# О двух подходах к классификации высших локальных полей

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**Аннотация**

Эта статья связывает классификацию Курихары о полных дискретных оценочных полях и теории устранения дикого ветвления Эппа.

Для любого полного дискретного поля оценки $K$ с произвольным полем вычетов простой характеристики можно определить некоторый численный инвариант $\Gamma(K)$, который лежит в основе классификации Курихары таких полей на 2 типа: поле $K$ имеет тип I тогда и только тогда, когда $\Gamma(K)$ положительно. Значение этого инварианта указывает, насколько далеко данное поле от стандартного, т. е. от поля, которое неразветвлено над его постоянным подполем $k$, которое является максимальным подполем с совершенным полем вычетов. (Стандартные 2-мерные локальные поля являются точными полями вида $k\{\{t\}\}$.)

Мы доказываем (при некотором мягком ограничении на $K$), что для смешанного характеристического 2-мерного локального поля типа I $K$ существует оценка снизу для $[l:k]$, где $l/k$ является расширением, таким что $lK$ является стандартным полем (существующим из-за теории Эпп); логарифм этой степени может быть оценен линейно в терминах $\Gamma(K)$ с коэффициентом, зависящим только от $e_{K/k}$.

*Ключевые слова:* Высшие локальные поля, дикое ветвление

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**Для цитирования:**

# On two approaches to classification of higher local fields[1]

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### Abstract

This article links Kurihara's classification of complete discrete valuation fields and Epp's theory of elimination of wild ramification.

For any complete discrete valuation field $K$ with arbitrary residue field of prime characteristic one can define a certain numerical invariant $\Gamma(K)$ which underlies Kurihara's classification of such fields into 2 types: the field $K$ is of Type I if and only if $\Gamma(K)$ is positive. The value of this invariant indicates how distant is the given field from a standard one, i.e., from a field which is unramified over its constant subfield $k$ which is the maximal subfield with perfect residue field. (Standard 2-dimensional local fields are exactly fields of the form $k\{\{t\}\}$.)

We prove (under some mild restriction on $K$) that for a Type I mixed characteristic 2-dimensional local field $K$ there exists an estimate from below for $[l:k]$ where $l/k$ is an extension such that $lK$ is a standard field (existing due to Epp's theory); the logarithm of this degree can be estimated linearly in terms of $\Gamma(K)$ with the coefficient depending only on $e_{K/k}$.

*Keywords:* Higher local fields, wild ramification

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## 1. Introduction

In the current paper we develop and compare two approaches to the classification of 2-dimensional local fields in the mixed characteristic case. Here a 2-dimensional local field is a complete discrete valuation field $K$ such that its residue field $\overline{K}$ has, in its turn, a structure of a complete discrete valuation field with perfect residue field of characteristic $p > 0$.

If $\operatorname{char} K = \operatorname{char} \overline{K}$, the field $K$ can be identified (non-canonically) with the field of formal Laurent series $\overline{K}((X))$. However, if $\operatorname{char} K = 0$ and $\operatorname{char} \overline{K} = p$, there is no explicit description and exhausting classification of such fields $K$. Here are some known results in this direction.

First of all, there is an important subclass of such fields $K$, so called standard fields. For any complete discrete valuation field $K$ with the residue field of characteristic $p > 0$, one can introduce its constant subfield $k$ which is a maximal subfield of $K$ with perfect residue field. It can be proved that in the mixed characteristic case such $k$ is unique. The field $K$ is said to be standard if $e_{K/k} = 1$, where $e_{K/k}$ is defined in 2.1.

This rather abstract definition working for any complete discrete valuation field with imperfect residue field, takes a very explicit form if $K$ is a 2-dimensional local field. Namely, if $K$ is standard and $k$ is its constant subfield, then

$$K \simeq k\{\{t\}\} = \Big\{ \sum_{i=-\infty}^{\infty} a_i t^i, \quad v(a_i) \gg -\infty, \quad v(a_i) \underset{i \to -\infty}{\longrightarrow} \infty \Big\};$$

conversely, if $K = k\{\{t\}\}$ for a (one-dimensional) local field $k$, then $K$ is standard, and $k$ is its constant subfield (see [8] or [14]). Note that in the very classical case, when the residue field of $\overline{K}$ is finite, $k$ can be constructed as the maximal algebraic extension of $\mathbb{Q}_p$ inside $K$.

Obviously, any 2-dimensional local field $K$ with local parameters $(\pi, t)$ is a finite totally ramified extension of its standard subfield $K_0 = k\{\{t\}\}$, where $k$ is the constant subfield of $K$, and $\pi$, $t$ are as in 2.2 A non-trivial result following from Epp's theorem on elimination of wild ramification (see [1], [13]) is that for any such $K$ there exists a constant (i. e. defined over $k$) finite extension $K'/K$ such that $K'$ is a standard field. In fact, there is a huge freedom in the choice of such $K'/K$, see [6]. However, the minimal degree $d_m(K)$ of such $K'/K$ can be arbitrarily large even in the simplest case $[K : K_0] = p$. Thus, $d_m(K)$ seems to be an interesting invariant in the classification of 2-dimensional local fields.

Another approach to classification of mixed characteristic complete discrete valuation fields was initiated by Kurihara in [7] to study Milnor $K$-groups (see [9] or [4]), These groups are applied in class field theory (see [10], [11], [4], [5]). Kurihara subdivides such fields into 2 types. For this, one considers any non-trivial relation $a \cdot d\pi + b \cdot dt$ in the module of differentials of the given field $K$ over its constant subfield $k$, where $(\pi, t)$ are any local parameters of $K$. The field $K$ belongs to Type I if $v_K(a) < v_K(b)$ and to Type II otherwise (see [7], corollary 1.2 and definition 1.3). In particular, all standard fields belong to Type I since $\pi$ can be chosen from $k$, and one can take $a = 1$, $b = 0$. Kurihara showed that the structure of extensions for the fields of Type I and Type II is very different. For example, $K$ has cyclic wild (resp. ferocious) $p$-extensions of any degree if and only if $K$ is of Type I (resp. Type II).

A refinement of this classification along with a number of new properties has been given in [2, 3]. It was suggested to consider values like

$$\Delta(\pi, t) = \frac{1}{e_K}\Big(v_K\big(\mathrm{d}\pi t_L\big) - v_K\big(\mathrm{d}t t_L\big)\Big),$$

where $t_L$ is a second local parameter in a certain standard field $L$ containing $K$, and the partial derivatives are used in the usual sense via identification $L = l\{\{t_L\}\}$. It is easy to see that

$$\Delta(\pi, t) = \frac{1}{e_K}\big(v_K(b) - v_K(a)\big),$$

so, the field $K$ is of Type I if and only if $\Delta(\pi, t) > 0$ for any choice of local parameters $\pi$, $t$. It can be shown that for the fields of Type I $\Delta(\pi, t)$ does not depend on the choice of $t$. For such fields, the value

$$\Gamma(K) = \sup_{v(\pi)=1} \Delta(\pi, t)$$

is an invariant of $K$ measuring resemblance between $K$ and standard fields. In particular, $\Gamma(K) = \infty$ if and only if $K$ is "almost standard": a certain unramified extension of $K$ is a standard field.

In this article we obtain a lower bound for $d_m(K)$ for a mixed characteristic 2-dimensional local field of Type I, in terms of $\Gamma(K)$ and ramification index of the field over its standard subfield. This is accomplished under a certain mild restriction on $K$ (Corollary 5.3.1).

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## 2. Notation and basic definitions

The following notation is used throughout the paper:

$p$ always denote a prime integer;

$v_p(x)$ is the $p$-adic exponent of an integer number $x$.

## 2.1. Discrete valuation fields

For a discrete valuation field $F$, we denote its valuation by $v_F$ and its residue field by $\overline{F}$. For any such $F$ it will be always assumed that char $\overline{F} = p > 0$. If char $\overline{F} = p > 0$, we put $e_F = v_F(p)$. An element $\pi_F$ such that $v_F(\pi_F) = 1$ is said to be a uniformizer or $F$.

Denote
$\mathcal{O}_F = \{x \in F \mid v_F(x) \geqslant 0\}$;
$U_F = \{x \in F \mid v_F(x) = 0\}$;
$U_F(n) = \{x \in F \mid v_F(x - 1) \geqslant n\}$ for $n \in \mathbb{N}$.

Let $L/F$ be an extension of valuation fields, $v_L$ be a valuation on $L$, and $v_L$ induces the valuation $w$ on $F$. We denote by $e_{L/F}$ the index of $w(F^*)$ in $v_L(L^*)$.

A finite extension $E/F$ of discrete valuation fields is said to be

• unramified, if $e_{E/F} = 1$, and $\overline{E}/\overline{F}$ is separable;

• tame, if $p \nmid e_{E/F}$, and $\overline{E}/\overline{F}$ is separable;

• ferocious, if $eE/F = 1$, and $\overline{E}/\overline{F}$ is purely inseparable;

• totally ramified, if $e_{E/F} = |E : F|$.

By $v_0$ we denote the valuation on any field normalized so that $v_0(p) = 1$.

For a Galois extension $L/F$ of degree $p$ we denote by $s(L/F)$ the (Swan) ramification number of any generator $\sigma$ of $\mathrm{Gal}(L/K)$:

$$s(L/F) = \inf_{x \in L^*} v_L(\sigma(x)x^{-1} - 1).$$

## 2.2. Two-dimensional local fields

Let $K$ be a two-dimensional local field; denote by $K^{(1)} = \overline{K}$ its first residue field, and by $K^{(0)} = \overline{K^{(1)}}$ its last residue field. It is always assumed in this article that char $K = 0$, char $\overline{K} = p > 0$, and $K^{(0)}$ is perfect.

Any two-dimensional mixed-characteristic local field K satisfies the conditions of 2.1. We will use the same notation, that is $e_K = v_K(p)$, $\mathcal{O}_K = \{x \in K \mid v_K(x) \geqslant 0\}$ and $v_0$ is such that $v_0(p) = 1$.

For the valuation of rank 2 on $K$ we use notation $\overline{v}_K = (v_{\overline{K}}, v_K) \colon K \to \mathbb{Z}^2$; here $\mathbb{Z}^2$ is linearly ordered as follows: $(a, b) < (c, d)$, if $b < d$ or $b = d$ and $a < c$.

Since $\theta \in K^{(0)}$ is a perfect subfield in $K^{(1)} = \overline{K}$, for $\theta \in K^{(0)}$, its Teichmüller representative in $\mathcal{O}_K$ is well defined. We denote it by $[\theta]$.

Given $\overline{v}_K$, we can define local parameters: a uniformizer $\pi$ with $\overline{v}_K(\pi) = (0, 1)$, and a "second local parameter"$t$ with $\overline{v}_K(t) = (1, 0)$.

The *constant subfield* of $K$ is its maximal subfield such that its residue field (with respect to $v_K$) is perfect. In particular, if the last residue field of $K$ is finite, the constant subfield of $K$ is the algebraic closure of $\mathbb{Q}_p$ in $K$.

In what follows $K$ denotes always a two-dimensional local field, and $k$ is its constant subfield. The field $K$ is said to be *standard*, if $e_{K/k} = 1$.

A finite extension $L/K$ is said to be *constant* if $L = lK$ where $l$ is an algebraic extension of $k$.

## 2.3. Kurihara's classification and related invariants

Let $K_0 = k\{\{t\}\}$ be a standard 2-dimensional field. For $x \in K_0$ its formal derivative $\frac{\partial x}{\partial t}$ is defined as follows. If $x = \sum a_i t^i$ with $a_i \in k$, then

$$\frac{\partial x}{\partial t} = \sum i a_i t^{i-1}.$$

It is easy to see that $\frac{\partial x}{\partial t}$ is a well defined element of $K_0$.

Let $K_0$ and $L_0$ be standard fields with $K_0 \subset L_0$, and let $t, t'$ be second local parameters of these fields. Then

$$\frac{\partial x}{\partial t'} = \frac{\partial x}{\partial t}\frac{\partial t}{\partial t'},$$

where the first factor in the right hand side is the image in $L_0$ of the respective element of $K_0$.

Let $K_0$ be a standard field, $t$ a second local parameter of $K_0$, and $a, b \in K_0^*$. Introduce

$$c(a, b) = v_0\Big(\frac{\partial a}{\partial t_0}\Big) - v_0\Big(\frac{\partial b}{\partial t_0}\Big) - v_0(a) + v_0(b).$$

Now we check that $c(a, b)$ is independent of the choice of $K_0$ and the second local parameter $t_0$. Let $K_1$ and $K_2$ be standard fields with the second local parameters $t_1$ and $t_2$, and let $c_1(a, b)$ and $c_2(a, b)$ be functions corresponding to these fields. There exists a standard field $E$ containing both $K_1$ and $K_2$. Let $t_E$ be any second local parameter of $E$. We have

$$v_0\Big(\frac{\partial x}{\partial t_i}\Big) = v_0\Big(\frac{\partial x}{\partial t_i}\Big) + v_0\Big(\frac{\partial t_i}{\partial t_L}\Big), \quad i = 1, 2;$$

therefore,

$$c_1(a, b) - c_2(a, b) = v_0\Big(\frac{\partial a}{\partial t_1}\Big) - v_0\Big(\frac{\partial b}{\partial t_1}\Big) - v_0\Big(\frac{\partial a}{\partial t_2}\Big) + v_0\Big(\frac{\partial b}{\partial t_2}\Big) =$$
$$= v_0\Big(\frac{\partial a}{\partial t_E}\Big) - v_0\Big(\frac{\partial b}{\partial t_E}\Big) - v_0\Big(\frac{\partial a}{\partial t_E}\Big) + v_0\Big(\frac{\partial b}{\partial t_E}\Big) = 0.$$

Note that for any $x, y, z$ we have

$$c(x, y) = c(x, z) - c(y, z), \qquad c(x, y) = -c(y, x).$$

In [2, 3] the notation $\Delta_K(\pi, t)$ was used for $v_K\big(\mathrm{d}\pi t_L\big) - v_K\big(\mathrm{d}t t_L\big)$, where $\pi$, $t$ are local parameters of $K$, and $t_L$ is a second local parameter of a standard field $L$ which is a finite extension of $K$. In this article we redefine $\Delta_K(\pi, t)$ using $v_0$ instead of $v_K$, i. e.,

$$\Delta(\pi, t) = \Delta_K(\pi, t) = v_0\Big(\mathrm{d}\pi t_L\Big) - v_0\Big(\mathrm{d}t t_L\Big).$$

It is shown in [7, 2] that if the condition $\Delta_K(\pi, t) > 0$ is satisfied for some local parameters $\pi$ and $t$ of $K$, then it is satisfied for any pair of local parameters. A field $K$ is of Type I if this condition is satisfied and $K$ is of Type II otherwise (see [2], proposition 4.3). For a field of Type I, $\Delta(\pi, t)$ is independent of the choice of the second local parameter $t$ (see [2], Cor. 4.4); its value will be denoted by $\Delta_K(\pi)$. Note that

$$\Delta_K(\pi, t) = c(\pi, t) + v_0(\pi) - v_0(t) = c(\pi, t) + \frac{1}{e_K}.$$

For a field $K$ of Type I, denote

$$\Gamma(K) = \max(\Delta_K(\pi)|\pi \in K^*, v_K(\pi) = 1),$$

$$\Gamma_c(K) = \max(\Delta_K(\pi)|\pi \in K^*, v_K(\pi) = 1) - \frac{1}{e_K}.$$

Then for any second local parameter $t$ of $K$ we have

$$\Gamma_c(K) = \max(c(\pi, t)|\pi \in K^*, v_K(\pi) = 1).$$

# 3. Properties of $c(a)$

**3.1 Proposition.** *Let $a, b \in K$. Then:*
   *1. $\min\{c(ab, a), c(ab, b)\} \geqslant 0$.*
   *2. $c(a^{-1}, a) = 0$.*
   *3. $c(a^p, a) = 1$.*

**Proof.** Direct calculation.

**3.2 Lemma.** *Let $K = k\{\{t\}\}$ be standard, and let $\pi_k$ be a uniformizing element of $k$. Then any $a \in K$ can be represented (non-uniquely) as*

$$a = a_\infty + \sum_{r=0}^{N} \pi_k^{\alpha_r} f_r, \tag{1}$$

*where $a_\infty \in k$, $N \geqslant 0$, $\alpha_r \in \mathbb{Z}$, and for each $r$ either $f_r = 0$ or*

$$f_r = \sum_{i \in \mathbb{Z}} [\theta_{r,i}] t^{p^r i},$$

*$\theta_{r,i} \in K^{(0)}$, exists $i$ such that $\theta_{r,i} \neq 0$, $p \nmid i$.*
   *For any such representation we have*

$$v_0\Big(\frac{\partial a}{\partial t}\Big) = \min_{r \,:\, f_r \neq 0} (\alpha_r e_K^{-1} + r).$$

**Proof.** See [2, Lemma 4.5].

**3.3 Proposition.** *Let $K$ be of Type I. Let $a \in \mathcal{O}_K$; assume*

$$a \equiv \pi^m f \bmod \pi^{m+1} O_K, \tag{2}$$

$$f = \sum_{i \in \mathbb{Z}} [\theta_i] t^i,$$

*$\theta_i \in K^{(0)}$, exists $i$ such that $\theta_i \neq 0$. Then*

$$\min\{c(\pi^m f, a), c(\pi^m f, \pi)\} \geqslant 0.$$

**Proof.** It is sufficient to prove that

$$c(\pi^m f, t) \geqslant \min(c(a, t), c(\pi, t)).$$

Let $L$ be a standard field, $L \supset K$, and let $t_L$ be a second local parameter of $L$. For any $x \in L$ let

$$d(x) = v_0\Big(\mathrm{d}x t_L\Big).$$

We have

$$d(a) = c(a, t) + v_0(a) - d(t) \geqslant M + m e_K^{-1} - d(t)$$

with $M = \min(c(a, t), c(\pi, t))$. Note that the value of $d$ for each term in the expansion (1) for $a$ cannot be less than $d(a)$; it follows

$$\overline{\pi_L^{-m e_{L/K}} a} \in \overline{k}\big( (\overline{t}^{\,p^{M-d(t)}}) \big). \tag{3}$$

In particular,

$$\overline{\pi_L^{-e_{L/K}}\pi} \in \overline{k}\big((\overline{t}^{p^{M-d(t)}})\big). \tag{4}$$

Let

$$r = \min\{v_p(i) \mid \theta_i \neq 0\}.$$

Combining (2), (3) and (4), we conclude that $r \geqslant M - d(t)$. Terefore,

$$c(f,t) = c(f,t_L) + d(t) \geqslant r + d(t) \geqslant M.$$

Applying Lemma 3.1, we obtain

$$c(\pi^m f, t) \geqslant \min(c(\pi,t), c(f,t)) \geqslant M.$$

Let us say that $f \in k\{\{T\}\}$ is *normalized* if either $f \in U_k$, or $f \in \mathcal{O}_{k\{\{T\}\}}$ and $\overline{f} \notin \overline{k}((\overline{T}^p))$.
Let $\pi$, $t$ be any local parameters of $K$.

**3.3.1 Corollary.** *Let $u \in U_K$, where $K$ is of Type I. Then*

$$u = \prod_{i \geqslant 0}(1 + \pi^i f_i(t^{p^{n_i}})),$$

*where for any $i$ either $f_i = 0$ or $f_i$ is normalized and $n_i \geqslant 0$, and for any such representation we have*

$$\min\{c(1 + \pi^i f_i(t^{p^{n_i}}), u), c(1 + \pi^i f_i(t^{p^{n_i}}), \pi)\} \geqslant 0$$

*for any $i$.*

**Proof.** This follows from Propositions 3.3 and 3.1 by induction.

# 4. Behavior of $c(a)$ in field extensions

**4.1 Lemma.** *Let $K'/K$ be a finite extension of 2-dimensional local fields, and let $x_1$, $x_2 \in K'$ be conjugate over $K$. Then $c(x_1, x_2) = 0$.*

**Proof.** Let $L_1/K(x_1)$ be a finite extension such that $L_1 = l_1\{\{t_1\}\}$ is a standard field. Then there exists a field $L_2 \supset K(x_2)$ and an isomorphism $\tau : L_1 \to L_2$ over $K$ such that $\tau(x_1) = x_2$.

The field $l_1$ is exactly the set of elements of $L_1$ algebraic over $k$. Therefore, $l_2 = \tau(l_1)$ is the constant subfield of $L_2$.

For any $z \in L_1$, we have $\overline{v}_{L_2}(\tau(z)) = \overline{v}_{L_1}(z)$, since for any $L/K$ the valuation $\overline{v}_K$ has a unique extension to $L$. Therefore, $e_{L_1} = e_{L_2}$, $e_{L_2/l_2} = e_{L_1/l_1} = 1$, $L_2$ is standard, and $t_2 = \tau(t_1)$ is a second local parameter of $L_2$.

Next, for any $z \in L_1$ it follows from $\tau(l_1) = l_2$ and $\tau(t_1) = t_2$ that

$$\tau\Big(\frac{\partial z}{\partial t_1}\Big) = \frac{\partial(\tau(z))}{\partial t_2},$$

and

$$\overline{v}_{L_1}\Big(\frac{\partial z}{\partial t_1}\Big) = \overline{v}_{L_2}\Big(\frac{\partial(\tau(z))}{\partial t_2}\Big).$$

Since $e_{L_1} = e_{L_2}$, the same relation is true for $v_0$ instead of $v_{L_1}$ and $v_{L_2}$. Applying this to $z = x_1$, we obtain $c(x_1, x_2) = 0$.

**4.1.1 Corollary.** *Let $K'/K$ be a finite Galois extension. Then the for any $x \in K'$ we have $c(N_{K'/K}(x), x) \geqslant 0$.*

**Proof.** This follows from Lemma 4.1 and Proposition 3.1.

**4.2 Lemma.** *Let $K'/K$ be a finite totally ramified Galois extension, and let $K'$ be of Type I. Then $K$ is of Type I, and $\Gamma_c(K') \leqslant \Gamma_c(K)$.*

**Proof.** The field $K$ is of Type I by [7, Proposition 1.7].

Set $s = \Gamma_c(K')$, if $\Gamma_c(K')$ is finite, and denote an arbitrary number by $s$ otherwise. We claim that $\Gamma_c(K) \geqslant s$. Let $t'$ be a common second local parameter of $K$ and $K'$. Let $\pi_{K'}$ be a uniformizer of $K'$ such that $c(\pi_{K'}, t') \geqslant s$. Then $\pi_K = \mathrm{N}_{K'/K}(\pi_{K'})$ is a uniformizer of $K$. Applying Corollary 4.1.1, we obtain

$$\Gamma_c(K) \geqslant c(\pi_K, t') = c(\pi_{K'}, t') + c(\pi_K, \pi_{K'}) = s + c(\mathrm{N}_{K'/K}(\pi_{K'}), \pi_{K'}) \geqslant s.$$

**4.3 Lemma.** *Let $K'/K$ be a tame extension. Then $K$ and $K'$ are of the same type, and, if they are of Type I, then $\Gamma_c(K') = \Gamma_c(K)$.*

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Our definition of $\Gamma(K)$ is tailored for fields of Type I only, and we do not know how a parallel result for Type II case can look like.

**Proof.** The fields $K$ and $K'$ are of the same type by [7, Corollary 1.6].

Assume they are of Type I. Let $M/K$ be the maximal unramified subextension in $K'/K$. Then $M/K'$ is totally ramified. We will prove that $\Gamma(M) = \Gamma(K), \Gamma(K') = \Gamma(M)$. It is sufficient to check the inequalities:

$$\Gamma_c(K) \leqslant \Gamma_c(M) \leqslant \Gamma_c(K') \leqslant \Gamma_c(M) \leqslant \Gamma_c(K).$$

Denote by $t_K$ and $t_M$ arbitrary second local parameters of $K$ and $M$. Then $t_M$ is also a second local parameter of $K'$. We will prove that

$$c(t_K, t_M) = 0. \tag{5}$$

Let $L$ be any standard field containing $M$, and $t_L$ be its second local parameter. The extension $\overline{M}/\overline{K}$ is separable; therefore,

$$\bar{t}_K = \alpha_i \bar{t}_M^i,$$

where $\alpha_i \in \overline{M}$, and there exists $i$ such that $p \nmid i$, $\alpha_i \neq 0$. It follows

$$v_0\Big(\frac{\partial t_K}{\partial t_L}\Big) = v_0\Big(\frac{\partial t_M}{\partial t_L}\Big),$$

and so $c(t_K, t_M) = c(t_K, t_L) - c(t_M, t_L) = 0$.

1) We prove $\Gamma_c(K) \leqslant \Gamma_c(M)$. Denote $s = \Gamma_c(K)$ if $\Gamma_c(K)$ is finite, and let $s$ be arbitrary otherwise.

Let $\pi_K$ be a uniformizer of $K$ such that $c(\pi_K, t_K) \geqslant s$. Then $\pi_K$ is also a uniformizer of $M$. Using 5 we obtain

$$\Gamma_c(M) \geqslant c(\pi_K, t_M) = c(\pi_K, t_K) \geqslant s.$$

2) Now we prove $\Gamma_c(M) \leqslant \Gamma_c(K'), \Gamma_c(M) \leqslant \Gamma_c(K)$. In view of 5, it is sufficient to prove that for any uniformizer $\pi_M$ of $M$ there exist uniformizers $\pi_K$ and $\pi_{K'}$ of $K$ and $K'$ such that $c(\pi_K, \pi_M) \geqslant 0$ and $c(\pi_{K'}, \pi_M) \geqslant 0$. Let $E$ be either $K$ or $K'$. Denote

$$q = \begin{cases} |M : K|, & E = K \\ |K' : M|, & E = K' \end{cases},$$

$$x = \begin{cases} \mathrm{N}_{M/K}\,\pi_M, & E = K \\ \pi_M, & E = K' \end{cases}.$$

In both cases we have $x \in E$ and $v_0(x) = qe_E$.

Let $\pi_{E,1}$, $t_E$ be arbitrary local parameters of $E$, and let $\theta \in E^{(0)}$, $s_1 \in \mathbb{Z}$, $u \in U_E(1)$ be such that $\pi_{E,1}^q = [\theta]t_E^{s_1}ux$. Denote by $t$ a second local parameter of any standard field which is a finite extension of $K'$, and denote by $r$ any integer number with $v_0(s_1 - rq) \geqslant v_0\big(\frac{\partial x}{\partial t}\big)$; put $s = s_1 - rq$. We will prove that the uniformizer

$$\pi_E = t_E^{-r}u^{-1/q}\pi_{E,1}$$

is appropriate. Since $\pi_E^q = [\theta]t_E^s x$, we have

$$q\pi_E^{q-1}\frac{\partial \pi_E}{\partial t} = \frac{\partial(\pi_E^q)}{\partial t} = \frac{\partial\big([\theta]t_E^s x\big)}{\partial t} = [\theta]st_E^{s-1}x\frac{\partial t_E}{\partial t} + [\theta]t_E^s\frac{\partial x}{\partial t}.$$

Taking into account

$$v_0\Big([\theta]st_E^{s-1}x\frac{\partial t_E}{\partial t}\Big) \geqslant v_0(s) \geqslant v_0\Big(\frac{\partial x}{\partial t}\Big),$$

we obtain

$$v_0\Big(\frac{\partial \pi_E}{\partial t}\Big) + (q-1)e_E \geqslant v_0\Big(\frac{\partial x}{\partial t}\Big)$$

and

$$c(\pi_E, x) = \Big(v_0\Big(\frac{\partial \pi_E}{\partial t}\Big) - v_0(\pi_E)\Big) - \Big(v_0\Big(\frac{\partial x}{\partial t}\Big) - v_0(x)\Big) \geqslant 0.$$

In the case $E = K'$ we obtained the desired inequality, whereas in the case $E = K$ it is a consequence of the above formula and Corollary 4.1.1.

3) It remains to prove $\Gamma_c(K') \leqslant \Gamma_c(M)$. This follows from Lemma 4.2.

# 5. Estimate

We generalize the notion of "being not in touch" introduced in [15] in the prime characteristic case. Let $L_1/F$ and $L_2/F$ be totally ramified Galois extensions of degree $p$, and denote $s_1 = s(L_1/F)$, $s_2 = s(L_2/F)$. The extensions $L_1/F$ and $L_2/F$ are said to be not in touch if either $s_1 \neq s_2$ or $s(L/F) = s_1 = s_2$ for any subextension $L/F$ in $L_1L_2/F$ of degree $p$.

Next, finite totally ramified Galois $p$-extensions $L_1/F$ and $L_2/F$ are not in touch, if for any intermediate fields $F \subset S_i \subset T_i \subset L_i$, where $S_i/F$ is normal and $T_i/S_i$ is a Galois extension of degree $p$ $(i = 1, 2)$, the extensions $T_1S_2/S_1S_2$ and $S_1T_2/S_1S_2$ are not in touch.

The idea behind this notion is that we consider extensions "in general position" such that the ramification of their compositum can be computed in terms of ramification of the original extensions, compare [12, 4.3].

We say that an extension $K'/K$ is *constant free*, if $K'/K$ is not in touch with any constant extension of $K$. For example, for $K = k\{\{t\}\}$, where $k$ contains a primitive $p$th root of unity, a Kummer extension $K\big(\sqrt[p]{1 + \pi_k^n a}\big)/K$ with $a \in U_K$ is constant free iff $\overline{a} \notin \overline{k}$.

**5.1 Lemma.** *Let $L_1/K$ and $L_2/K$ be Galois extensions of degree $p$ that are not in touch. Assume that $L_1L_2/K$ is totally ramified. Then:*

    *1. $s(L_1L_2/L_2) \geqslant s(L_1/K)$;*

    *2. If $s(L_1/K) = \frac{pe_K}{p-1}$, then $s(L_1L_2/L_2) > s(L_1/K)$.*

**Proof.** The first part follows immediately from Lemma 3.3.1 in [12]. (It is assumed there that the residue field is perfect but the proof goes through assuming only that $L_1L_2/K$ is totally ramified.)

For the second part it is sufficient to notice that $s(L_2/K) < \frac{pe_K}{p-1}$. Indeed, if

$$s(L_1/K) = s(L_2/K) = \frac{pe_K}{p-1},$$

and $L_1L_2/K$ is totally ramified, then $L_1/K$ and $L_2/K$ are always in touch; this can be seen from the explicit form of Kummer equations (after adjoining a primitive $p$th root of unity).

**5.2 Lemma.** *Let $K$ contain a primitive $p$th root of unity. Let $K_1/K$ be a totally ramified extension of degree $p$; denote by $\pi_1$ any uniformizer of $K_1$. Let $u \in K$ be such that $K(\sqrt[p]{u})$ is not in touch with $K_1/K$ and either $v_K(u) = 1$ or $0 < v_K(u-1) < \frac{pe_K}{p-1}$; $p \nmid v_K(u-1)$. Assume that $c(u, t_L) \geqslant N$ and $c(\pi_1, t_L) \geqslant N$ for some integer $N \geqslant 3$ and for some standard field $L = l\{\{t_L\}\}$ containing $K_1$. Then $u = u_1 b^p$, where $u_1, b \in K_1$ are such that $p \nmid v_{K_1}(u_1 - 1)$ and $c(u_1, t_L) \geqslant N - 1$.*

**Proof.** Let $c(x) = c(x, t_L)$ and let $i_0 = v_K(u-1)$. By Corollary 3.3.1

$$u = \pi_1^{pv_K(u)} \prod_{i \geqslant pi_0} (1 + \pi_1^i f_i(t^{p^{n_i}}))$$

with $f_i$ normalized or $f_i = 0$, and $c(1 + \pi_1^i f_i(t^{p^{n_i}})) \geqslant N$. It follows for $f_i \neq 0$ that

$$c(1 + \pi_1^i f_i(t^{p^{n_i}})) \geqslant N$$

and by Lemma 3.2

$$n_i = c(f_i(t^{p^{n_i}})) \geqslant \min(c(\pi_1), c(\pi_1^i f_i(t^{p^{n_i}}))) \geqslant N - ie_{K_1}^{-1}.$$

We conclude that $n_i > 0$ for $i < Ne_{K_1}$.

Denote $i_1 = \min\{i : f_i \neq 0, p \nmid i\}$; we have $i_1 \leqslant pe_{K_1}/(p-1) < Ne_{K_1}$ since $N \geqslant 3$. Introduce

$$u_1 = \prod_{i \geqslant i_1} (1 + \pi_1^i f_i(t^{p^{n_i}})) \times \prod_{i_0 \leqslant i < \frac{i_1}{p}} \frac{1 + \pi_1^{pi} f_{pi}(t^{p^{n_{pi}}})}{\left(1 + \pi_1^i f_{pi}^{\phi^{-1}}(t^{p^{n_{pi}-1}})\right)^p}$$

$$= (1 + S_1)(1 + S_2).$$

Here $\phi$ denotes application of Frobenius automorphism to the coefficients of a power series from $k\{\{T\}\}$.

We see from Lemma 5.1 and [12] that

$$
\begin{aligned}
v(u_1 - 1) &\leqslant \frac{pe_{K_1}}{p-1} - s(K_1(\sqrt[p]{u_1})/K_1) \\
&\leqslant \frac{pe_{K_1}}{p-1} - s(K(\sqrt[p]{u_1})/K) \\
&= \frac{pe_{K_1}}{p-1} - \left(\frac{pe_K}{p-1} - i_0\right) = e_{K_1} + i_0.
\end{aligned}
\tag{6}
$$

Since $v_{K_1}(S_1) = i_1$ and $v_{K_1}(S_2) = e_{K_1} + i_0$, we obtain that the initial terms in $S_1$ and $S_2$ do not cancel, whence $v_{K_1}(S_1 + S_2)$ is exactly $\min(v_{K_1}(S_1), v_{K_1}(S_2))$. If $i_0 \neq 0$, this gives $p \nmid v(u_1 - 1)$, since both $i_1$ and $e_{K_1} + i_0$ are not divisible by $p$. If $i_0 = 0$, we still have $p \nmid v(u_1 - 1)$. Indeed, in this case we have a strict inequality in (6) by the second part of Lemma 5.1, whence $v(u_1 - 1) = i_1$.

Obviously, we have $u = u_1 b^p$ with

$$b = \pi_1^{v_K(u)} \prod_{i_0 \leqslant i < \frac{i_1}{p}} \left(1 + \pi_1^i f_{pi}^{\phi^{-1}}(t^{p^{n_{pi}-1}})\right)^{-1}.$$

It remains to estimate $c\big(1 + \pi_1^i f_{pi}^{\phi^{-1}}(t^{p^{n_{pi}-1}})\big)$. Denoting $f_{pi}^{\phi^{-1}}$ by $g$, by definition we have

$$
\begin{aligned}
c\big(1 + \pi_1^i g(t^{p^{n_{pi}-1}})\big) &\geqslant \min\Big(v_0\Big(\pi_1^{i-1} g(t^{p^{n_{pi}-1}}) \mathrm{d}\pi_1 t_L\Big), \\
&\qquad\qquad v_0\Big(p^{n_{pi}-1}\pi_1^i g'(t^{p^{n_{pi}-1}})t^{p^{n_{pi}-1}} \mathrm{d}t t_L\Big)\Big) \\
&\geqslant \min(ie_{K_1}^{-1} + c(\pi_1), (n_{pi}-1) + ie_{K_1}^{-1} + c(t)) \\
&\geqslant \min(ie_{K_1}^{-1} + N, N - pie_{K_1}^{-1} - 1 + ie_{K_1}^{-1} + c(t)) > N - 2.
\end{aligned}
$$

It follows $c(b) \geqslant N - 2$, and $c(u_1) \geqslant \min(c(u), c(b^{-p})) \geqslant N - 1$.

**5.3 Proposition.** *Let $K$ be of Type I, not almost standard, with $\Gamma_c(K) > n+3$, where $n = v_p(e_{K/k})$. Assume that $K/k\{\{t\}\}$ is constant free (for some choice of $t$). Let $K'/K$ be a constant extension of degree $p$. Then $K'$ is of Type I, not almost standard, and $\Gamma_c(K') \geqslant \Gamma_c(K) - n - 3$.*

**Proof.** Let $K' = k'K$, where $k'/k$ is an extension of degree $p$.

By Lemma 4.3 the proof is reduced to the case of cyclic totally ramified $k'/k$ and $K'/K$. We have $K' = K(x_0)$, where $x_0^p = a \in k^*$.

Consider a chain of subfields

$$
K_0 \subset K_1 \subset \cdots \subset K_n = K,
$$

where each $K_{i+1}/K_i$ is totally ramified of degree $p$, and $K_0/k\{\{t\}\}$ is tame.

Denote by $L$ any standard field containing $K'$; put $c(x) = c_L(x, t_L)$.

Let $\pi_n = \pi$ be a uniformizer of $K$ with $c(\pi) = N$, where $N = \Gamma_c(K)$, and let $\pi_i = N_{K/K_i}\pi$; we have $c(\pi_i) \geqslant N$, $0 \leqslant i \leqslant n$ (by Corollary 4.1.1).

Applying Lemma 5.2 to $K_1/K_0, \ldots, K_n/K_{n-1}$, we obtain that $K' = K(x)$, $x^p = u \in K$, $p \nmid v(u-1)$, $c(u) \geqslant N - n$. In particular, $K'/K$ is totally ramified, whence $K'$ is not almost standard. We have

$$
\begin{aligned}
c(x-1) &= c(x) + v_0(x) - v_0(x-1) = \\
&= c(u) - 1 - v_0(x-1) \geqslant N - n - 1 - \frac{p}{p-1} \geqslant N - n - 3.
\end{aligned}
$$

Pick integers $i$ and $j$ such that $v_{K'}(\pi') = 1$, where $\pi' = (x-1)^i \pi^j$. Then

$$
c(\pi') \geqslant \min(c(x-1), c(\pi)) \geqslant N - n - 3,
$$

and this proves that $K'$ is of Type I with

$$
\Gamma_c(K') \geqslant c(\pi') \geqslant \Gamma_c(K) - n - 3.
$$

**5.3.1 Corollary.** *Let $K$ be as in Proposition 5.3, with $\Gamma(K) > m(n+3)$, where $n = v_p(e_{K/k})$, $m$ a positive integer. Assume that $l/k$ is an extension such that $lK$ is almost standard. Then the inequality $[l : k] \geqslant p^m$ holds.*

## Afterword

Thus, we have established, under some restrictions, a relation between two invariants measuring how far is a given 2-dimensional local field from being standard. We expect that this relation, in some refined form, can be extended to all higher local fields.

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