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Точные оценки для специального класса целочисленных многочленов с заданным дискриминантом

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Аннотация

В статье получена верхняя и нижняя оценка для числа целочисленных многочленов, которые имеют только два близких корня и малый дискриминант в терминах Евклидовой метрики.

Ключевые слова: диофантовы приближения, дискриминант многочлена.

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Exact bounds for the special class of integer polynomials with given discriminant

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Abstract

An upper bound and lower bound for the number of integer polynomials which have only two close to each other roots, and small discriminant in terms of the Euclidean metric is obtained.

Keywords: Diophantine approximation, discriminant of polynomial.

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1. Introduction

Denote by $\mathcal{P}_n$ the class of integer polynomials P of degree n. For a given polynomial $P = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in \mathcal{P}_n$ define $H(P) = \max_{0 \leqslant j \leqslant n} |a_j|$ to be the height of P.

Given a parameter $Q \in \mathbb{N}_{>1}$, let

$$\mathcal{P}_n(Q) = \{P(x) \in \mathbb{Z}[x], \deg P = n,\ H(P) \leqslant Q\}$$

denote the set of integer polynomials P of degree n and height $H(P) \leqslant Q$. Throughout, $D(P)$ will stand for the discriminant of a polynomial P which is defined by

$$D(P) = a_n^{2n-2} \prod_{1 \leqslant i < j \leqslant n} (\alpha_i - \alpha_j)^2$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ are the roots of P (see [15]). The discriminant contains the information regarding the distance between different algebraic numbers, and it was important tool for the solving Mahler's conjecture by Sprindzuk [14] as well for a various generalisations of Sprindzuk's techniques [3, 4, 5, 8, 11].

In what follows we will use the Vinogradov symbols $\ll$ (and $\gg$) where $a \ll b$ implies that there exists a constant C such that $a \leqslant Cb$. If $a \ll b \ll a$ then we write $a \asymp b$. The cardinality of a set B will be denoted by $\#B$. Positive constants which depend only on n will be denoted by $c(n)$; where necessary these constants will be numbered $c_j(n)$, $j = 1, 2, \ldots$.

Given $v \in \mathbb{R}_{\geqslant 0}$, define the subset of $\mathcal{P}_n(Q)$ as follows:

$$\mathcal{P}_n(Q, v) = \{P(x) \in \mathcal{P}_n(Q) :\ 1 \leqslant |D(P)| < Q^{2n-2-2v}\}.$$

Establishing exact upper bounds and lower bounds for $\#\mathcal{P}_n(Q, v)$ have been the subject of numerous papers in recent years, and became a new branch of Diophantine approximation. We now briefly recall the results that have been obtained to date. In the case of quadratic polynomials it was shown in [13] that $\#\mathcal{P}_2(Q, v) \asymp Q^{3-2v}$ for $0 < v < 3/4$ and in the case of cubic polynomials it was established in [12] that $\#\mathcal{P}_3(Q, v) \asymp Q^{4-5v/3}$ for $0 \leqslant v < 3/5$. Establishing the lower bounds for an arbitrary n has been the subject of numerous papers including [1, 2, 6]. The most general and best estimate for the lower bound with arbitrary n was found in [2] where it was shown that $\#\mathcal{P}_n(Q, v) \gg Q^{n+1-(n+2)v/n}$, $0 \leqslant v \leqslant n - 1$. It is much harder to get the upper bounds for $\#\mathcal{P}_n(Q, v)$ with arbitrary n.

There are also p-adic [7] and mixed analogues [9, 10] of the above problem, which along with the size of the discriminant take into account their arithmetic structure.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ be the roots of $P \in \mathcal{P}_n(Q, v)$ ordered so that

$$|\alpha_1 - \alpha_2| \leqslant |\alpha_1 - \alpha_3| \leqslant \ldots \leqslant |\alpha_1 - \alpha_n|, \tag{1}$$

and satisfy

$$|\alpha_1 - \alpha_j| \ll 1,\ \ 3 \leqslant j \leqslant n. \tag{2}$$

Also, define the real number ρ such that

$$|\alpha_1 - \alpha_2| = Q^{-\rho},\ \ \rho \geqslant 0. \tag{3}$$

Let $\mathcal{P}'_n(Q, v)$ denote the set of irreducible polynomials $P \in \mathcal{P}_n(Q, v)$ which have only one root α_2 close to α_1. Thus, we investigate the set of irreducible polynomials $P \in \mathcal{P}_n(Q, v)$ with the roots satisfy (1)–(3). In this paper we obtain an upper bound and lower bound for the number of polynomials $P \in \mathcal{P}'_n(Q, v)$.

Теорема 1. *Let $n \in \mathbb{N}$, $n \geqslant 2$ be given. For any $\epsilon > 0$ and for any sufficiently large Q the following estimates*

$$\begin{array}{ll}\#\mathcal{P}'_n(Q,v) \gg Q^{n+1-2v} & \\ \#\mathcal{P}'_n(Q,v) \ll Q^{n+1-2v} & \text{for} \quad \rho \geqslant \frac{n-1+2v}{3}+\epsilon\end{array}$$

hold if

$$0 \leqslant v \leqslant (n+2)/4 - \epsilon,$$

where the constant implied by the Vinogradov symbol depends on n only.

2. Lower Bound

2.1. Preliminary results

Now we give several lemmas which are used to obtain lower bounds for the Lebesgue measure of certain sets. In what follows given a Lebesgue measurable set $A \subset \mathbb{R}$, $|A|$ stand for its Lebesgue measure.

Lemma 1. *[2] Let $n \geqslant 2$ and $v_0, v_1, \dots, v_n$ be a collection of real numbers such that*

$$v_0 + v_1 + \ldots + v_n = 0 \ \text{ and } \ v_0 \geqslant v_1 \geqslant \ldots \geqslant v_n \geqslant -1. \tag{4}$$

Then there are positive constants δ_0 and c_0 depending on n only with the following property. For any interval $J \subset [-1/2, 1/2]$ there is a sufficiently large Q_0 such that for all $Q > Q_0$ there is a measurable set $G_J \subset J$ satisfying $|G_J| \geqslant |J|/2$ such that for every $x \in G_J$ there are $n+1$ linearly independent primitive irreducible polynomials $P \in \mathbb{Z}[x]$ of degree exactly n such that

$$\delta_0 Q^{-v_0} \leqslant |P(x)| \leqslant c_0 Q^{-v_0}, \ \ \delta_0 Q^{-v_j} \leqslant |P^{(j)}(x)| \leqslant c_0 Q^{-v_j} \quad (1 \leqslant j \leqslant n). \tag{5}$$

Lemma 2. *[2] Let n and v_j be the same as in Lemma 1. Let*

$$d_j = v_{j-1} - v_j \ \ (1 \leqslant j \leqslant n). \tag{6}$$

Suppose that

$$d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n \geqslant 0 \tag{7}$$

and that for some $x \in \mathbb{C}$ and $Q > 1$ inequalities (5) are satisfied by some polynomial P over $\mathbb{C}$ of degree $\deg P = n$. Then there are roots $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ of P such that

$$|x - \alpha_j| \leqslant c_j Q^{-d_j} \ \ (1 \leqslant j \leqslant n) \tag{8}$$

where

$$c_1 = nc_0\delta_0^{-1} \text{ and } \ c_{j+1} = \max\left(\frac{2c_0 n!}{\delta_0 (j+1)!(n-j-1)!}, \frac{2c_j n!}{j!(n-j!)}\right) \ \ (1 \leqslant j \leqslant n-1).$$

2.2. Obtaining a lower bound in Theorem 1

Let $v_0, v_1, \dots, v_n$ be given and satisfy (4) and let the parameters d_j be given by (6) and (7). Consider the system

$$\delta_0 Q^{-v_0} \leqslant |P(x)| \leqslant c_0 Q^{-v_0}, \ \delta_0 Q^{-v_1} \leqslant |P'(x)| \leqslant c_0 Q^{-v_1} \delta_0 Q \leqslant |P^{(j)}(x)| \leqslant c_0 Q \ \ (2 \leqslant j \leqslant n). \tag{9}$$

Therefore, we have $v_2 = v_3 = \ldots = v_n = -1$. From (4), we have

$$v_0 + v_1 = -\sum_{i=2}^{n} v_i = n - 1. \tag{10}$$

Let $J=[-\frac{1}{2},\frac{1}{2}]$, Q be sufficiently large and $x\in G_J$, where G_J is the same as in Lemma 1. By Lemma 1, inequalities (9) are satisfied for some irreducible polynomial $P\in\mathbb{Z}[x]$ of degree n. Then by Lemma 2 we have (8). Hence, for any pair of integers (i,j) satisfying $1\leqslant i<j\leqslant n$ we have by (8) that

$$|\alpha_i-\alpha_j|\leqslant|x-\alpha_i|+|x-\alpha_j|\ll Q^{-d_j}. \tag{11}$$

By (9), we have that $H(P)\ll Q$, where the implicit constant depends on n only. Therefore, using $D(P)=a_n^{2n-2}\prod_{1\leqslant i<j\leqslant n}(\alpha_i-\alpha_j)^2$, we have that

$$1\leqslant|D(P)|\ll Q^{2n-2-2\sum_{i=2}^n(i-1)d_i}. \tag{12}$$

Note that the left hand side inequality is due to the irreducibility of P. By (12), inequalities $1\leqslant|D(P)|\ll Q^{2n-2-2v}$ are fulfilled if we impose the condition

$$\sum_{i=2}^n(i-1)d_i=v. \tag{13}$$

By (9), we have that

$$d_1=v_0-v_1,\ d_2=v_1+1,\ d_i=0,\,3\leqslant i\leqslant n. \tag{14}$$

Thus, from (13), we get $d_2=v$. By (10) and (14), we have that

$$v_1=v-1,\quad v_0=n-v. \tag{15}$$

Next we obtain that $d_1=v_0-v_1=n+1-2v$. A quick check shows that $d_1\geqslant d_2\geqslant 0$ for $0\leqslant v\leqslant\frac{n+1}{3}$. Using (3), (11), (14) and (15), it can be shown that $\rho\geqslant v$.

Now, we estimate the number of polynomials that can obtain this way. By (8) and Lemma 1, for every $x\in G_J$ we have that $|x-\alpha_1(P)|\ll Q^{-d_1}$, where P arises from Lemma 1. Therefore, we have that

$$G_J\subset\cup_{P\in\mathcal{P}'_n(Q,v)}\cup_{j=1}^n\{|x-\alpha_j(P)|\ll Q^{-d_1}\}.$$

Hence,

$$\frac{3}{4}=\frac{3}{4}|J|\ll Q^{-d_1}\#\mathcal{P}'_n(Q,v).$$

This results in the required lower bound

$$\#\mathcal{P}'_n(Q,v)\gg Q^{d_1}\gg Q^{n+1-2v}$$

for $0\leqslant v\leqslant\frac{n+1}{3}$.

3. Upper Bound

3.1. Auxiliary results

The following Lemma is a quantitative description of the fact that two relatively prime integer polynomials cannot both have very small absolute values in an interval.

Lemma 1. *Let $\delta,\eta,\mu\in\mathbb{R}^+$ and let $Q_0(\delta,n)$ be a sufficiently large real numbers. Furthermore, let $P(x),T(x)\in\mathbb{Z}[x]$ be polynomials of degree $n>1$ without common roots such that $\max(H(P),H(T))=Q^\mu$, where $Q>Q_0(\delta,n)$. Assume that the interval $I\subset(-n,n)\subset\mathbb{R}$ with $|I|=Q^{-\eta}$. If there exists $\tau>0$ such that for all $x\in I$*

$$\max(|P(x)|,|T(x)|)<Q^{-\tau},$$

then

$$\tau+\mu+2\max(\tau+\mu-\eta,0)<2\mu n+\delta. \tag{16}$$

Lemma 1 is proved in [3].

For a given number $\epsilon_1 > 0$ let $T = [\epsilon_1^{-1}] + 1$, where $[a]$ is the integer part of $a \in \mathbb{R}$. For a polynomial $P \in \mathcal{P}'_n(Q,v)$ the real number ρ was defined in (3). Also define the integer l by

$$(l-1)/T < \rho \leqslant l/T.$$

It is not difficult to show that the number of integers l is finite and depends only on ϵ_1 and does not depend on Q and $H(P)$. Define the class $\mathcal{P}'_{n,l}(Q,v)$ which consists of the polynomials $P \in \mathcal{P}'_n(Q,v)$ corresponding to an integer l.

In order for the polynomial $P(x)$ to belong to the class $\mathcal{P}'_n(Q,v)$ it is necessary and sufficient that the inequality

$$\rho \geqslant v$$

holds. By (2)–(3) and using the fact that $|D(P)| \geqslant 1$ for the irreducible polynomial P, we have

$$\rho \leqslant n-1.$$

3.2. Obtaining an upper bound in Theorem 1

In this section we are going to obtain the upper bound

$$\#\mathcal{P}'_n(Q,v) \ll Q^{n+1-2v} \tag{17}$$

for the number of polynomials $P \in \mathcal{P}_n(Q,v)$ with only two close roots α_1 and α_2.

Assume that the estimate (17) does not hold, so

$$\#\mathcal{P}'_n(Q,v) \gg Q^{n+1-2v}.$$

Then there must exist an interval $I \subset \mathbb{R}$ of size $|I| = Q^{-\gamma}$, $\gamma \geqslant 0$, containing a root of P such that

$$\#\mathcal{P}'_n(Q,v,I) \gg Q^{n+1-2v-\gamma}$$

where $\mathcal{P}'_n(Q,v,I)$ is the set of polynomials $P \in \mathcal{P}'_n(Q,v)$ which have a root in the interval I. Since $\#l \ll 1$ there exist an integer l such that

$$\#\mathcal{P}'_{n,l}(Q,v,I) \gg Q^{n+1-2v-\gamma}$$

where $\mathcal{P}'_{n,l}(Q,v,I)$ denotes the subset of $\mathcal{P}'_{n,l}(Q,v)$ which have a root in the interval I.

Expand the polynomial $P \in \mathcal{P}'_{n,l}(Q,v,I)$ into its Taylor series in the neighbourhood of α_1 to obtain

$$P(x) = P(\alpha_1) + \sum_{i=1}^{n} \frac{P^{(i)}(\alpha_1)(x-\alpha_1)^i}{i!}.$$

Using (1)–(3) and estimating each term, gives

$$\begin{array}{lcl} |P'(\alpha_1)(x-\alpha_1)| & \ll & Q^{1-\rho-\gamma}, \\ |P^{(i)}(\alpha_1)(x-\alpha_1)^i| & \ll & Q^{1-i\gamma},\ \ 2 \leqslant i \leqslant n, \end{array}$$

for $x \in I$. Thus

$$\begin{array}{ll} |P(x)| \ll Q^{1-2\gamma} & if \quad \gamma \leqslant \rho \\ |P(x)| \ll Q^{1-\rho-\gamma} & if \quad \gamma \geqslant \rho \end{array} \tag{18}$$

for $x \in I$.

For sufficiently large Q, we have that $\#\mathcal{P}'_{n,l}(Q,v,I) \geqslant 2$ for $\gamma \leqslant n+1-2v$. Next we show that the assumption that at least two irreducible polynomials without common roots have small values in the interval I will lead to a contradiction. Suppose that two polynomials $P_1, P_2 \in \mathcal{P}'_{n,l}(Q, v)$ without common roots belong to I, i.e. $\alpha_1(P_1) \in I$ and $\alpha_1(P_2) \in I$. Then the estimates (18) hold for P_i, $i = 1, 2$, on the interval I.

Consider the following three cases.

Case A: $\frac{n-1+2v}{3} + \epsilon \leqslant \rho \leqslant n/2 + 3\epsilon/4$.

Choose $\gamma = 2n - 3\rho + 3\epsilon$. Applying Lemma 1 to polynomials P_1 and P_2 with $\tau = -1 + \rho + \gamma$, $\eta = \gamma$, $\mu = 1$, leads to a contradiction in (16) for $\delta \leqslant 3\epsilon$.

Case B: $n/2 + 3\epsilon/4 < \rho < n + 1 - 2v \leqslant n - 1$ **for** $v \geqslant 1$ **or** $n/2 + 3\epsilon/4 < \rho \leqslant n - 1$ **for** $v \leqslant 1$.

Choose $\gamma = \rho$. Applying Lemma 1 to polynomials P_1 and P_2 with $\tau = -1 + 2\gamma$, $\eta = \gamma$, $\mu = 1$, leads to a contradiction in (16) for $\delta \leqslant 3\epsilon$.

Case C: $n + 1 - 2v \leqslant \rho \leqslant n - 1$ **for** $v \geqslant 1$.

Choose $\gamma = n + 1 - 2v$. Applying Lemma 1 to polynomials P_1 and P_2 with $\tau = -1 + 2\gamma$, $\eta = \gamma$, $\mu = 1$, leads to a contradiction in (16) for $\delta \leqslant 8\epsilon$ and $v \leqslant \frac{n+2}{4} - \epsilon$.

Thus it has been shown that $\#\mathcal{P}'_n(Q, v) \ll Q^{n+1-2v}$ for $\rho \geqslant \frac{n-1+2v}{3} + \epsilon$ and $v \leqslant \frac{n+2}{4} - \epsilon$.

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