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p-адические L-функции и p-адические кратные дзета значения

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Аннотация

Статья посвящена памяти Георгия Вороного. Описываются новые избранные результаты о рядах Эйзенштейна, о (мотивных), ($p-$адических), (кратных) значениях (круговых) дзета и $L-$функций, и их приложения, полученные ниже перечисляемыми авторами, а также элементарное введение в эти результаты. Дан краткий обзор новых результатов о (мотивных), ($p-$адических), (кратных) значениях (круговых) дзета функциях, $L-$функциях и рядах Эйзенштейна. Статья ориентирована на избранные задачи и не является исчерпывающей. Начало статьи содержит краткое изложение результатов о числах Бернулли, связанных с исследованиями Георгия Вороного. Результаты о кратных значениях дзета функций были представлены Д. Загиром, П. Делинем и А. Гончаровым, А. Гончаровым, Ф. Брауном, К. Глэносом (Glanois) и другими. С. Унвер ("Unver) исследовал кратные p-адические дзета-значения глубины два. Таннакиева интерпретация кратных p-адических дзета-значений дана Х. Фурушо. Краткая история и связи между группами Галуа, фундаментальными группами, мотивами и арифметическими функциями представлены в докладе Ю. Ихара. Результаты о кратных дзета-значениях, группах Галуа и геометрии модулярных многообразий представлены Гончаровым. Интересная унипотентная мотивная фундаментальная группа определена и исследована Делинем и Гончаровым. В данной работе мы кратко упоминаем в рамках (p -адических) L -функций и (p -адических) (кратных) дзета-значений применения подходов Куботы-Леопольдта и Ивасавы, которые основанны на P -адических L -функциях Куботы-Леопольда, и арифметических $p-$ адических L -функциях Ивасавы. Прореферирован ряд недавних работ (и соответствующих результатов): кратные дзета-значения в корнях из единицы, построение семейств мотивных итерированных интегралов с предписанными свойствами по Глэносу (Glanois); явные выражения для круговых $p-$ адических кратных дзета-значений глубины два по Унверу (Unver); связи арифметических степеней циклов Кудлы-Рапопорта на интегральной модели многообразия Шимуры, соответствующей унитарной группе сигнатуры (1,1), с коэффициентами Фурье центральных производных рядов Эйзенштейна рода 2 по Санкарану (Sankaran). Более полно с содержанием статьи можно ознакомиться по приводимому ниже оглавлению: Введение. 1. Сравнения типа Вороного для чисел Бернулли. 2. Римановы дзета-значения. 3. О группах классов колец с теорией дивизоров. Мнимые квадратичные и круговые поля. 4. Ряды Эйзенштейна. 5. Группы классов, поля классов и дзета-функции. 6. Кратные дзета-значения. 7. Элементы неархимедовых локальных полей и неархимедова анализа. 8. Итерированные интегралы и (кратные) дзета-значения. 9. Формальные и $p-$делимые группы. 10. Мотивы и ($p-$адические) (кратные) дзета-значения. 11. О рядах Эйзенштейна, ассоциированных с многообразиями Шимуры. Разделы 1-9 и подраздел 11.1 (О некоторых многообразиях Шимуры и модулярных формах Зигеля) можно рассматривать как элементарное введение в результаты раздела 10 и подраздела 11.2 (О несобственном пересечении дивизоров Кудлы-Рапопорта и рядах Эйзенштейна).

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Ключевые слова: p -адическая интерполяция; (p-адическая) L-функция; ряд Эйзенштейна; изоморфизм сравнения; кристаллический морфизм Фробениуса; фундаментальная группа де Рама; (p-адическое) кратное дзета-значение; геория Ивасавы; многообразие Шимуры; арифметические циклы.

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p-adic L-functions and p-adic multiple zeta values

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Abstract

The article is dedicated to the memory of George Voronoi. It is concerned with (p-adic) L-functions (in partially (p-adic) zeta functions) and cyclotomic (p-adic) (multiple) zeta values. The beginning of the article contains a short summary of the results on the Bernoulli numbers associated with the studies of George Voronoi. Results on multiple zeta values have presented by D. Zagier, by P. Deligne and A.Goncharov, by A. Goncharov, by F. Brown, by C. Glanois and others. S. Ünver have investigated p-adic multiple zeta values in the depth two. Tannakian interpretation of p-adic multiple zeta values is given by H. Furusho. Short history and connections among Galois groups, fundamental groups, motives and arithmetic functions are presented in the talk by Y. Ihara. Results on multiple zeta values, Galois groups and geometry of modular varieties has presented by Goncharov. Interesting unipotent motivic fundamental group is defined and investigated by Deligne and Goncharov. The framework of (p-adic) L-functions and (p-adic) (multiple) zeta values is based on Kubota-Leopoldt p-adic L-functions and arithmetic p-adic L-functions by Iwasawa. Motives and (p-adic) (multiple) zeta values by Glanois and by Ünver, improper intersections of Kudla-Rapoport divisors and Eisenstein series by Sankaran are reviewed. More fully the content of the article can be found at the following table of contents: Introduction. 1. Voronoi-type congruences for Bernoulli numbers. 2. Riemann zeta values. 3. On class groups of rings with divisor theory. Imaginary quadratic and cyclotomic fields. 4. Eisenstein Series. 5. Class group, class fields and zeta functions. 6. Multiple zeta values. 7. Elements of non-Archimedean local fields and $p-$adic analysis. 8. Iterated integrals and (multiple) zeta values. 9. Formal groups and p-divisible groups. 10. Motives and (p-adic) (multiple) zeta values. 11. On the Eisenstein series associated with Shimura varieties. Sections 1-9 and subsection 11.1 (On some Shimura varieties and Siegel modular forms) can be considered as an elementary introduction to the results of section 10 and subsection 11.2 (On improper intersections of Kudla-Rapoport divisors and Eisenstein series). Numerical examples are included.

Keywords: p-adic interpolation; (p-adic) L-function; Eisenstein Series; comparison isomorphism; crystalline Frobenius morphism; de Rham fundamental group; (p-adic) multiple zeta value; Iwasawa theory; Shimura variety; arithmetic cycles.

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Introduction

The article is dedicated to the memory of George Voronoi. It is concerned with (p-adic) L-functions (in partially (p-adic) zeta functions) and cyclotomic (p-adic) (multiple) zeta values. The beginning of the article contains a short summary of the results on the Bernoulli numbers associated with the studies of George Voronoi. Results on multiple zeta values have presented by D. Zagier [1], by P. Deligne and A.Goncharov [5], by A. Goncharov [6], by F. Brown [7], by C. Glanois [8] and others. Tannakian interpretation of p-adic multiple zeta values is given by H. Furusho [10]. Short history and connections among Galois groups, fundamental groups, motives and arithmetic functions are presented in the talk by Y. Ihara [12]. Results on multiple zeta values, Galois groups and geometry of modular varieties has presented by Goncharov [6]. Interesting unipotent motivic fundamental group is defined and investigated by Deligne and Goncharov [5]. S. Ünver [9, 11] have investigated p-adic multiple zeta values in the depth two. The framework of (p-adic) L-functions and (p-adic) (multiple) zeta values is based on Kubota-Leopoldt p-adic L-functions [13] and arithmetic p-adic L-functions by Iwasawa [14]. Motives and (p-adic) (multiple) zeta values, improper intersections of Kudla-Rapoport divisors and Eisenstein series by Sankaran [37] are reviewed. More fully the content of the article can be found at the following table of contents: Introduction. 1. Voronoi-type congruences for Bernoulli numbers. 2. Riemann zeta values. 3. On class groups of rings with divisor theory. Imaginary quadratic and cyclotomic fields. 4. Eisenstein Series. 5. Class group, class fields and zeta functions. 6. Multiple zeta values. 7. Elements of non-Archimedean local fields and $p-$adic analysis. 8. Iterated integrals and (multiple) zeta values. 9. Formal groups and p-divisible groups. 10. Motives and (p-adic) (multiple) zeta values. 11. On the Eisenstein series associated with Shimura varieties. Sections 1-9 and subsection 11.1 (On some Shimura varieties and Siegel modular forms) can be considered as an elementary introduction to the results of section 10 and subsection 11.2 (On improper intersections of Kudla-Rapoport divisors and Eisenstein series). Numerical examples are included.

The subject matter of this review has deep historical roots, with contributions of many mathematiciens. I apologize for any oversights and any misrepresentations, which are not intentional but rather due to my ignorance.

Remark 1. *Let me now present very briefly the background of my interest on the subject of the values of zeta and L−functions. In 1970-1971 years Yu. Manin gave courses of lectures and seminars on Algebraic Geometry, Diophantine Geometry in MGU and in Steklov mathematical institute. In his lectures and talks Yu. Manin presented and discussed the Birch-Swinnerton-Dyer conjecture concerning L− functions of elliptic curves and abelian varieties. In particular Yu. Manin have proposed in these talks modular symbols for computation of values of L−functions of elliptic curves at $s = 1$ [2, 3]. Author of the text attended the lectures and seminars of Yu. Manin. Following of the kind conversation with Yu. Manin the author has implemented the computer program and has computed Manin's modular symbols [39] for elliptic curve $E_{\Gamma_0(11)}$ follow to Manin article [2].*

1. Voronoi-type congruences for Bernoulli numbers

We follow to [18, 19].

1.1. Bernoulli numbers

Bernoulli numbers B_m are determined for integers $m \geqslant 0$ by the expansion

$$\frac{t}{\exp(t)-1} = 1 + \sum_{m=1}^{\infty} \frac{B_m}{m!} t^m.$$

Remark 2. *For $m > 0, B_{2m+1} = 0$.*

So we have $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ldots$

1.2. Voronoi's congruences

Let N be an natural number (the modulus), a coprime with N and let $B_{2m} = \frac{P_{2m}}{Q_{2m}}$ be the Bernoulli number with coprime P_{2m} and Q_{2m}. Then

$$(a^{2m}-1)P_{2m} \equiv 2ma^{2m-1}Q_{2m} \sum_{s=1}^{N-1} s^{2m-1} \left[\frac{sa}{N}\right] \bmod N.$$

1.3. Kummer congruences

If p is prime and $p-1$ not divide even positive m then the number $\frac{B_m}{m}$ is p-integer and there is the congruence

$$\frac{B_{m+p-1}}{m+p-1} \equiv \frac{B_m}{m} \bmod p.$$

2. Riemann zeta values

Here we follow to [15, 16, 17, 18].

Let $s = \sigma + it$ be a complex number and let $\zeta(s)$ be the Riemann zeta function which is presented for $\sigma > 1$ by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

;

By Euler for $m \geqslant 1$

$$\zeta(2m) = (-1)^{m-1} \frac{(2\pi)^{2m}}{2(2m)!} B_{2m}$$

where B_{2m} are Bernoulli numbers; recall also that

$$\zeta(-n) = -\frac{B_{n+1}}{n+1},$$

for odd $n = 1, 3, 5, \ldots$.

$$\zeta(1-2m) = -\frac{B_{2m}}{2m}, \text{ if } m > 0.$$

Remark 3. *(By Euler),*

$$\zeta(2) = \frac{\pi^2}{6}. \zeta(4) = \frac{\pi^4}{90}, \ \zeta(6) = \frac{\pi^6}{3^3 5 \cdot 7},$$

$$\zeta(-1) = -\frac{B_2}{2} = -\frac{1}{12}, \zeta(-3) = \frac{1}{120}.$$

Define polylogarithm

$$L_m(z) = \sum_{n=1}^{\infty} z^n n^{-m}.$$

Remark 4.

$$\zeta(2) = L_2(1).$$

3. On class groups of rings with divisor theory. Imaginary quadratic and cyclotomic fields

The study of class groups of rings and corresponding schemes is an actual scientific problem (see [18, 20] and references therein). For regular local rings, according to the Auslander-Buchsbaum theorem, the (divisors) class group is trivial. But in most interesting cases the group is nontrivial. The Heegner approach, together with the results of Weber, Birch, Baker and Stark, makes it possible to calculate and even parametrize rings with a given (small) class number in some cases. Let R be a commutative ring with identity for which there exists the theory of divisors [18]. The order of the class group is calculated on the basis of the use of L-functions. We investigate one of the aspects of this problem, consisting in finding the moduli spaces of elliptic curves defined over the rings R with the given class number.

Problem. To investigate the case of elliptic curves over rings of integers of quadratic fields (rings of integers $\mathcal{O}$ of quadratic algebraic extensions k of the field of rational numbers $\mathbb{Q}$) with a small class number, see [18].

In some cases, for instance under computer algebra computations, we have to enumerate investigated objects. Some simple parametric spaces and moduli spaces in the case of imaginary quadratic fields are presented below [40]. We present an elementary introduction to this problem and give the moduli spaces as trivial bundles over affine part of the groups of rational points of some elliptic curves over the ring of integers $\mathbb{Z}$. Below we present parameter spaces and moduli for class number one and two. Let

$$E : y^2 = x^3 + ax + b,\ Disc(E) = 4a^3 + 27b^2, Disc(E) \neq 0, \quad (*)$$

be an elliptic curve over the ring $\mathcal{O}$. Let A_1 be the affine part of the group of rational points over $\mathbb{Z}$ of the Heegner elliptic curve $y^2 = 2x(x^3 + 1)$. With results by Heegner, Deuring, Birch, Baker, Stark, Kenku, Abrashkin, we deduce

Proposition 1. *Let $\mathcal{O}$ be the ring of integers of the imaginary quadratic field with class number one. Then the parameter space of elliptic curves of the form (*) is the trivial bundle*

$$(\mathcal{O} \times \mathcal{O}/(Disc(E) = 0)) \times A_1.$$

Proposition 2. *Let k be the imaginary quadratic field with class number one. Then the moduli space of elliptic curves of the form (*) is the trivial bundle*

$$k \times A_1.$$

Let A_2 be the affine part of the group of rational points over $\mathbb{Z}$ of the elliptic curve $X^3 + 3X = -Y^2$, let A_3 be the affine part of the group for the elliptic curve $X^3 - 3X = 2Y^2$, and A_4 respectively for $9X^4 - 1 = 2Y^2$.

Proposition 3. *Let $\mathcal{O}$ be the ring of integers of the imaginary quadratic field with class number two. Then the parameter spaces of elliptic curves of the form (*), without an exceptional case, are trivial bundles*

$(\mathcal{O}\times\mathcal{O}/(Disc(E)=0))\times A_2$, $(\mathcal{O}\times\mathcal{O}/(Disc(E)=0))\times A_3$, $(\mathcal{O}\times\mathcal{O}/(Disc(E)=0))\times A_4$.

Proposition 4. *Let k be the imaginary quadratic field with class number two. Then the moduli spaces of elliptic curves of the form (*), without an exceptional case, are the trivial bundles*

$$k\times A_2,\ k\times A_3,\ k\times A_4.$$

Theorem 1. *(The Kronecker-Weber theorem) Every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field.*

With results by Heegner, Deuring, Birch, Baker, Stark, Shafarevich we have

Proposition 5. *Imaginary quadratic fields with class number one and with descriminants $-D=4,8,3,7,11,19,43,67,163$ are contained, respectively, in cyclotomic fields*

$$\begin{gathered}\mathbb{Q}(\sqrt[4]{1}),\mathbb{Q}(\sqrt[8]{1}),\mathbb{Q}(\sqrt[3]{1}),\mathbb{Q}(\sqrt[7]{1}),\mathbb{Q}(\sqrt[11]{1}),\\ \mathbb{Q}(\sqrt[19]{1}),\mathbb{Q}(\sqrt[43]{1}),\mathbb{Q}(\sqrt[67]{1}),\mathbb{Q}(\sqrt[163]{1}).\end{gathered}\tag{1}$$

4. Eisenstein Series

Here we follow to [15, 16, 17, 18].
Let τ belong to the modular figure of the modular group $\Gamma=\Gamma(1)$.

Definition 1. *In these notations with $k>1$ the Eisenstein series is defined as*

$$c_k=\sum_{m\neq 0,k>1}\frac{1}{(n+m\tau)^{2k}}.$$

Proposition 6. *Eisenstein series have the representation*

$$c_k=2\zeta(2k)+\frac{2(-2\pi i)^{2k}}{(2k-1)!}\sum_{n>0,m>0}n^{2k-1}q^{nm},$$

where $q=e^{2\pi i\tau}\neq 0$.
If we will use functions of the sums of divisors σ_{2k-1} we obtain
$c_k=2\zeta(2k)+\frac{2(-2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n$
or shortly
$c_k=2\zeta(2k)+\frac{2(-2\pi i)^{2k}}{(2k-1)!}S_{2k-1}$.

As $\zeta(2k)=(-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}$ we have

Corollary 1. $c_k=2\zeta(2k)(1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n)$.

Put $g_2=60c_2$, $g_3=140c_3$.

Proposition 7. $\Delta=g_2^3-27g_3^2\neq 0$.

As $\Delta\neq 0$ it is possible to define $J=\frac{g_2^3}{\Delta}$.

Definition 2. *Modular invariant of the elliptic curve $y^2=4x^3-g_2x-g_3$ is equal to $j=2^6 3^3 J$.*

Proposition 8. *$j=\frac{1}{q}+u_1q+\cdots$ where u_i are integers, $u_0=0$.*

Let us transform c_k in such a way that corresponding Fourier coefficients under $q^n, n \geqslant 1$ will rational numbers. Dividing c_k on $2\zeta(2k)$ and denoting the obtained result as E_k we have by the Corollary 1

Proposition 9. $E_k = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$.

Remark 5.

$$E_2 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_3 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n.$$

5. Class group, class fields and zeta functions

Here we follow to [16, 18].

Let K be an imaginary quadratic field and let Cl_K be its class group.

Definition 3. *Let $N(\mathfrak{a})$ be the norm of the ideal $\mathfrak{a}$. The Dedekind ζ-function for K is defined for all $s > 1$ by the series*

$$\zeta_K(s) = \sum \frac{1}{N(\mathfrak{a})^s},$$

where the sum is taken over all nonzero ideals $\mathfrak{a} \in \mathcal{O}_K$.

Let R be a subring ($R \neq \mathbf{Z}$) of the ring of integers $\mathcal{O}_K$ of the imaginary quadratic field K.

Let $M_1, \dots M_h$ be pairwise nonequivalent modules of K with the same ring of multipliers R.

Proposition 10. *$j(M_1), \dots, j(M_h)$ are integer algebraic numbers which are conjugate over K.*

Proposition 11. *The field $K(j(M_i))/K$ is the normal field.*

Definition 4. *The field $K(j(M_i))/K$ is called the ring class field.*

Follow to [16] it is possible to define ray class field. As in an imaginary quadratic field there is no real infinite primes so modulus of the field is an ideal of the ring of integers of the field.

Let $\mathfrak{m}$ be a modulus of the an imaginary quadratic field K, let $Cl_K^{\mathfrak{m}}$ be the ray class group, let τ_W be the Weber function .

Let $\mathfrak{R} \in Cl_K$ and let $\mathfrak{R}^* \in Cl_K^{\mathfrak{m}}$ be the ideal class whose image in Cl_K is equal to $(\mathfrak{m})\mathfrak{R}^{-1}$.

Proposition 12. *The field $K(j(\mathfrak{R}), \tau_W(\mathfrak{R}^*))/K$ is the ray class field.*

Let C be an ideal class.

Definition 5. *The ideal class zeta function is the expression of the form*

$$\zeta_C(s) = \sum_{\substack{\mathfrak{a} \in C \\ \mathfrak{a} \text{ integral}}} \frac{1}{N(\mathfrak{a})^s}$$

$$s = \sigma + it, \ \sigma > 1.$$

Below we present values of zeta and L-functions connecting with imaginary quadratic fields. Let d be a squarefree integer number, $K = \mathbb{Q}(\sqrt{d})$ a quadratic field, χ be the character of the quadratic field K. Let $L(s, \chi)$ be the $L-$series with a nonunit character χ modulo $|D|$. Here D is the discriminant of the field K.

Proposition 13.

$$\zeta_K(s) = \zeta(s)L(s,\chi) = \zeta(s)\prod_p(1-\frac{\chi(p)}{p^s}).$$

Let m be the number of roots of unity of the imaginary quadratic field $[K:\mathbb{Q}]$.

Remark 6. *$m=4$ for $K=\mathbb{Q}(\sqrt{-1})$, $m=6$ for $K=\mathbb{Q}(\sqrt{-3})$, $m=2$ for all other imaginary quadratic fields.*

Let h be the class number of the field K.

Proposition 14.

$$L(1,\chi) = \frac{2\pi h}{m\sqrt{|D|}}.$$

Corollary 2. *For imaginary quadratic fields with class number one ($h=1$) we have*

$$L(1,\chi) = \begin{cases} \frac{\pi}{4}, K=\mathbb{Q}(\sqrt{-1}) \\ \frac{\pi}{3\sqrt{3}}, K=\mathbb{Q}(\sqrt{-3}) \\ \frac{\pi}{\sqrt{|D|}}, -D = 8,7,11,19,43,67,163. \end{cases}$$

6. Multiple zeta values

Definition 6. *Let $x_1,\dots x_p$ be natural numbers with $x_p \geqslant 2$. The multiple zeta value of the weight w and the depth p is called the expression of the form*

$$\zeta(x_1,\dots x_p) = \sum_{0<n_1<\dots<n_r} \frac{1}{n_1^{x_1}\cdots n_p^{x_p}}, w = \sum x_i.$$

Remark 7.

$$\zeta(2,2) = \sum_{0<n_1<n_2} \frac{1}{n_1^2 n_2^2}, w = \sum x_i = 4.$$

Remark 8.

$$\zeta(2,2) = \frac{1}{2}(\zeta(2)\zeta(2) - \zeta(4)).$$

Let μ_N be the group of roots of unity.

Definition 7. *Let $x_1,\dots x_p$ be natural numbers with $x_p \geqslant 2$. The multiple zeta value relative to μ_N of the weight w and the depth p is called the expression of the form*

$$\zeta\binom{x_1,\dots x_p}{\epsilon_1,\dots,\epsilon_p} = \sum_{0<n_1<\dots<n_p} \frac{\epsilon_1^{n_1}\dots,\epsilon_p^{n_p}}{n_1^{x_1}\cdots n_p^{x_p}}, \epsilon_i \in \mu_N,$$

$$w = \sum x_i, (x_p,\epsilon_p) \neq (1,1).$$

7. Elements of non-Archimedean local fields and $p-$adic analysis

Here we present elements of $p-$adic local fields, their algebraic extensions and $p-$adic interval analysis. We follow to [18, 21].

7.1. Elements of non-Archimedean local fields

A non-Archimedean local field is a complete discrete valuation field with finite residue field. Further, for brevity, we call these fields local. In other words, a field K is called local if it is complete in a topology determined by the valuation of the field and if its residue field k is finite. We assume further that the valuation ν is normalized, i.e. the homomorphism of the multiplicative group of the field to the additive group of rational integers $\nu : K^* \to \mathbb{Z}$ is surjective.

The structure of such fields is known: if the field K has the characteristic zero, then it is a finite extension of the $p-$adic field $\mathbb{Q}_p$, which is the completion of the field of rational numbers with respect to the $p-$adic valuation.

If $[K : \mathbb{Q}_p] = n$, then $n = ef$, where f is the degree of classes of residues, (i.e. $f = [k : \mathbb{F}_p]$) and $e = \nu_K(p)$ is the ramification index of K..

If the field K has the characteristic p, then it is isomorphic to the field $k((T))$ of formal power series, where T is a uniformizing parameter.

Let L be a finite extension of a local field K with their residue fields l and k, $p = char\ k$ and $e_{L/K}$ be the ramification index of L over K.

An extension L/K is called unramified if a) $e_{L/K} = 1$; b) the extension l/k is separable. An extension L/K is called tamely ramified if a) p does not divide $e_{L/K}$; b) the extension l/k is separable.

An extension L/K is called wildly ramified if $e_{L/K} = p^s,\ s \geqslant 1$;

Denote by $Tr_{L/K}$ and by $Norm_{L/K}$ respectively the trace and the norm of the extension L/K. We drop indices, when it is clear what kind of extension we are talking about.

Denote by K_{nr} the maximal unramified extension of the field K (in a fixed algebraic closure of the field K) with a residue field k_s, which is the algebraic closure of a field k.

In a non-Archimedean local field K each of its elements α has a representation $\alpha = \epsilon\pi^m$, where ϵ is a unit of the ring of integers of the field K and π its uniformizing element, that is $\nu(\pi) = 1$, m is an integer rational number. A unit is called principal if $\epsilon \equiv 1 \pmod{\pi}$.

LEMMA 1. *If the local field contains a primitive $p-$th root ξ_p of unity, then $\nu(\xi_p - 1) = \frac{e}{p-1}$ is an integer number.*

Proof. $\xi_p - 1$ is the root of the equation $(x+1)^{p-1}+(x+1)^{p-2}+\cdots+(x+1)+1 = x^{p-1}+p(\cdots)+p$. The value of the $p-$adic valuation at the root of this equation is $\frac{e}{p-1}$ which proves the required. □

A complete discrete valuation field with an algebraically closed residue field is called a quas-ilocal field.

7.2. $p-$adic intervals and $p-$adic distributions

Let X be a topological space. A distribution on X with values in an abelian group A is a finitely additive function from the compact-open subsets of X to A. Let $|\ |_p$ be the $p-$adic norm.

Define $[\alpha, N]_p = \{x \in \mathbb{Q}_p || x - \alpha|_p \leqslant \frac{1}{p^N}\}$, $\alpha \in \mathbb{Q}_p, N \in \mathbb{N}$.

DEFINITION 8. *We call sets $[\alpha, N]_p$ the $p-$adic intervals (disks) and define by these $p-$adic intervals the basis of open sets on $\mathbb{Q}_p$.*

It is easy to test that axioms of open sets are satisfied.

REMARK 9. *$p-$adic intervals $[\alpha, N]_p$ open and closed simultaneously.*

Proof. Any union of open $p-$adic intervals is open. Intervals $[\alpha, N]_p$ are closed, because $[\alpha, N]_p$ is an addition to the union of open intervals $[\alpha', N]_p$ for all $\alpha' \in \mathbb{Q}_p$ for which $\alpha' \neg \in [\alpha, N]_p$. □

Further we will call $[\alpha, N]_p$ as intervals. More generally we will consider compact-open sets. Let X be a compact-open set. Recall that a function $f : X \to \mathbb{Q}_p$ is is locally constant if and only if

f has a representation as a finite linear combination of characteristic functions of compact-open subsets.

Let $U = U_1 \cup U_2 \cup \cdots \cup U_n$ be a partition of $U \subset X$. Recall that the additive mapping μ of a set of compact-open subsets of X with value in $\mathbb{Q}_p$ is called the $p-$adic distribution on X:

$$\mu(U) = \mu(U_1) + \mu(U_2) + \cdots + \mu(U_n).$$

7.2.1. Bernoulli distributions.

Let $B_m(x)$ be the $m-$Bernoulli polynomial. These polynomials are defined by the decomposition

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}.$$

We have:

$$B_0(x)=1,\ B_1(x)=x-\frac{1}{2},\ B_2(x)=x^2-x+\frac{1}{6},\ B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x,\ B_4(x)=x^4-2x^3+x^2-\frac{1}{30},\ldots$$

Remark 10. *If we substitute $x = 0$ in the $m-$Bernoulli polynomial we obtain $m-$Bernoulli number:*

$$B_0(0) = 1,\ B_1(0) = -\frac{1}{2},\ B_2(0) = \frac{1}{6},\ B_3(0) = 0,\ B_4(0) = -\frac{1}{30},\ldots$$

Let now for α the inequality $0 \leqslant \alpha \leqslant p^N - 1$ is satisfied. Define the function $\mu_{B,m}$ by the formula

$$\mu_{B,m}([\alpha, N]_p) = p^{N(m-1)}B_m(\alpha/p^N).$$

Proposition 15. *The function $\mu_{B,m}$ is expanded to the distribution on $\mathbb{Z}_p$. This distribution for the given m is called the $m-$th Bernoulli distribution.*

8. Iterated integrals and (multiple) zeta values

Here we follow to[22, 23].

Let $\mathbb{C}$ be the complex plane and $f_i(z)$ be the holomorphic function on $\mathbb{C}$. Let $f_i(z)dz$ be the differential of the first kind on $\mathbb{C}$. Let S be a Riemann surfaces and w be the differential of the first kind on S. Parshin has considered iterated integrals of this type on Riemann surfaces [22]. Chen [23] for smooth paths on a manifold M and respective path spaces have investegated iterated (path) integrals. For differential forms $w_1, \ldots, w_r$ on M he has constructed the iterated integrals by repeating r times the integration of the path space differential forms (and their linear combinations). Chen [23] has denoted the iterated integrals as $\int w_1 w_2 \cdots w_r$ and set $\int w_1 w_2 \cdots w_r = 1$ when $r = 0$ and $\int w_1 w_2 \cdots w_r = 0$ when $r < 0$.

Remark 11.

$$\zeta(2) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t1} \frac{dt_2}{1 - t_2} = \frac{\pi^2}{6}.$$

More generally iterated integrals are path space differential forms which permit further integration.

9. Formal groups and p-divisible groups

Recall some definitions. Let K be a complete discrete variation field with the ring of integers O_K and the maximal ideal M_K. A complete discrete variation field with finite residue field is called a *local* field [24]. A complete discrete variation field K with algebraically closed residue field k is called a *quasi-local* field [26]. Below we will suppose that in the case the characteristic of k satisfies $p > 0$. Let K be a local or quasi-local field. If K is a local field [24] and has the characteristic 0 then it is a finite extension of the field of p-adic numbers $\mathbb{Q}_p$. Let ν_K be the normalized exponential valuation of K. If $[K : \mathbb{Q}_p] = n$ then $n = e \cdot f$, where $e = \nu_K(p)$ and $f = [k : \mathbb{F}_p]$, where k is the residue field of K (always assumed perfect). If K has the characteristic $p > 0$ then it isomorphic to the field $k((T))$ of formal power series, where T is uniformizing parameter. Let L be a finite extension of a local field K, k, l their residue fields, $p = char\ k$ and $e_{L/K}$ ramification index of L over K. An extension L/K is said to be *unramified* if $e_{L/K} = 1$ and extension l/k is separable. An extension L/K is said to be *tamely ramified* if p not devides $e_{L/K}$ and the residue extension l/k is separable. An extension L/K is said to be *totally ramified* if $e_{L/K} = [L : K] = (char\ k)^s, s \geqslant 1$.

Let L/K be the finite Galois extension of quasi-local field K with Galois group G, $F(x, y)$ one dimensional formal group low over the ring of integers O_K of the field K, $F(M_K)$ be the G - module, that is defined by the group low $F(x, y)$ on the maxilal ideal M_K of the ring O_K, $M_K^t (t \in \mathbb{Z}, t \geqslant 1)$ be the subgroup of t-th degrees of elements from $M_K, F_K^t := F(M_K^t)$.

Definition 9. *For $n \in \mathbb{Z}$ the function $\mu(n)$, $N_{L/K}(F_L^n) \subset F_K^{\mu(n)}$ is defined by the condition: $F_K^{\mu(n)}$ is the least of subgroups F_K^t $(t = 1, 2, \ldots)$ containes $N_{L/K}(F_L^n)$.*

Remark 12. *Please do not confuse with the measure μ.*

Below we will suppose that $char\ k > 3$.

9.1. Norm Maps

Here we use results on formal groups from [27, 25]. Let $F_L = F(M_L)$ be the G - module that is defined by the n-dimentional group low $F(x, y)$ on the product $(M_L)^n := M_L \times \cdots \times M_L$, ($n$ times) of maximal ideals of the ring O_L of any finite Galois extension L of the field K.

Definition 10. *The norm map $N : F_L \to F_K$ of the module F_L to F_K is defined by the formula $N(a) = (((a +_F \sigma a) +_F \cdots) +_F \sigma_s a)$, where $a +_F b$ denotes the addition of points in the sense of group structure of the module F_L, $a, b \in M_L$, $G = Gal(L/K)$, $\sigma_s \in G$, $[G : 1] = s$.*

Let $p := char\ k$, $e := \nu_K(p)$, ($e = +\infty$, if characteristic of the field K is equal p and e is positive integer in the opposide case), L/K be the Galois extension of the prime degree q, $F(x, y)$ be the one dimensional group low over O_K. Let $p := char\ k > 0$.

Lemma 2. *If $\Pi_s \in \pi_L^s \cdot O_L$, $s \geqslant 1$ then*
$N(\Pi_s) \equiv Tr(\Pi_s) + \sum_{n=1}^{\infty} c_n [Norm\ \Pi_s]^n (mod\ Tr(\pi_L^{2s} \cdot O_L))$
where $c_n \in O_K$ are coefficients of the p - iteration of the group low.

Let R be a commutative ring. Let A, B, C be finite group schemes over R. The sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is called exact if $Im\ f = Ker\ g$.

Let p be a prime number and h be an integer, $h \geqslant 0$.

Recall the definition of the p-divisible group by J. Tate.

Definition 11. *A p-divisible group over R of height h is an inductive system*

$$G = (G_\nu, i_\nu), \nu \geqslant 0,$$

where
(i) G_ν is a finite group scheme over R of order $p^{\nu h}$,
(ii) for each $\nu \geqslant 0$,

$$0 \longrightarrow G_\nu \xrightarrow{i_\nu} G_{\nu+1} \xrightarrow{p^\nu} G_{\nu+1}$$

is exact.

Remark 13. $(G_\nu = (\mathbb{Z}/p^\nu\mathbb{Z}))^h$, $G = (\mathbb{Q}_p/\mathbb{Z}_p)^h$.

10. Motives and (p-adic) (multiple) zeta values

Glanois in paper [8] presents the revised and expanded version of his Doctoral thesis [Periods of the motivic fundamental groupoid of $\mathbb{P}^1\{0, \mu_N, \infty\}$, Pierre and Marie Curie University, 2016;], written under F. Brown.

Let $k_N = \mathbb{Q}(\xi_N)$ be the cyclotomic field, $\xi_N \in \mu_N$ be a primitive Nth root of unity and $\mathcal{O}_N$ be the ring of integers of k_N. The corresponding multiple zeta values at arguments $x_i \in \mathbb{N}, \epsilon_i \in \mu_N$ can be expressed in terms of the coefficients of a version of Drinfeld's assosiators by Drinfeld [28], which in turn, can be expressed in terms of periods of the corresponding motivic multiple zeta values (MMZV).

These MMZV $\zeta^{\mathfrak{m}}\binom{x_1,\dots x_p}{\epsilon_1,\dots,\epsilon_p}$, $\epsilon_i \in \mu_N$, $(x_p, \epsilon_p) \neq (1,1)$ relative to μ_N (of the weight $w = \sum x_i$ and the depth p), are elements of an algebra $\mathcal{H}^N$ over $\mathbb{Q}$ and span the algebra.

The algebra $\mathcal{H}^N$ carries an action of the motivic Galois group of the category of mixed Tate motives over $\mathcal{O}_N[1/N]$. The author studies the Galois action on the motivic unipotent fundumental groupoid of $\mathbb{P}^1\backslash\{0, \mu_N, \infty\}$ (or of $\mathbb{G}_m\backslash\mu_N$) for next values of N:

$$N \in \{2^a 3^b, a + 2b \leqslant 3\} = \{1, 2, 3, 4, \text{`}6\text{`}, 8\}.$$

His results include: bases of multiple zeta values via multiple zeta values at roots of unity μ_N for the above N; more generally, constructing of families of motivic iterated integrals with prescribed properties; the new proof, via the coproduct by Goncharov [29] and its extension by Brown [7], of the results by Deligne [30] that the Tannakian category of mixed Tate motives over $\mathcal{O}_N[1/N]$ 'for $N = \{2, 3, 4, 8\}$ is spanned by the motivic fundumental groupoid of $\mathbb{P}^1\backslash\{0, \mu_N, \infty\}$ with an explicit basis'.

In article [11] Ünver continues his investigation of p-adic multiple zeta values[9], presenting a computation of values of the p-adic multiple polylogarithms at roots of unity. The main result of the paper [11] (Theorem 6.4.3 with Propositions 6.4.1 and 6.3.1) is to give explicit expression for the cyclotomic p-adic multi-zeta values $\zeta_p(s_1, s_2; i_1, i_2)$ of depth two. The result is far too technical to state here.

The proof of the theorem is rather technical; it is based on rigid analytic function arguments and a long distance analysis of group-like elements of related algebras.

For number fields the category of realizations has defined and investigated by Deligne [4]. Results on multiple zeta values, Galois groups and geometry of modular varieties has presented by Goncharov [6]. Interesting unipotent motivic fundamental group is defined and investigated by Deligne and Goncharov [5]. Tannakian interpretation of p-adic multiple zeta values is given by Furusho [10].

Results obtained in the paper [11] may be applied to the problems of the p-adic theory of higher cyclotomy.

11. On the Eisenstein series associated with Shimura varieties

Interesting classes of Shimura varieties form varieties which have an interpretation as moduli spaces of abelian varieties. Moduli spaces of corresponding $p-$divisible groups over perfect fields of characteristic p are used for investigation of the local structure of these Shimura varieties.

11.1. On some Shimura varieties and Siegel modular forms

Let $\mathbb{C}^n$ be $n-$dimensional complex vector space, $\{e_1, \dots, e_{2n}\}$ be $2n-$linear independent vectors and

$$\Lambda = \{e_1\mathbb{Z}, \dots, e_{2n}\mathbb{Z}\}$$

be a lattice. Then

$$\mathbb{C}^n/\Lambda$$

is a compact commutative topological group. If $\alpha \in GL_n(\mathbb{C})$ and $\Lambda_1 = \alpha\Lambda$ then

$$\mathbb{C}^n/\Lambda = \mathbb{C}^n/\Lambda_1.$$

If $n > 1$ then not for every lattice Λ there exists an abelian variety.

Proposition 16. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ and let $F(\mathbf{x}, \mathbf{y})$ be $\mathbb{R}-$bilinear form such that*

$$(i) F(\mathbf{x}, \mathbf{y}) = -F(\mathbf{y}, \mathbf{x}),$$

(ii) $F(\mathbf{x}, i\mathbf{x})$ is the Hermitian positive defined form
and for
(iii) $\mathbf{x}, \mathbf{y} \in \Lambda$ it takes integer values: $F(\Lambda, \Lambda) \in \mathbb{Z}$.
Then for this lattice Λ there exists the abelian variety.

Definition 12. *The pair (Λ, F) is called the polarized abelian variety.*

Let $M = M(F)$ be the matrix of the form F.

Definition 13. *The abelian variety $A = (\Lambda, F)$ is called principally polarized if the bilinear form F is unimodular or, equivalently, $det(M) = det(M(F)) = 1$.*

Denote by Π the period matrix of the abelian variety A. This is $n \times 2n$ complex matrix $\Pi = (M_1, M_2)$ with nondegenerate $n \times n$ matrices M_1 and M_2.

Definition 14. *The period matrix Π is called normalized if it has the form (E_n, Z) where E_n is the unit $n \times n$ matrix and $Z \in \mathbb{H}_n$, where.*

$$\mathbb{H}_n = \{n \times n \text{ matrices } Z | Z^T = Z \text{ and } ImZ > 0\},$$

is the Siegel upper half-plane. Here Z^T is the matrix transposed to Z.

Remark 14. *It is clear that the Siegel matrix $Z \in \mathbb{H}_n$ defines the normalized period matrix Π.*

Let

$$J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

Definition 15. *Siegel modular group $\Gamma_n = Sp_n(\mathbb{Z})$ is the set of matrices*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

such that

$$M^T J M = J.$$

Definition 16. *Siegel modular group Γ_n acts on the the Siegel upper half-plane $\mathbb{H}_n$ by the formula*

$$Z \mapsto (AZ + B)(CZ + D)^{-1}.$$

Proposition 17. *In the framework of Definitions 13, 14 two Siegel matrices define isomorphic principally polarized abelian varieties if and only if one of them can be obtained from the other by the transformation of the Definition 15.*

Sometimes we will use for Siegel matrices Z an equivalent notations: $Z = X + iY$, X, Y are real matrices, $X^T = X$, $Y^T = Y$, $Y > 0$; Y is the matrix of the positive definite quadratic form.

Definition 17. *Let f be an analytic function on the Siegel upper half-plane that satisfies the equality*

$$f((AZ + B)/(CZ + D)) = (CZ + D)^h f(Z),$$

and is bounded on the domains of the form

$$\{Z = X + iY,\ Z \in \mathbb{H}_n, Y \geqslant cE_n,\ c > 0.\}$$

Then the function f is called Siegel modular form of the genus n and the weight k.

Remark 15. *As the matrix*

$$J = \begin{pmatrix} E_n & B \\ 0 & E_n \end{pmatrix}.$$

belong to Γ_n (its determinant is equal 1) and

$$f(Z + B) = f(Z),$$

so $f(Z)$ has a representation by the Fourier series.

11.2. On improper intersections of Kudla-Rapoport divisors and Eisenstein series

Let k be an imaginary quadratic field, o_k its ring of integers and $o_{k,p}$ be the ring of integers of the completion k_p of k at p. Sankaran [37] proves that the arithmetic degrees of Kudla-Rapoport cycles on an integral model of a Shimura variety attached to a unitary group of signature $(1, 1)$ are Fourier coefficients of the central derivative of an Eisenstein series of genus 2. The main results of the paper are the following Theorem 4.13 on the value of the Eisenstein series and the Corollary 4.15 on the relation between the arithmetic degree of special cycle and the Eisenstein series. These results confirm conjectures by Kudla [31] and by Kudla, Rapoport [32] on relations between intersection numbers of special cycles and the Fourier coefficients of automorphic forms in the degenerate setting and for dimension 2. As have pointed out by Kudla [33] and others 'these relations may be viewed as an arithmetic version of the classical Siegel-Weil formula, which identifies the Fourier coefficients of values of Siegel-Eisenstein series with representation numbers of quadratic forms'. In the paper by Kudla, Rapoport [34] 'the Shimura variety is replaced by a formal moduli space of p-divisible groups, the special arithmetic cycles are replaced by formal subvarieties, and the special values of the derivative of the Eisenstein series are replaced by the derivatives of representation densities of hermitian forms.' Sankaran defines the local Kudla-Rapoport cycles $Z(b)$ and gives some applications of results obtained in his earlier paper [38] where he proved the Theorem 3.14 on cycles $Z(b)$. He allows 'the polarizations to be non-principal in a controlled way'. An unpolarized case of p-divisible groups with the given p^m-kernel type and with applications to their Newton polygons has considered in the paper by Harashita [35]. Sankaran's paper [37] consists of four sections. The first section presents the purpose of the paper and short description of ideas and

results of next sections. Second section concerns with local Kudla-Rapoport cycles on the Drinfeld upper half-plane. The main result of this section is the Theorem 2.14 on values of local intersection numbers of these cycles. The third section is devoted to the prove of the closed-form formula for representation densities $\alpha(S,T)$. Author specializes the explicit formula on Hermitian representation densities by Hironaka [36] to the case at hand: $F(S,T,X) \in Q[X], T \in Herm_2(o_{k,p}), ord_p det(T)$ is even, $S = diag(p,1), \alpha(S_r,T) = F(S,T,(-p)^{-r})$. In the last section global aspects are discussed and main result is presented. Let $M^d_{(1,1)}$ denote the Deligne-Mumford (DM) stack over o_k of almost-principally polarized abelian surfaces and $\mathcal{E}$ the DM stack over o_k of principally polarized elliptic curves with multiplication by o_k. In conditions of the subsection 4.1 of the paper [37] author sets $\mathcal{M} = \mathcal{E} \times_{Spec o_k} M^d_{(1,1)}$ and define for $T \in Herm_2(o_k)$ cycles $\mathcal{Z}(T)$. Then in subsection 4.3 the author prove Theorem 4.13 and Corollary 4.15.

Conclusions

Classical and novel results on (p-adic) L-functions, (p-adic) (multiple) zeta values and Eisenstein series are presented.

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