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Нет строго регулярного графа локально Хивуд

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Аннотация

Мы исследуем, когда сильно регулярный граф локально Хивуд. Мы фокусируемся на предполагаемом сильно регулярном графе с параметрами $(v, k, \lambda, \mu) = (85, 14, 3, 2)$, который является единственным кандидатом на такой график. Предполагая, что граф является локально Хивудом, мы анализируем его структуру, в конце концов приходя к противоречию, которое позволяет нам заключить, что никакой сильно регулярный граф не является локально Хивудом.

Ключевые слова: сильно регулярные графы, локальные графы, граф Хивуд.

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No strongly regular graph is locally Heawood

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Abstract

We investigate when a strongly regular graph is locally Heawood. We focus on a putative strongly regular graph with parameters $(v, k, \lambda, \mu) = (85, 14, 3, 2)$, which is the only candidate for such a graph. Assuming that the graph is locally Heawood, we analyze its structure, finally arriving to a contradiction, which allows us to conclude that no strongly regular graph is locally Heawood.

Keywords: strongly regular graphs, local graphs, Heawood graph.

Bibliography: 6 titles.

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1. Introduction

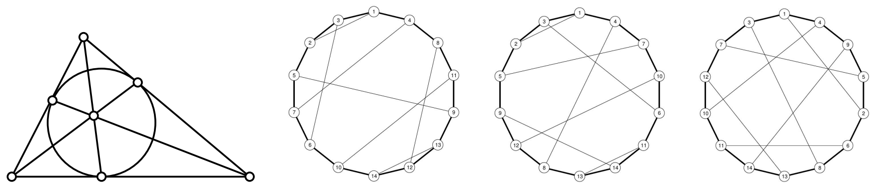
Let Γ be a k-regular graph with v vertices such that each pair of adjacent vertices has λ common neighbours, and each pair of nonadjacent vertices has μ common neighbours. Such a graph is said to be *strongly regular* with parameters (n, k, λ, μ), or $\mathrm{SRG}(n, k, \lambda, \mu)$ for short. Note that for a given parameter set, there may be one strongly regular graph, or there may be more or none at all. Looking at the tables of feasible parameters of strongly regular graphs by Andries E. Brouwer [2], one finds only 10 open cases on at most 100 vertices, see Table 1.

	n	k	λ	μ	σ	τ	m_σ	m_τ	graph	# of edges
O1	65	32	15	16	3.531	-3.531	32	32	2-graph*?	1040
O2	69	20	7	5	5	-3	23	45		690
O3	85	14	3	2	4	-3	34	50		595
O4	85	30	11	10	5	-4	34	50	$S(2,6,51)$?	1275
O5	85	42	20	21	4.110	-5.110	42	42	2-graph*?	1785
O6	88	27	6	9	3	-6	55	32		1188
O7	96	35	10	14	3	-7	63	32	$\mathrm{pg}(5,6,2)$	2030
O8	99	14	1	2	3	-4	55	44		693
O9	99	42	21	15	9	-3	21	77		2079
O10	100	33	8	12	3	-7	66	33		1650

Таблица 1: Feasible parameters of strongly regular graphs on at most 100 vertices, with their parameters, spectrum, some graph information and the number of edges.
Related to the open case O8, John H. Conway asked if there is a graph on 99 vertices in which every edge belongs to a unique triangle and every nonedge to a unique quadrilateral (see [5], cf. [1])

Out of all these 10 cases, O3 has the smallest number of edges. The graph induced by the neighbours of a vertex is a cubic graph (i.e., all vertices have degree 3) on 14 vertices. It is known that there are precisely 509 connected cubic graphs on 14 vertices [6]. If we add the restriction that each pair of nonadjacent vertices has at most one common neighbour, then we are left with only 36 potential candidates. Among them, the Heawood graph has the largest girth, namely 6. One can describe it as the cycle C_{14} (labeled with elements of $\mathbb{Z}_{14}$), where we add the following chords: $x \sim x + 5$ for all even $x \in \mathbb{Z}_{14}$, and $x \sim x - 5$ for all odd $x \in \mathbb{Z}_{14}$. Three more possibilities are shown in Figure 1(b, c, d). There are also three more possibilities in the case the local graph is disconnected. In each case, one component is K_4, while the other component is either (a) the Petersen graph, (b) the cycle C_9 (labeled with the elements of $\mathbb{Z}_9$), with 3 triangles ($x \sim x + 2$ for $x = 0, 3, 6$) and the vertex 9 adjacent to 1, 4 and 7, (c) the 3-prism with vertical edges subdivided and the new vertices adjacent to one more vertex.

We will prove that no strongly regular graph is locally Heawood.



(a) (b) (c) (d)

Рис. 1: (a) The Fano plane. (b, c, d) Some trivalent graphs on 14 points.

2. Preliminaries

It is easy to see that a strongly regular graph is either a disjoint sum of complete graphs (if $\lambda = k - 1$), or a connected graph with diameter 2. We may generalize the latter case.

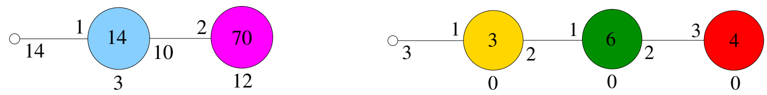
Let Γ be a connected graph of diameter d and assume that there exist constants a_i, b_i, c_i ($0 \leqslant i \leqslant d$) such that for each pair of vertices u, v at distance i, there are precisely c_i neighbours of v at distance $i - 1$ from u, a_i neighbours of v at distance i from u, and b_i neighbours of v at distance $i + 1$ from u. Such a graph is said to be *distance-regular* with *intersection array* $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$. Clearly, such a graph is k-regular for $k = b_0$, and $k = a_i + b_i + c_i$ for all i ($0 \leqslant i \leqslant d$). Furthermore, we see that we always have $a_0 = c_0 = b_d = 0$ and $c_1 = 1$. Equivalently, we could say that a graph is distance-regular if all of its *distance partitions* (the partitions of vertices with respect to the distance from a fixed vertex) are *equitable* (the number of neighbours a vertex u has in a part C only depends on C and the part containing u) with the same parameters (which are precisely the numbers a_i, b_i, c_i).

Given an intersection array of a distance-regular graph, it is possible to compute the *intersection numbers* p^h_{ij} ($0 \leqslant h, i, j \leqslant d$) counting the number of vertices at distances i and j from any pair of vertices at distance h [3]. Just like in the case of strongly regular graphs, there might be zero, one or multiple distance-regular graphs with a given intersection array. It is easy to see that a graph $\mathrm{SRG}(n, k, \lambda, \mu)$ of diameter 2 is distance-regular with intersection array $\{k, k - \lambda - 1; 1, \mu\}$ – conversely, every strongly regular graph with $\lambda \neq k - 1$ arises in this way. For more on strongly regular and distance-regular graphs, see Brouwer, Cohen & Neumaier [3].

Let Γ be a graph with n vertices, and let u be a vertex of Γ. We define $\Gamma_i(u)$ as the set of vertices at distance i from u in Γ (also called the *i-th subconstituent*). We abbreviate $\Gamma_1(u)$ as $\Gamma(u)$. For vertices u, v of Γ, we also define $\Gamma(u, v) = \Gamma(u) \cap \Gamma(v)$. By abuse of notation, we will also denote by $\Gamma(u)$ the *local graph at* u, i.e., the graph induced by the set $\Gamma(u)$. If there is a graph Δ such that $\Gamma(u)$ is isomorphic to Δ for every vertex u of Γ, then the graph Γ is said to be *locally* Δ. If u and v are vertices of Γ at distance 2, then the graph induced by $\Gamma(u, v)$ is called a *μ-graph* of Γ.

Let $A \in \{0, 1\}^{n \times n}$ be the adjacency matrix of Γ. An *eigenvalue* of the graph Γ is a number θ such that there exists a vector x for which $Ax = \theta x$ holds. The subspace of all such vectors is called the *eigenspace* of the eigenvalue θ; its dimension is the *multiplicity* of θ. The multiset of all eigenvalues of Γ counted with their multiplicity is called the *spectrum* of Γ and is usually written in the form ${\theta_1}^{m_1}\ {\theta_2}^{m_2}\ \dots\ {\theta_\ell}^{m_\ell}$, where the eigenvalues θ_i ($1 \leqslant i \leqslant \ell$) are given in decreasing order, and the numbers m_i represent their multiplicities.

For sets $\mathbb{P}$ of *points* and $\mathbb{L}$ of *lines* and an *incidence relation* $\mathcal{I} \subseteq \mathbb{P} \times \mathbb{L}$, we define an *incidence structure* as the triple $(\mathbb{P}, \mathbb{L}, \mathcal{I})$. A pair (p, ℓ) ($p \in \mathbb{P}$, $\ell \in \mathbb{L}$) is called a *flag* if $p \mathcal{I} \ell$ and an *antiflag* otherwise. A *projective geometry* $\mathrm{PG}(d, q)$ is an incidence structure $(\mathbb{P}, \mathbb{L}, \in)$ in which the points are 1-dimensional subspaces of $\mathbb{F}_q^d$ (i.e., the d-dimensional vector space over the finite field of size q), the lines are 2-dimensional subspaces of $\mathbb{F}_q^d$, and the incidence relation is inclusion (i.e., for $p \in \mathbb{P}$ and $\ell \in \mathbb{L}$, $p \in \ell$ holds whenever $p \leqslant \ell$). Such an incidence relations has precisely $q^2 + q + 1$ points and $q^2 + q + 1$ lines.



(a) (b)

Рис. 2: Distance partitions of (a) SRG$(85, 14, 3, 2)$, spectrum $14^1\ 4^{34}\ {-3}^{50}$, $b_1 = 10$, $a_2 = 12$, $k_2 = 70$, $p^2_{22} = 57$, $p^1_{22} = 60$, $\ldots$, and (b) the Heawood graph, which is the $(3, 6)$-cage and is bipartite with spectrum $3^1\ \sqrt{2}^6\ {-\sqrt{2}}^6\ {-3}^1$.

3. Locally Heawood

Let $F = (\mathbb{P}, \mathbb{L}, \mathcal{I})$ be the Fano plane PG$(2, 2)$ and H its incidence graph, which is known as the Heawood graph (see Figure 1(a)). It is the unique distance-regular graph with intersection array $\{b_0, b_1, b_2; 1, c_2, c_3\} = \{3, 2, 2; 1, 1, 3\}$. We try to construct a strongly regular graph with parameters $(n, k, \lambda, \mu) = (85, 14, 3, 2)$ denoted by Γ that is locally Heawood. Figure 2 shows the distance partitions of Γ and H.

Let ∞ be a vertex of Γ. As Γ is locally Heawood, we identify the vertices of the local graph $\Gamma(\infty)$ with points and lines of F, with a point and line adjacent when they are incident in F. Now consider the second subconstituent graph $\Gamma_2(\infty)$. It has 70 vertices that are divided corresponding to their μ-graphs into

(i) $42 = 2 \cdot 7 \cdot 3$ vertices that correspond to signed flags of F: by $\lambda = 3$, there are 2 vertices per each flag, which we denote by $(p, \ell)^+$ and $(p, \ell)^-$, where $p \in \ell$, and

(ii) $28 = 7 \cdot 4$ vertices that correspond to the antiflags of F: by $\mu = 2$, there is one such vertex for each antiflag, and we denote it by (p, ℓ), where $p \notin \ell$.

Therefore, $V\Gamma = \{\infty\} \cup \big(\mathbb{P} \cup \mathbb{L}\big) \cup \big(A \cup B\big)$, where

$$A := \{(p, \ell) \mid p \in \mathbb{P}, \ell \in \mathbb{L}, p \notin \ell\} \qquad \text{(the antiflags), and}$$
$$B := \left\{(p, \ell)^\delta \;\middle|\; p \in \mathbb{P}, \ell \in \mathbb{L}, p \in \ell, \delta \in \{+, -\}\right\} \qquad \text{(the signed flags)}$$

and we have established all adjacencies when both vertices do not belong to $\Gamma_2(\infty)$:

$$\begin{aligned}
\forall\, p \in \mathbb{P} &: \infty \sim p \\
\forall\, \ell \in \mathbb{L} &: \infty \sim \ell \\
\forall\, p \in \mathbb{P}, \ell \in \mathbb{L} &: \ p \sim \ell && \Leftrightarrow p \in \ell \\
\forall\, p \in \mathbb{P}, (q, m) \in A &: \ p \sim (q, m) && \Leftrightarrow p = q \\
\forall\, \ell \in \mathbb{L}, (q, m) \in A &: \ \ell \sim (q, m) && \Leftrightarrow \ell = m \\
\forall\, p \in \mathbb{P}, (q, m)^\delta \in B &: \ p \sim (q, m)^\delta && \Leftrightarrow p = q \\
\forall\, \ell \in \mathbb{L}, (q, m)^\delta \in B &: \ \ell \sim (q, m)^\delta && \Leftrightarrow \ell = m
\end{aligned}$$

However, we can also say something about adjacencies in $\Gamma_2(\infty)$.

Lemma 1. *Let $\boldsymbol{e} = (p, k)$ and $\boldsymbol{f} = (q, \ell)$ be flags of F, and $\boldsymbol{g} = (r, m)$ and $\boldsymbol{h} = (s, n)$ be antiflags of F. Then the following statements (i)-(iv) hold:*

(i) signed flags corresponding to the same point or line (or both) are not adjacent, i.e., $(p = q \vee k = \ell) \Rightarrow \boldsymbol{e}^\delta \not\sim \boldsymbol{f}^\varepsilon$ for all $\delta, \varepsilon \in \{+, -\}$,

(ii) *antiflags corresponding to the same point or line are not adjacent, i.e.,* $(r = s \vee m = n) \Rightarrow \boldsymbol{g} \not\sim \boldsymbol{h}$,

(iii) *a signed flag is adjacent to two of the four antiflags corresponding to the same point or line,*

(iv) *any antiflag is adjacent to precisely one of the signed flags* $\boldsymbol{f}^+$ *and* $\boldsymbol{f}^-$ *corresponding to the same point or line, i.e.,* $(q = r \vee \ell = m) \Rightarrow (\boldsymbol{f}^+ \sim \boldsymbol{g} \Leftrightarrow \boldsymbol{f}^- \not\sim \boldsymbol{g})$,

(v) *two distinct antiflags corresponding to the same point or line have precisely one common neighbour in* $\Gamma_2(\infty)$*, which is a signed flag corresponding to the common point or line.*

(vi) *the signed flags* $\boldsymbol{f}^+$ *and* $\boldsymbol{f}^-$ *have no common neighbours in* $\Gamma_2(\infty)$,

In particular, the second subconstituent graph $\Gamma_2(\infty)$ *has diameter* 3 *and only the opposite signed flags are at distance* 3 *in it.*

Proof. Let us consider the local graph $\Gamma(p)$ (resp. $\Gamma(\ell)$). Then

$$\{\infty\} \quad \cup \quad (\Gamma(p) \cap \Gamma(\infty)) \quad \cup \quad (\Gamma(p) \cap B) \quad \cup \quad (\Gamma(p) \cap A)$$

is its distance partition as a Heawood graph H, where the last two sets consist of signed flags and antiflags corresponding to the point p, respectively. A similar distance partition can be obtained for a line ℓ. As H is bipartite, (i) and (ii) follow immediately. The parameter $b_2 = 2$ implies (iii), and the parameters $c_2 = 1$ and $c_3 = 3$ of H then imply (iv). Considering (ii), (v) also follows. Finally, the signed flags $\boldsymbol{f}^+$ and $\boldsymbol{f}^-$ are nonadjacent by (i) and their common neighbours are $q, \ell \in \Gamma(\infty)$, so (vi) follows by $\mu = 2$. Let $\boldsymbol{f} = (p, \ell)$ be an antiflag or a flag of F, i.e., $\boldsymbol{f}^\varepsilon \in \Gamma_2(\infty)$, where $\varepsilon \in \{\emptyset, +, -\}$. Suppose that there exists an antiflag or signed flag $\boldsymbol{g}^\delta \in \Gamma_2(\infty)$ (where $\delta \in \{\emptyset, +, -\}$) that is not adjacent to $\boldsymbol{f}^\varepsilon$ and has no common neighbours with it in $\Gamma_2(\infty)$. Then $\Gamma(\boldsymbol{f}^\varepsilon, \boldsymbol{g}^\delta) = \{p, \ell\} \subset \Gamma(\infty)$ and $\Gamma(p, \ell) = \{\infty, \boldsymbol{f}^\varepsilon, \boldsymbol{g}^\delta\}$. Since $|\Gamma(p, \ell)| = 3$, we conclude $\boldsymbol{f}$ is a flag and $\boldsymbol{g} = \boldsymbol{f}$, $\delta = -\varepsilon$. Therefore, each antiflag is at distance at most 2 from every other vertex in $\Gamma_2(\infty)$, so the diameter of the latter graph is 3 by (vi). □

Lemma 2. *Let* $\boldsymbol{g} = (q, m)$ *be an antiflag of* F.

(i) *The graph induced on* $\Gamma(\boldsymbol{g}) \cap A$ *is an induced subgraph of a hexagon.*

(ii) *Let* $\boldsymbol{f} = (p, \ell)$ *be a flag of* F *such that* $\boldsymbol{f}^\delta$ *is adjacent to* $\boldsymbol{g}$ *for some* $\delta \in \{+, -\}$*. Then there is a signed flag* $(r, n)^\varepsilon \in B$ *with* $q = r$ *or* $m = n$ *for some* $\varepsilon \in \{+, -\}$ *that is adjacent to* $\boldsymbol{f}^\delta$ *and* $\boldsymbol{g}$.

(iii) *Let* $\boldsymbol{f} = (p, \ell)$ *be an antiflag of* Γ *such that* $\boldsymbol{g} \sim \boldsymbol{f}$*. Then* $p \notin m$ *or* $q \notin \ell$.

(iv) *Let* $\boldsymbol{e} = (s, k)$ *and* $\boldsymbol{f} = (p, \ell)$ *be flags of* F *such that* $\boldsymbol{e}^\varepsilon$, $\boldsymbol{f}^\delta$ *and* $\boldsymbol{g}$ *are mutually adjacent for some* $\delta, \varepsilon \in \{+, -\}$*. If* $s = q$*, then* $p \notin m$*. If* $k = m$*, then* $q \notin \ell$.

(v) *Let* $\boldsymbol{e} = (s, k)$ *and* $\boldsymbol{e'} = (s', k')$ *be distinct flags of* F*, and let* $\boldsymbol{f} = (p, \ell)$ *and* $\boldsymbol{f'} = (p', \ell')$ *be flags or antiflags of* F *such that* $\boldsymbol{g}, \boldsymbol{e}^\varepsilon, \boldsymbol{f}^\delta$ *and* $\boldsymbol{g}$, $\mathbf{e}'^{\varepsilon'}$, $\mathbf{f}'^{\delta'}$ *induce triangles for some* $\delta, \delta' \in \{\emptyset, +, -\}$ *and* $\varepsilon, \varepsilon' \in \{+, -\}$*. If* $s = s' = q$ *and* $\ell \neq m$*, or* $k = k' = m$ *and* $p \neq q$*, then* $p \neq p'$ *and* $\ell \neq \ell'$.

Proof. Consider the local graph $\Gamma(\boldsymbol{g}) \cong H$. It contains the nonadjacent vertices q and m, whose common neighbours ∞ and $\boldsymbol{g}$ are both outside $\Gamma(\boldsymbol{g})$. Therefore, q and m are at distance 3 in $\Gamma(\boldsymbol{g})$. Let D^i_j be the set of vertices at distances i and j from q and m in $\Gamma(\boldsymbol{g})$. Since the Heawood graph has $p^3_{12} = p^3_{23} = 3$, we have $|D^1_2| = |D^2_1| = |D^2_3| = |D^3_2| = 3$, and these four sets cover all the vertices of $\Gamma(\boldsymbol{g})$ except q and m. By $c_2 = 1$, there are matchings between D^1_2 and D^2_1, D^1_2 and D^2_3, and between D^2_1 and D^3_2. $b_2 = 2$ implies that each vertex from D^2_3 or D^3_2 has two neighbours in D^3_2 or D^2_3, respectively.

By Lemma 1(iv), the antiflag $\boldsymbol{g}$ is adjacent to six signed flags $(q, n_i)^{\varepsilon_i} \in D^1_2$ and $(r_i, m)^{\zeta_i} \in D^2_1$ for $i = 1, 2, 3$ with $n_i \neq n_j$ and $r_i \neq r_j$ if $i \neq j$. Since, by Lemma 1(ii), the vertices in $D^1_2 \cup D^2_1$

are all flags, it follows that $\Gamma(\boldsymbol{g}) \cap A \subseteq D_3^2 \cup D_2^3$. The graph induced on $D_3^2 \cup D_2^3$ is bipartite on 6 vertices of valency 2, i.e., a hexagon. This proves (i).

Let $\boldsymbol{f}$ be a flag of F such that $\boldsymbol{f}^\delta \sim \boldsymbol{g}$ for some $\delta \in \{+,-\}$. Then we have $\boldsymbol{f}^\delta \in D_2^1 \cup D_1^2 \cup D_3^2 \cup D_2^3$, so $\boldsymbol{f}^\delta$ must be adjacent to a vertex in $D_2^1 \cup D_1^2$ and (ii) follows.

Let $\boldsymbol{f} = (p,\ell)$ be a flag or antiflag such that $\boldsymbol{f}^\delta$ is in D_3^2 (resp. D_2^3) for some $\delta \in \{\emptyset,+,-\}$. Then there is a flag $\boldsymbol{e} = (s,k)$ such that $s = q$ and $\boldsymbol{e}^\varepsilon \in D_2^1$ (resp. $k = m$ and $\boldsymbol{e}^\varepsilon \in D_1^2$) and $\boldsymbol{e}^\varepsilon$ is adjacent to $\boldsymbol{f}^\delta$ for some $\varepsilon \in \{+,-\}$. Each signed flag in D_1^2 (resp. D_2^1) is nonadjacent to $\boldsymbol{f}^\delta$, and their $\mu = 2$ common neighbours are $\boldsymbol{g}$ and another flag or antiflag of $\Gamma(\boldsymbol{g})$. Therefore, $p \neq r_i$ (resp. $\ell \neq n_i$) for $i = 1,2,3$, i.e., $p \notin m$ (resp. $q \notin \ell$), so (iii) and (iv) follow.

Now let $\boldsymbol{f'} = (p',\ell')$ be another flag or antiflag such that $\mathbf{f'}^{\delta'}$ is in D_3^2 (resp. D_2^3) for some $\delta' \in \{\emptyset,+,-\}$. Then $\boldsymbol{f}^\delta$ and $\mathbf{f'}^{\delta'}$ are at distance 2 in $\Gamma(\boldsymbol{g})$, so their $\mu = 2$ common neighbours are $\boldsymbol{g}$ and another flag or antiflag of $\Gamma(\boldsymbol{g})$. Therefore, $p \neq p'$ and $\ell \neq \ell'$, which proves (v). □

LEMMA 3. *Let $\boldsymbol{f} = (p,\ell)$ be a flag of F and $\delta \in \{+,-\}$.*

(i) The graph induced on $\Gamma(\boldsymbol{f}^\delta) \cap B$ is an induced subgraph of an octagon.

(ii) Let $\boldsymbol{g} = (q,m)$, $\boldsymbol{g'} = (q',m')$ and $\boldsymbol{h} = (r,n)$ be three distinct antiflags adjacent to $\boldsymbol{f}^\delta$ with either $p = q = q'$ and $m = n$, or $\ell = m = m'$ and $q = r$. Then $\boldsymbol{h} \sim \boldsymbol{g'}$.

(iii) Let $\boldsymbol{g} = (q,m)$ and $\boldsymbol{g'} = (q',m')$ be two distinct antiflags adjacent to $\boldsymbol{f}^\delta$, and $\boldsymbol{e} = (s,k)$ and $\boldsymbol{e'} = (s',k')$ be two flags with $\boldsymbol{f}^\delta, \boldsymbol{g} \sim \boldsymbol{e}^\varepsilon$ and $\boldsymbol{f}^\delta, \boldsymbol{g'} \sim \mathbf{e'}^{\varepsilon'}$ for some $\varepsilon, \varepsilon' \in \{+,-\}$. If either $p = q = q'$, $m = k$ and $m' = k'$, or $\ell = m = m'$, $q = s$ and $q' = s'$, then $s \neq s'$ and $k \neq k'$.

PROOF. Let $\boldsymbol{f} = (p,\ell)$ be a flag of F, and consider the local graph $\Gamma(\boldsymbol{f}^\delta) \cong H$. It contains the adjacent vertices p and ℓ, two antiflags $(p,m),(p,m')$ adjacent to p, and two antiflags $(q,\ell),(q',\ell)$ adjacent to ℓ. Let D_j^i be the set of vertices at distances i and j from p and ℓ in $\Gamma(\boldsymbol{f}^\delta)$. Then $D_2^1 = \{(p,m),(p,m')\}$, $D_1^2 = \{(q,\ell),(q',\ell)\}$ and $|D_3^2| = |D_2^3| = 4$ by $p_{23}^1 = 4$. These sets cover all vertices of $\Gamma(\boldsymbol{f}^\delta)$. By $b_2 = 2$ and $c_2 = 1$, each vertex in D_3^2 or D_2^3 has two neighbours in D_2^3 or D_3^2 and one neighbour in D_2^1 or D_1^2, respectively.

Since the vertices in $D_2^1 \cup D_1^2$ are all antiflags, it follows that $\Gamma(\boldsymbol{f}^\delta) \cap B \subseteq D_3^2 \cup D_2^3$. The graph induced on $D_3^2 \cup D_2^3$ is bipartite on 8 vertices of valency 2, and since $\Gamma(\boldsymbol{f}^\delta)$ is square-free by $c_2 = 1$, it follows that it must be an octagon. This proves (i).

Let $\boldsymbol{g}$, $\boldsymbol{g'}$ be the two antiflags in D_2^1 (resp. D_1^2), and $\boldsymbol{h}$ an antiflag in $\boldsymbol{f}^\delta$ corresponding to the same line (resp. point) as $\boldsymbol{g}$. Since $\boldsymbol{g}$ and $\boldsymbol{h}$ are nonadjacent by Lemma 1 and their $\mu = 2$ common neighbours are $\boldsymbol{f}^\delta$ and the common corresponding line (resp. point), it follows that they must be at distance 3 in $\Gamma(\boldsymbol{f}^\delta)$. The vertices at distance 3 from $\boldsymbol{g}$ are in $D_1^2 \cup D_3^2$ (resp. $D_2^1 \cup D_2^3$). As $r \neq p$ and $n \neq \ell$ by Lemma 1(iii), we have $\boldsymbol{h} \notin D_2^1 \cup D_1^2$. Therefore, $\boldsymbol{h}$ must be adjacent to $\boldsymbol{g'}$ and (ii) follows

Let $\boldsymbol{g}$, $\boldsymbol{g'}$ be the two antiflags in D_2^1 (resp. D_1^2), and $\boldsymbol{e}^\varepsilon$ and $\mathbf{e'}^{\varepsilon'}$ be two signed flags in D_3^2 (resp. D_2^3) adjacent to $\boldsymbol{g}$ and $\boldsymbol{g'}$, respectively, with each adjacent pair corresponding to the same point (resp. line). As $\Gamma(\boldsymbol{f}^\delta)$ is bipartite, $\boldsymbol{e}^\varepsilon$ and $\mathbf{e'}^{\varepsilon'}$ must be at even distance in the octagon induced on $D_3^2 \cup D_2^3$. But as $\Gamma(\boldsymbol{f}^\delta)$ has girth 6 and $c_3 = 3$, two vertices at distance 4 in this octagon must have a common neighbour in $D_2^1 \cup D_1^2$. Therefore, $\boldsymbol{e}^\varepsilon$ and $\mathbf{e'}^{\varepsilon'}$ are at distance 2 in the octagon. They are nonadjacent, with their $\mu = 2$ common neighbours being $\boldsymbol{f}^\delta$ and an antiflag or signed flag in the octagon. It follows that $\boldsymbol{e}^\varepsilon$ and $\mathbf{e'}^{\varepsilon'}$ cannot correspond to the same point or line, so (iii) holds. □

LEMMA 4. *The following hold.*

(i) Let $\boldsymbol{e} = (p,\ell)$ and $\boldsymbol{f} = (q,m)$ be flags and $\boldsymbol{g} = (r,n)$ be an antiflag such that $\boldsymbol{e}^\varepsilon \sim \boldsymbol{g} \sim \boldsymbol{f}^\delta$ for some $\delta, \varepsilon \in \{+,-\}$. If $p = r$ or $\ell = n$, and $q = r$ or $m = n$, then $\Gamma(\boldsymbol{e}^\varepsilon) \cap \Gamma(\boldsymbol{f}^\delta) \cap \Gamma(\boldsymbol{g}) \cap A = \emptyset$.

(ii) Each antiflag $\boldsymbol{g} \in A$ has 3 neighbours in A and 9 neighbours in B.

(iii) *Each signed flag $\boldsymbol{f}^\delta \in B$ has 6 neighbours in A and 6 neighbours in B, of which no two correspond to the same point or line.*

(iv) *Let $\boldsymbol{g} = (r, n)$ and $\boldsymbol{h} = (s, k)$ be two adjacent antiflags. Then there are flags $\boldsymbol{e} = (p, \ell)$ with $p = r$ or $\ell = n$ and $\boldsymbol{f} = (q, m)$ with $q = s$ or $m = k$ such that $\boldsymbol{g}$ and $\boldsymbol{h}$ are adjacent to both $\boldsymbol{e}^\varepsilon$ and $\boldsymbol{f}^\delta$ for some $\delta, \varepsilon \in \{+, -\}$.*

Proof. Let $\boldsymbol{e} = (p, \ell)$ and $\boldsymbol{f} = (q, m)$ be flags and $\boldsymbol{g} = (r, n)$ be an antiflag such that $\boldsymbol{e}^\varepsilon \sim \boldsymbol{g} \sim \boldsymbol{f}^\delta$ for some $\delta, \varepsilon \in \{+, -\}$. If $p = q = r$ (resp. $\ell = m = n$), the common neighbours of $\boldsymbol{e}^\varepsilon$ and $\boldsymbol{f}^\delta$ are p (resp. ℓ) and $\boldsymbol{g}$ – in particular, no antiflag adjacent to $\boldsymbol{g}$ is among them. If $p = r \neq q$ and $m = n \neq \ell$, then $\boldsymbol{e}^\varepsilon$ and $\boldsymbol{f}^\delta$ are at odd distance in $\Gamma(\boldsymbol{g})$, so they do not share a common neighbour with $\boldsymbol{g}$. This proves (i).

Consider two distinct signed flags $(p, \ell)^\delta$ and $(p, m)^\varepsilon$. By Lemma 1(i), they are nonadjacent and their $\mu = 2$ common neighbours are p and an antiflag corresponding to p (or ℓ if $\ell = m$ and $\delta = -\varepsilon$). Therefore, a signed flag can be adjacent to at most one signed flag corresponding to a chosen point (and by a similar argument, a chosen line), i.e., a signed flag can have at most 6 neighbours in B and therefore at least 6 neighbours in A. Thus, there are at least $42 \cdot 6 = 252$ edges between A and B.

Now consider the local graph $\Gamma(\boldsymbol{f}^\delta)$. By Lemma 3(i), the at most 6 adjacent signed flags of $\boldsymbol{f}^\delta$ are on an octagon, and by Lemma 1(i), they are not adjacent to q and m, which are also in the local graph. The remaining vertices must then be antiflags. Consider only the four of them that are adjacent to q or m. There are at least 2 antiflags in $\Gamma(\boldsymbol{f}^\delta)$ adjacent to one of them. By Lemma 1(iv) and picking all possibilities for $\boldsymbol{f}^\delta$, each antiflag occurs six times in this position (for signed flags corresponding to the same point or line), and in each local graph $\Gamma(\boldsymbol{f}^\delta)$ it has distinct adjacent antiflags by (i). Therefore, there are at least $42 \cdot 2 = 84$ ordered pairs of adjacent antiflags, so the average valency on the graph induced on A is at least $84/28 = 3$. Thus, there are at most $28 \cdot (12 - 3) = 252$ edges between A and B.

We have thus established that there are precisely 252 edges between A and B, i.e., on average, an antiflag $\boldsymbol{g} \in A$ has 3 neighbours in A and 9 neighbours in B, while each signed flag $\boldsymbol{f}^\delta \in B$ has precisely 6 neighbours in each of A and B – this proves (iii). Consider the quotient matrix M of the partition of the vertices of Γ into $\{\infty\}$, $\Gamma(\infty)$, A and B

$$M = \begin{pmatrix} 0 & 14 & 0 & 0 \\ 1 & 3 & 4 & 6 \\ 0 & 2 & 3 & 9 \\ 0 & 2 & 6 & 6 \end{pmatrix}$$

with average valencies between parts as entries. Its spectrum is $14^1\ 4^1\ {-3}^2$, which interlaces the spectrum of Γ tightly. The aforementioned partition is therefore equitable (see [4]). Hence, (ii) holds.

Let $\boldsymbol{g} = (r, n)$ be an antiflag. The above argument shows that each edge between $\boldsymbol{g}$ and another antiflag lies in the local graph $\Gamma(\boldsymbol{e}^\varepsilon)$ for precisely one choice of $\boldsymbol{e} = (p, \ell)$ with $p = r$ or $\ell = n$, and $\varepsilon \in \{+, -\}$. Let $\boldsymbol{h} = (s, k)$ be an antiflag that is adjacent to $\boldsymbol{g}$ and one of the aforementioned signed flags $\boldsymbol{e}^\varepsilon$. By applying the same argument to $\boldsymbol{h}$, we see that there is precisely one flag $\boldsymbol{f} = (q, m)$ with $q = s$ or $m = k$ such that $\boldsymbol{f}^\delta$ is adjacent to both $\boldsymbol{g}$ and $\boldsymbol{h}$ for some $\delta \in \{+, -\}$. This proves (iv). □

We are now ready to prove the main result.

Theorem 1. *No strongly regular graph is locally Heawood.*

Proof. We have $k = 14$ and $\lambda = 3$, therefore $\mu \mid k(k - 1 - \lambda) = 140$ and hence $\mu \in \{2, 4, 5, 7, 10\}$, i.e., $n \in \{85, 50, 43, 35, 29\}$. By examining the tables by Brouwer [2], we see that only the first possibility is feasible. We therefore continue with the labels introduced above.

Consider an antiflag $\boldsymbol{g} = (p, \ell)$ and a point $q \neq p$ such that $q \notin \ell$ (resp. a line $m \neq \ell$ such that $p \notin m$). They are not adjacent, and no point or line or ∞ is among their common neighbours, which must then be two antiflags or signed flags corresponding to the point q (resp. line m). Since these are not adjacent to p or ℓ and they are distinct for the 3 possible choices of q (resp. m), it follows that the 6 neighbours of $\boldsymbol{g}$ not equal or adjacent to p or ℓ are, by Lemma 4(ii, iii), three antiflags and three signed flags, each one of which corresponds to one choice of q and one choice of m. Therefore, any antiflag (q, m) adjacent to $\boldsymbol{g}$ must have $p \notin m$ and $q \notin \ell$, and there are three such neighbours (actually, all three antiflags satisfying the condition), so the graph induced on the antiflags is the Coxeter graph with intersection array $\{3, 2, 2, 1; 1, 1, 1, 2\}$.

Since the Coxeter graph is triangle-free, the three common neighbours of two adjacent antiflags $\boldsymbol{g} = (p, \ell)$ and $\boldsymbol{h} = (q, m)$ are all signed flags: by Lemma 4(iv), one corresponding to p or ℓ and one corresponding to q or m, and by Lemma 4(i), another signed flag $\boldsymbol{f}^\delta$ for $\boldsymbol{f} = (r, n)$, $r \neq p, q$, $n \neq \ell, m$ and some $\delta \in \{+, -\}$. In the local graph $\Gamma(\boldsymbol{f}^\delta)$, the antiflags $\boldsymbol{g}$ and $\boldsymbol{h}$ are not adjacent to r or n, so they are adjacent to an antiflag $\boldsymbol{g'} = (r, \ell')$ and an antiflag $\boldsymbol{h'} = (q', n)$, thus forming a path $\boldsymbol{g'} \sim \boldsymbol{g} \sim \boldsymbol{h} \sim \boldsymbol{h'}$ or $\boldsymbol{g'} \sim \boldsymbol{h} \sim \boldsymbol{g} \sim \boldsymbol{h'}$. Since there are $28 \cdot 3/2 = 42$ edges in the Coxeter graph, it follows that such a path of four antiflags occurs in the local graphs at each of the $|B| = 42$ signed flags.

Consider a signed flag $\boldsymbol{f}^\delta$ for $\boldsymbol{f} = (r, n)$ and some $\delta \in \{+, -\}$, and a line ℓ' such that $r \notin \ell'$ (resp. a point q' such that $q' \notin n$). They are not adjacent, and by Lemma 4(iii), precisely one of their two common neighbours is a signed flag corresponding to ℓ' (resp. q'), so the second common neighbour is an antiflag $\boldsymbol{g'}$, also corresponding to ℓ' (resp. $\boldsymbol{h'}$ corresponding to q'). For a fixed choice of $\boldsymbol{f}^\delta$, Lemma 1(iii) implies that the antiflag $\boldsymbol{g'}$ is (r, ℓ') (resp. $\boldsymbol{h'} = (q', n)$) for precisely two of the four possible choices of ℓ' (resp. q'). Therefore, the two antiflags $\boldsymbol{g} = (p, \ell)$ and $\boldsymbol{h} = (q, m)$ adjacent to $\boldsymbol{f}^\delta$ with $r \neq p, q$, $n \neq \ell, m$ have $r \notin \ell, m$ and $p, q \notin n$. By the above argument, $\boldsymbol{g}$ and $\boldsymbol{h}$ are adjacent, so $p \notin m$ and $q \notin \ell$ also holds. As p, q, ℓ, m cover all points and lines of the Fano plane F, we are forced to conclude that r and n are not on the plane F, contradiction. Thus, the statement follows. □

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