathbf{Q}(\sqrt{-\mathbf{d}})$ of χ such that $Cl_K/Cl_K^2 = 1$.*

Proof. It follows from [12] that $Cl_K/Cl_K^2 = 1$ for quadratic fields $K = \mathbf{Q}(\sqrt{-\mathbf{p}})$ for $-p \equiv 1(mod 4)$. Let $T = \cup_i q_i$ be the set of rational primes such that Schur indices of χ at q_i are 2. An extension K of the character field $\mathbf{Q}$ is a splitting field of if all places of K above the $p \in T$ have inertia degrees divisible by 2. The Legendre symbol $\left(\frac{d_K}{q_i}\right) = (-1)^{(q_i-1)/2}\left(\frac{p}{q_i}\right)$ for the discriminant $d_K = -p$. It follows from proposition 4 that for primes $q_i \neq 2$ the character χ splits iff $\left(\frac{p}{q_i}\right) = (-1)^{(q_i+1)/2}$. For $q_i = 2$ we can see that the inertia degree is 2 iff $d_k = -p \equiv 1(mod 8)$. Now we can use Dirichlet's theorem on primes in arithmetic progression to conclude that there are infinitely many primes p satisfying the above congruence conditions for all p_i. This completes the proof of lemma 3.

Theorem 5. *Let χ be an irreducible character of a finite group with $m_\mathbf{Q}(\chi) > 1, \mathbf{Q}(\chi) = \mathbf{Q}$ and $deg(\chi) = 2$. Then there exist infinitely many integral minimal splitting fields of χ, and there is infinitely many nonintegral minimal splitting fields of χ.*

Proof.

1) We can use proposition 3, (iv) together with lemma 3 to prove that there exist infinitely many integral minimal splitting fields of χ. It follows from proposition 4 that the infinite number of K from lemma 3 are minimal splitting fields of χ.

2) Let P be the set of all finite rational primes and ∞. Let $Ram(D)$ be the set of all finite ramified primes in D together with -1 in the case if D is ramified at ∞. Let for the elements $p_i \in S \cup Ram(D)$ the set $\epsilon_i = \pm 1$ be prescribed elements. It follows from [11], ch 5, sect. 6, and Dirichlet's theorem on primes in arithmetic progressions that there is an infinite number of primes q and integers x such that Hilbert symbols $(p_i, x)_q = \epsilon_i$ for all indices i.

According to proposition 3, (iv) and lemma 1, it is sufficient to prove that the unit class is not contained in the set of classes $(D) \cdot \{\Pi_{p \in S_0}(p, -d) | S_0 \subset S\}$ for any $S_0 \subset S_1 = S \cup Ram(D)$ and for an infinite number of d; D is a non-split algebra, and we can assume that S_0 is not empty. For any $S_0 \subset S \cup Ram(D)$ take elements $\epsilon_p \pm 1, p \in S_1$ such that $\Pi_{p \in S_1} \epsilon_p = -1$; according to the above argument there are integers x and a prime $q_1 \notin S_1$ such that D splits at q_1 and $(p, x)_{q_1} = \epsilon_p$ for all $p \in S_1$, thus $((D) \cdot \Pi_{p \in S_0}(p, -x))_{q_1} = \Pi_{p \in S_0}(p, x)_{q_1} = -1$ since $(p, -1)_{q_1} = 1$. Also we can take $q_1 \neq q_1''$ if the corresponding $S_1 \neq S_1''$. Now we can use lemma 2 for $I = S_1, \{a_i\}_{i \in I} = \{p\}_{p \in S_1}, \epsilon_{i,v} = \epsilon_p, P_0 = \{q_1\}$, where q_1 corresponds to $S_0 \subset S$.

The sufficient conditions for an imaginary quadratic field $K = \mathbf{Q}(\sqrt{-\mathbf{d}})$, where $d > 0$, to be a splitting field of D is that for all $q \in Ram(D)$ the condition $(q, d)_q = (-1)^{(q+1)/2}$ if $q \neq 2$, or the condition $(q, d)_2 = 1$ if $q = 2$ hold true; since K is imaginary, the sufficient condition at the infinite place is also satisfied. We can also assume that since the conditions for $K = \mathbf{Q}(\sqrt{-\mathbf{d}})$ to be a splitting field affect only $v \in Ram(D)$ which do not intersect P_0, and the second claim of theorem 5 follows from lemma 2.

Remark 1. *A similar theorem holds in a more general settings, we have can find minimal integral and nonintegral splitting fields for a large number of characters of various groups assuming that χ is an irreducible character of G with $m_{\mathbf{Q}}(\chi) > 1$.*

Remark 2. In some earlier papers, see e.g [21], the author considered the conditions of integrality for representations of finite groups together with conditions of stability of Galois action. The following question has a deep topological motivation, see [1].

Let $\rho : G \to GL_n(\mathbf{C})$ be a complex n-dimensional representation of a finite group G. Let τ be an automorphism of the field $\mathbf{C}$, not necessarily continuous. For $g \in G$, we act by τ on the matrix coefficients of $\rho(g)$ and obtain a new matrix $\tau(\rho(g))$.

We obtain a new subgroup $\tau(\rho(G))$ in $GL_n(\mathbf{C})$. Is it possible that the subgroup $\tau(\rho(G))$ is not conjugate to $\rho(G)$ in $GL_n(\mathbf{C})$, i.e. there is no matrix $X \in GL_n(\mathbf{C})$ such that $\tau(\rho(G)) = X\rho(G)X^{-1}$?

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3. Conclusion

Theory of characters of finite and infinite groups plays the central role in the group theory and the theory of representations of finite groups and associative algebras. The classical results are related to some arithmetic problems: the description of integral representations are essential for finite groups over rings of integers in number fields, local fields, or, more generally, for Dedekind rings. In this paper we are interested to study the integrality of characters of discrete and finite subgroups of $SL_2(\mathbf{C})$ and related lattices. A substantial part of this paper is devoted to the following question, coming back to W. Burnside: whether every representation over a number field can be made integral. To study this question, it is possible to translate integrality into the setting of lattices.

Question. *(W. Burnside, I. Schur, later W. Feit, J.-P. Serre). Given a linear representation $\rho : G \to GL_n(K)$ of finite group G over a number field $K/\mathbf{Q}$, is it conjugate to a representation $\rho : G \to GL_n(O_K)$ over the ring of integers O_K?*

This question is closely related to globally irreducible representations; the concept introduced by J. G. Thompson and B. Gross, was developed and generalized by Pham Huu Tiep, F. Van Oystaeyen and A.E. Zalesskii, and there are still many open questions. We are interested in the arithmetic aspects of the integral realizability of representations of finite groups, and, in particular, prove the existence of infinite number of splitting fields where the representations are not realizable.

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