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Расширение теоремы Лауринчикаса — Матсумото

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Аннотация

В 1975 г. С. М. Воронин открыл замечательное свойство универсальности дзета функции Римана $\zeta(s)$. Он показал, что широкого класса аналитические функции могут быть приближены с желаемой точностью сдвигами $\zeta(s+i\tau)$, $\tau\in\mathbb{R}$, одной и той же функции $\zeta(s)$. Открытие Воронина вдохновило продолжить исследования в этом направлении. Оказалось, что универсальность является свойством многих других дзета и L-функций, а также некоторых классов рядов Дирихле. Среди них L-функции Дирихле, дзета функции Дедекинда, Гурвица и Лерха. В 2001 г. А. Лауринчикас и К. Матсумото получили универсальность дзета-функций $\zeta(s,F)$, связанных с некоторыми параболическими формами F. В статье получено расширение теоремы Лауринчикаса–Матсумото с использованием для приближения аналитических функций сдвигов $\zeta(s+i\varphi(\tau),F)$. Здесь $\varphi(\tau)$ – дифференцируемая функция, при $\tau\geqslant\tau_0$, имеющая непрерывную монотонную положительную производную $\varphi'(\tau)$, удовлетворяющую при $\tau\to\infty$ оценкам $\frac{1}{\varphi'(\tau)}=o(\tau)$ и $\varphi(2\tau)\max_{\tau\leqslant t\leqslant 2\tau}\frac{1}{\varphi'(t)}\ll\tau$. Более точно, в статье доказано, что если κ – вес параболической формы F, K – компактное множество полосы $\left\{s\in\mathbb{C}:\frac{\kappa}{2}<\sigma<\frac{\kappa+1}{2}\right\}$, обладающее связным дополнением, и $f(s)$ – непрерывная, неимеющая нулей в K и аналитическая внутри K функция, то для всякого $\varepsilon>0$ множество $\{\tau\in\mathbb{R}:\sup_{s\in K}|\zeta(s+i\varphi(\tau),F)-f(s)|<\varepsilon\}$ имеет положительную нижнюю плотность.

Ключевые слова: дзета-функция параболической формы, параболическая форма Гекке, универсальность.

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Extention of the Laurinčikas — Matsumoto theorem

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Abstract

In 1975, S. M. Voronin discovered the remarkable universality property of the Riemann zeta-function $\zeta(s)$. He proved that analytic functions from a wide class can be approximated with a given accuracy by shifts $\zeta(s+i\tau)$, $\tau\in\mathbb{R}$, of one and the same function $\zeta(s)$. The Voronin discovery inspired to continue investigations in the field. It turned out that some other zeta and $L-$functions as well as certain classes of Dirichlet series are universal in the Voronin sense. Among them, Dirichlet L-functions, Dedekind, Hurwitz and Lerch zeta-functions. In 2001, A. Laurinčikas and K. Matsumoto obtained the universality of zeta-functions $\zeta(s,F)$ attached to certain cusp forms F. In the paper, the extention of the Laurinčikas-Matsumoto theorem is given by using the shifts $\zeta(s+i\varphi(\tau),F)$ for the approximation of analytic functions. Here $\varphi(\tau)$ is a differentiable real-valued positive increasing function, having, for $\tau\geqslant\tau_0$, the monotonic continuous positive derivative, satisfying, for $\tau\to\infty$, the conditions $\frac{1}{\varphi'(\tau)}=o(\tau)$ and $\varphi(2\tau)\max_{\tau\leqslant t\leqslant 2\tau}\frac{1}{\varphi'(t)}\ll\tau$. More precisely, in the paper it is proved that, if κ is the weight of the cusp form F, K is the compact subset of the strip $\left\{s\in\mathbb{C}:\frac{\kappa}{2}<\sigma<\frac{\kappa+1}{2}\right\}$ with connected complement, and $f(s)$ is a continuous non-vanishing function on K which is analytic in the interior of K, then , for every $\varepsilon>0$, the set $\{\tau\in\mathbb{R}:\sup_{s\in K}|\zeta(s+i\varphi(\tau),F)-f(s)|<\varepsilon\}$ has a positive lower density.

Keywords: zeta-function of cusp forms, Hecke-eigen cusp form, universality.

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Dedicated to Professor Antanas LAURINČIKAS on the occasion of his 70th birthday

1. Introduction

In 1975 the remarkable property of universality was discovered by Voronin [25]. By analyzing the Riemann zeta-function, he noticed that with certain shifts of one and the same function a whole class of analytic functions can be approximated. This fact inspired further research of functions with similar properties and became a subject of interest for number theory specialists, among them Reich [14], Gonek [4], Good [6], Bagchi [1], Laurinčikas [8], [9] and others. The aim of this paper is certain extended results on the universality for zeta-functions attached to certain cusp forms.

Denote by $s=\sigma+it$ a complex variable. Let

$$SL(2,\mathbb{Z}):=\left\{\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}:a,b,c,d\in\mathbb{Z},ad-bc=1\right\}$$

be the full modular group. We say that the function $F(z)$, $z\in\mathbb{C}$, is a holomorphic cusp form of weight κ for $SL(2,\mathbb{Z})$ if it is holomorphic for $\mathrm{Im}(z)>0$, for all $\gamma\in SL(2,\mathbb{Z})$ satisfies the functional equation

$$F\left(\frac{az+b}{cz+d}\right)=(cz+d)^{\kappa}F(z)$$

and at infinity has the Fourier series expansion

$$F(z)=\sum_{m=1}^{\infty}c(m)e^{2\pi imz}.$$

We assume additionally that $F(z)$ is an eigen form of all Hecke operators

$$T_mF(z) = m^{\kappa-1} \sum_{\substack{a,d>0 \\ ad=m}} \frac{1}{d^\kappa} \sum_{b \pmod d} F\left(\frac{az+b}{d}\right), \quad m \in \mathbb{N}.$$

Then $c(m) \neq 0$, and, therefore, $F(z)$ can be normalized to have the Fourier coefficient $c(1) = 1$.

Having all the aforementioned assumptions, the zeta-function $\zeta(s, F)$ associated with the cusp form $F(z)$ of weight κ is defined, for $\sigma > \frac{\kappa+1}{2}$, by absolutely convergent Dirichlet series

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

It is proved [5] that $\zeta(s, F)$ is analytically continued to an entire function. Moreover, for $\sigma > \frac{\kappa+1}{2}$, the function $\zeta(s, F)$ has the Euler product expansion over primes, i. e.,

$$\zeta(s, F) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $\mathbb{P}$ is the set of all prime numbers, and $\alpha(p)$ and $\beta(p)$ are complex conjugate numbers satisfying $\alpha(p) + \beta(p) = c(p)$.

The first result on the universality of $\zeta(s, F)$ was obtained by the Laurinčikas and Matsumoto in 2001 [10]. For the formulation of the theorem, we need some notation.

Let $D = D_F = \left\{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\right\}$, $\mathcal{K} = \mathcal{K}_F$ be the class of compact subsets in the strip D with connected complements, and $H_0(K)$, $K \in \mathcal{K}$, stand for the class of continuous non-vanishing functions on K that are analytic in the interior of K. The Lebesgue measure of a measurable set $A \subset \mathbb{R}$ is denoted by meas$A$. Then the Laurinčikas-Matsumoto universality theorem for $\zeta(s, F)$ can be formulated as follows.

Theorem 1 ([10]). *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$, the following inequality*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, F) - f(s)| < \varepsilon\right\} > 0$$

holds.

In the theorem, the shifts τ take arbitrary real values. However, it turns out that more general shifts can be considered. The aim of this paper is taking shifts for the universality theorem from a certain class of functions $U(\tau_0)$. We say that a function $\varphi(\tau) \in U(\tau_0)$, $\tau_0 > 0$, if the following conditions are satisfied:

1. $\varphi(\tau)$ is a differentiable real-valued positive increasing function on $[\tau_0, \infty)$;
2. $\varphi'(\tau)$ is monotonic, continuous, positive on $[\tau_0, \infty)$ satisfying $\frac{1}{\varphi'(\tau)} = o(\tau)$, $\tau \to \infty$;
3. $\varphi(2\tau) \max_{\tau\leqslant t\leqslant 2\tau} \frac{1}{\varphi'(t)} \ll \tau$, $\tau \to \infty$.

Then the following result is true.

Theorem 2. *Suppose that $\varphi(\tau) \in U(\tau_0)$, $K \in \mathcal{K}$, $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T - \tau_0} \text{meas} \left\{\tau \in [\tau_0, T] : \sup_{s\in K} |\zeta(s + i\varphi(\tau), F) - f(s)| < \varepsilon\right\} > 0.$$

It is known [11], [12] that universality theorems can be stated in a slightly different form. Theorem 2 has the following modification which will be proved in the paper.

THEOREM 3. *Suppose that* $\varphi(\tau) \in U(\tau_0)$, $K \in \mathcal{K}$, $f(s) \in H_0(K)$. *Then the limit*

$$\lim_{T\to\infty} \frac{1}{T-\tau_0} \text{meas} \left\{ \tau \in [\tau_0, T] : \sup_{s\in K} |\zeta(s+i\varphi(\tau), F) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

In the following section, some lemmas necessary for the proof of the above mentioned theorems will be introduced.

2. Auxiliary results

Denote by $\mathcal{B}(X)$ the Borel σ-field of the space X, and by γ the unit circle on the complex plane. Define

$$\Omega = \prod_{p\in\mathbb{P}}^{\infty} \gamma_p,$$

where $\gamma_p = \gamma$ for all primes $p \in \mathbb{P}$. By the Tikhonov theorem, with product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ can be defined, and so we have a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of an element $\omega \in \Omega$ to the coordinate space γ_p, $p \in \mathbb{P}$, by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta, and on probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define the $H(D)$-valued random element $\zeta(s, \omega, F)$ by the formula

$$\zeta(s, \omega, F) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}.$$

Denote by $P_{\zeta,F}$, the distribution of $\zeta(s, \omega, F)$, i. e.,

$$P_{\zeta,F}(A) = m_H\{\omega \in \Omega : \zeta(s, \omega, F) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

Proof of the universality theorem is based on the weak convergence, as $T \to \infty$, for

$$P_{T,F}(A) = \frac{1}{T-\tau_0} \text{meas} \{\tau \in [\tau_0, T] : \zeta(s+i\varphi(\tau), F) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

THEOREM 4. *Suppose that* $\varphi(\tau) \in U(\tau_0)$. *Then* $P_{T,F}$ *converges weakly to* $P_{\zeta,F}$ *as* $T \to \infty$. *Moreover, the support of* $P_{\zeta,F}$ *is the set* $S_F = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

We divide the proof of Theorem 4 into several lemmas. The first of them is a limit theorem on the torus Ω. For the proof of this lemma, properties of the function $\varphi(\tau)$ are needed.

For $A \in \mathcal{B}(\Omega)$, define

$$Q_T(A) = \frac{1}{T-\tau_0} \text{meas} \left\{ \tau \in [\tau_0, T] : (p^{-i\varphi(\tau)} : p \in \mathbb{P}) \in A \right\}.$$

LEMMA 1. *Suppose that* $\varphi(\tau) \in U(\tau_0)$. *Then* Q_T *converges weakly to the Haar measure* m_H *as* $T \to \infty$.

Доказательство. [Proof] For the proof, we will apply the Fourier transform method. Let $g_T(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z},\, p \in \mathbb{P})$, be the Fourier transform of Q_T, i. e.,

$$g_T(\underline{k}) = \int_\Omega \left({\prod_{p\in\mathbb{P}}}' \omega^{k_p}(p) \right) dQ_T,$$

where "$\,'\,$" means that only a finite number of k_p are distinct from zero. Thus, from the definition of Q_T, we have

$$g_T(\underline{k}) = \frac{1}{T-\tau_0} \int_{\tau_0}^{T} \left({\prod_{p\in\mathbb{P}}}' p^{-ik_p\varphi(\tau)} \right) d\tau = \frac{1}{T-\tau_0} \int_{\tau_0}^{T} \exp\left\{ -i\varphi(\tau) {\sum_{p\in\mathbb{P}}}' k_p \log p \right\} d\tau, \tag{1}$$

Obviously,

$$g_T(\underline{0}) = 1. \tag{2}$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$, we have that

$$a := {\sum_{p\in\mathbb{P}}}' k_p \log p \neq 0$$

for all $\underline{k} \neq 0$.

Clearly,

$$\int_{\tau_0}^{T} \exp\{-ia\varphi(\tau)\}\, d\tau = \int_{\tau_0}^{T} \cos(a\varphi(\tau)) d\tau - i \int_{\tau_0}^{T} \sin(a\varphi(\tau)) d\tau. \tag{3}$$

Suppose that $\varphi'(\tau)$ is decreasing. Then, $\frac{1}{\varphi'(\tau)}$ is increasing, and therefore, by the mean value theorem,

$$\begin{aligned} \int_{\tau_0}^{T} \cos(a\varphi(\tau)) d\tau &= \frac{1}{a} \int_{\tau_0}^{T} \frac{a\varphi'(\tau)\cos(a\varphi(\tau))}{\varphi'(\tau)} d\tau = \frac{1}{a\varphi'(T)} \int_{\xi}^{T} a\varphi'(\tau)\cos(a\varphi(\tau)) d\tau \\ &= \frac{1}{a\varphi'(T)} \int_{\xi}^{T} d \sin(a\varphi(\tau)) = o(\tau) \end{aligned}$$

as $T \to \infty$, where $\tau_0 \leqslant \xi \leqslant T$. The same is also true for the second integral in (3). Thus, by (3),

$$\int_{\tau_0}^{T} \exp\{-ia\varphi(\tau)\}\, d\tau = o(\tau), \qquad T \to \infty. \tag{4}$$

Similarly, if $\varphi'(\tau)$ is increasing, then

$$\int_{\tau_0}^{T} \exp\{-ia\varphi(\tau)\}\, d\tau \ll \frac{1}{|a|\varphi(\tau_0)}. \tag{5}$$

From (4) and (5) together with (1), we get that

$$\lim_{T\to\infty} g_T(\underline{k}) = 0,$$

whenever $\underline{k} \neq \underline{0}$. Therefore, in view of (2),

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

The right-hand side of the latter equality is the Fourier transform of the Haar measure m_H. Therefore, the lemma follows from the continuity theorem for probability measures on compact groups. $\square$

Now, some absolutely convergent Dirichlet series will be analysed. Let $\theta > \frac{1}{2}$ be a fixed number, and $m, n \in \mathbb{N}$. We define series

$$\zeta_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m)v_n(m)}{m^s}$$

and

$$\zeta_n(s, \omega, F) = \sum_{m=1}^{\infty} \frac{c(m)\omega(m)v_n(m)}{m^s},$$

where

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\theta}\right\} \quad \text{and} \quad \omega(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

The latter series are absolutely convergent for $\sigma > \frac{\kappa}{2}$ [10]. Define the function $u_{n,F} : \Omega \to H(D)$ by the formula $u_{n,F}(\omega) = \zeta_n(s, \omega, F)$. Due to absolute convergence of $\zeta_n(s, \omega, F)$, we have that the function $u_{n,F}(\omega)$ is continuous, hence $(\mathcal{B}(\Omega), \mathcal{B}(H(D)))$-measurable. Therefore, the Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ induces the unique probability measure $\hat{P}_{n,F}$ on $(H(D), \mathcal{B}(H(D)))$ defined by

$$\hat{P}_{n,F}(A) = m_H u_{n,F}^{-1}(A) = m_H(u_{n,F}^{-1}A), \qquad A \in \mathcal{B}(H(D))).$$

Lemma 2. *Suppose that $\varphi(\tau) \in U(\tau_0)$. Then*

$$P_{T,n,F}(A) := \frac{1}{T - \tau_0} \text{meas}\left\{\tau \in [\tau_0, T] : \zeta_n(s + i\varphi(\tau), F) \in A\right\}, \qquad A \in \mathcal{B}(H(D))),$$

converges weakly to $\hat{P}_{n,F}$ as $T \to \infty$.

Доказательство. [Proof] The lemma is derived by standard arguments from Lemma 1 and the continuity of the function $u_{n,F}$. $\square$

Our aim is to prove that $P_{T,F}$ converges weakly to the limit measure P_F of the measure $\hat{P}_{n,F}$ as $n \to \infty$. For the proof of Theorem 4, approximation in the mean of $\zeta(s, F)$ by $\zeta_n(s, F)$ is used. Thus, the following estimate of the mean square is needed.

Lemma 3. *Suppose that $\varphi(\tau) \in U(\tau_0)$, and σ, $\frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}$, is fixed. Then, for all $t \in \mathbb{R}$,*

$$\int_{\tau_0}^{T} |\zeta(\sigma + it + i\varphi(\tau), F)|^2 d\tau \ll T(1 + |t|).$$

Доказательство. [Proof] It is known that, for fixed σ, $\frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}$,

$$\int_{\tau_0}^{T} |\zeta(\sigma + it, F)|^2 dt \ll T. \tag{6}$$

For $X > \tau_0$, we get

$$\begin{aligned}
\int_X^{2X} |\zeta(\sigma + it + i\varphi(\tau), F)|^2 d\tau &= \int_X^{2X} \frac{1}{\varphi'(\tau)} |\zeta(\sigma + it + i\varphi(\tau), F)|^2 d\varphi(\tau) \\
&\ll \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\varphi'(\tau)} \int_X^{2X} d\left(\int_0^{|t|+\varphi(\tau)} |\zeta(\sigma + iu, F)|^2 du\right) \\
&= \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\varphi'(\tau)} \left(\int_0^{|t|+\varphi(\tau)} |\zeta(\sigma + iu, F)|^2 du\right)\Bigg|_X^{2X}.
\end{aligned}$$

Consequently, by (6),

$$\int_0^{|t|+\varphi(\tau)} |\zeta(\sigma+iu,F)|^2 du\Big|_X^{2X} \ll_\sigma |t|+\varphi(2X),$$

and thus,

$$\int_X^{2X} |\zeta(\sigma+it+i\varphi(\tau),F)|^2 d\tau \ll (|t|+\varphi(2X)) \max_{X\leqslant\tau\leqslant 2X} \frac{1}{\varphi'(\tau)}$$
$$\ll X+|t| \max_{X\leqslant\tau\leqslant 2X} \frac{1}{\varphi'(\tau)} \ll X(1+|t|).$$

Taking $X=2^{-k-1}T$ and summing over $k=0,1,\dots$ prove the lemma. □

Now, we can approximate $\zeta(s,F)$ by $\zeta_n(s,F)$ in the mean. For $g_1, g_2 \in H(D)$, take

$$\rho(g_1,g_2)=\sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|},$$

where $\{K_l : l\in\mathbb{N}\}\subset D$ is a sequence of compact subsets such that

$$D=\bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l\in\mathbb{N}$, and if $K\subset D$ is a compact subset, then $K\subset K_l$ for some $l\in\mathbb{N}$. Then ρ is the metric in $H(D)$ inducing its topology of uniform convergence on compacta.

Lemma 4. *Suppose that $\varphi(\tau)\in U(\tau_0)$. Then*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T-\tau_0}\int_{\tau_0}^{T} \rho\left(\zeta(s+i\varphi(\tau),F),\zeta_n(s+i\varphi(\tau),F)\right)d\tau=0.$$

Доказательство. [Proof] Let θ be from the definition of $v_n(m)$, and

$$l_n(s)=\frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)n^s, \quad n\in\mathbb{N},$$

where $\Gamma(s)$ denotes the Euler gamma-function. Then the function $\zeta_n(s,F)$ has the representation [10]

$$\zeta_n(s,F)=\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z,F)l_n(z)\frac{dz}{z}, \quad \sigma>\frac{\kappa}{2}.$$

Let K be an arbitrary compact subset of D. Then, from the residue theorem and the above equality, we get

$$\frac{1}{T-\tau_0}\int_{\tau_0}^{T}\sup_{s\in K}\left(\zeta(s+i\varphi(\tau),F),\zeta_n(s+i\varphi(\tau),F)\right)d\tau$$
$$\ll \int_{\infty}^{\infty}|l_n(\hat\sigma+iu)|\left(\frac{1}{T-\tau_0}\int_{\tau_0}^{T}|\zeta(\sigma+it+iu+i\varphi(\tau),F)|d\tau\right)du,$$

as $T\to\infty$, where $\hat\sigma<0$, $\frac{\kappa}{2}<\sigma<\frac{\kappa+1}{2}$, and t is bounded by a constant depending on K. Lemma 3 implies that with $t\in\mathbb{R}$, for $\frac{\kappa}{2}<\sigma<\frac{\kappa+1}{2}$,

$$\int_{\tau_0}^{T}|\zeta(\sigma+it+iu+i\varphi(\tau),F)|d\tau \ll \left(T\int_{\tau_0}^{T}|\zeta(\sigma+it+iu+i\varphi(\tau),F)|^2 d\tau\right)^{1/2} \ll_{\sigma,K} T(1+|u|).$$

Therefore,

$$\frac{1}{T-\tau_0}\int_{\tau_0}^{T}\sup_{s\in K}\left(\zeta(s+i\varphi(\tau),F),\zeta_n(s+i\varphi(\tau),F)\right)d\tau \ll_{\sigma,K}\int_{\infty}^{\infty}|l_n(\hat{\sigma}+iu)|(1+|u|)du,$$

as $T\to\infty$. Hence,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T-\tau_0}\int_{\tau_0}^{T}\sup_{s\in K}\left(\zeta(s+i\varphi(\tau),F),\zeta_n(s+i\varphi(\tau),F)\right)d\tau=0.$$

So, the lemma follows from the definition of the metric ρ. □

Доказательство. [Proof of Theorem 4] Let ξ be a random variable uniformly distributed on $[0,1]$ and defined on a certain probability space with measure μ. Define the $H(D)$-valued random element $X_{T,n,F}$ by the formula

$$X_{T,n,F}=X_{T,n,F}(s)=\zeta_n(s+i\varphi(\xi T),F).$$

Then the assertion of Lemma 2 can be written as

$$X_{T,n,F}\xrightarrow[T\to\infty]{\mathcal{D}}\hat{X}_{n,F},\tag{7}$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution, and $\hat{X}_{n,F}$ is the $H(D)$-valued random element with the distribution $\hat{P}_{n,F}$. Here $\hat{P}_{n,F}$ is the same limit probability measure as in Lemma 2.

Now, we will prove that the family $\{\hat{P}_{n,F}:n\in\mathbb{N}\}$ is tight, i.e., for every $\varepsilon>0$, there exists a compact set $K=K(\varepsilon)\subset H(D)$ such that $\hat{P}_{n,F}(K)>1-\varepsilon$ for all $n\in\mathbb{N}$. Let $K\subset D$ be a compact set. Then, by the integral Cauchy formula,

$$\sup_{s\in K}|\zeta(s+i\varphi(\tau),F)|\ll\frac{1}{\delta_K}\int_{L_K}|\zeta(z+i\varphi(\tau),F)||dz|,$$

where L_K is a simple closed contour lying in D and enclosing the set K, and δ_K is the distance of L_K from the set K. Hence,

$$\int_{\tau_0}^{T}\sup_{s\in K}|\zeta(s+i\varphi(\tau),F)|d\tau\ll\frac{1}{\delta_K}\int_{L_K}|dz|\int_{\tau_0}^{T}|\zeta(\mathrm{Re}(z)+\mathrm{Im}(z)+i\varphi(\tau),F)|d\tau\ll_K T.$$

This with Lemma 4 shows that

$$\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{T-\tau_0}\int_{\tau_0}^{T}\sup_{s\in K_l}|\zeta_n(s+i\varphi(\tau),F)|d\tau\leqslant C_l<\infty,\tag{8}$$

where $\{K_l:l\in\mathbb{N}\}$ is the sequence of compact subsets of D from the definition of metric ρ.

Now, let the ε be an arbitrary positive number, and $M_l=M_l(\varepsilon)=C_l2^l\varepsilon^{-1}$. Then, from (8), we have

$$\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\mu\left\{\sup_{s\in K_l}|X_{T,n,F}(s)|>\varepsilon\right\}\leqslant\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{T-\tau_0}\int_{\tau_0}^{T}\sup_{s\in K_l}|\zeta_n(s+i\varphi(\tau),F)|d\tau\leqslant\frac{\varepsilon}{2^l},$$

and, by (7),

$$\mu\left\{\sup_{s\in K_l}|\hat{X}_{n,F}(s)|>\varepsilon\right\}\leqslant\frac{\varepsilon}{2^l}\tag{9}$$

for all $n \in \mathbb{N}$. Define the set $K = K(\varepsilon) = \left\{g \in H(D) : \sup_{s\in K_l} |g(s)| \leqslant M_l,\, l \in \mathbb{N}\right\}$. Then K is a compact set in $H(D)$, and, by (9),

$$\mu\left\{\hat{X}_{n,F} \in K\right\} \geqslant 1-\varepsilon$$

for all $n \in \mathbb{N}$, or, by definition of $\hat{X}_{n,F}$,

$$\hat{P}_{n,F}(K) \geqslant 1-\varepsilon$$

for all $n \in \mathbb{N}$, thus the family $\left\{\hat{P}_{n,F} : n \in \mathbb{N}\right\}$ is tight. Therefore, by the Prokhorov theorem (see Theorem 6.1 in [2]), it is relatively compact, i. e., every sequence of $\{\hat{P}_{n,F}\}$ contains a weakly convergent subsequence. Thus, there exists $\{\hat{P}_{n_r,F}\} \subset \{\hat{P}_{n,F}\}$ such that $\{\hat{P}_{n_r,F}\}$ converges weakly to a certain probability measure P_F on $(H(D), \mathcal{B}(H(D)))$ as $r \to \infty$, or, in terms of convergence in distribution, we say

$$\hat{X}_{n_r,F} \xrightarrow[r\to\infty]{\mathcal{D}} P_F \tag{10}$$

Define one more $H(D)$-valued random element

$$X_{T,F} = X_{T,F}(s) = \zeta_n(s+\varphi(\xi T), F).$$

Then, in view of Lemma 4, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mu\left\{\rho(X_{T,F}, X_{T,n,F}| \geqslant \varepsilon\right\} \\
&\quad = \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-\tau_0} \operatorname{meas}\left\{\tau \in [\tau_0, T] : \rho\left(\zeta(s+i\varphi(\tau), F), \zeta_n(s+i\varphi(\tau), F)\right) \geqslant \varepsilon\right\} \\
&\quad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{(T-\tau_0)\varepsilon} \int_{\tau_0}^{T} \rho\left(\zeta(s+i\varphi(\tau), F), \zeta_n(s+i\varphi(\tau), F)\right) d\tau = 0.
\end{aligned}$$

This together with (7) and (10) shows that all hypotheses of Theorem 4.2 of [2] are fulfilled, therefore,

$$X_{T,F} \xrightarrow[T\to\infty]{\mathcal{D}} P_F,$$

or $P_{T,F}$ converges weakly to the limit measure P_F of $\hat{P}_{n,F}$ as $T \to \infty$.

The final step is to identify the measure P_F. For this, we will use a simple observation. It is known [3], [7] that

$$\frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] : \zeta(s+i\tau, F) \in A\right\}, \qquad A \in \mathcal{B}(H(D))),$$

as $T \to \infty$, converges weakly to the limit measure P_F of $\hat{P}_{n,F}$, and that $P_F = P_{\zeta,F}$. Moreover, the support of $P_{\zeta,F}$ is the set S_F. Therefore, $P_{T,F}$ also converges weakly to $P_{\zeta,F}$ as $T \to \infty$. □

3. Proofs of universality theorems

Доказательство. [Proof of Theorem 2] Define the set

$$G_\varepsilon = \left\{g \in H(D) : \sup_{s\in K} |g(s) - e^{p(s)}| \leqslant \frac{\varepsilon}{2}\right\},$$

where $p(s)$ is a polynomial satisfying

$$\sup_{s\in K} \left|f(s) - e^{p(s)}\right| < \frac{\varepsilon}{2}. \tag{11}$$

The existence of $p(s)$ follows from the Mergelyan theorem on the approximation of analytic functions by polynomials (see [13]).

By the second part of Theorem 4, the function $e^{p(s)}$ belongs to the support of the measure $P_{\zeta,F}$. Therefore,

$$P_{\zeta,F}(G_\varepsilon) > 0. \tag{12}$$

Since G_ε is an open set, by the first part of Theorem 4 and the equivalent of weak convergence of probability measures in terms of open sets, we have that

$$\liminf_{T\to\infty} P_{T,F}(G_\varepsilon) \geqslant P_{\zeta,F}(G_\varepsilon).$$

This, the definition of $P_{T,F}$ and inequality (12) give

$$\liminf_{T\to\infty} \frac{1}{T-\tau_0} \text{meas}\left\{\tau \in [\tau_0, T] : \sup_{s\in K} \left|\zeta(s+i\varphi(\tau), F) - e^{p(s)}\right| < \frac{\varepsilon}{2}\right\} > 0.$$

This together with (11) proves the theorem. □

Доказательство. [Proof of Theorem 3] Define the set

$$\hat{G}_\varepsilon = \left\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\right\}.$$

Then the boundary $\partial\hat{G}_\varepsilon$ of $\hat{G}_\varepsilon$ lies in the set

$$\left\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon\right\}.$$

Therefore, $\partial\hat{G}_{\varepsilon_1} \cap \partial\hat{G}_{\varepsilon_2} = \varnothing$ for $\varepsilon_1 \neq \varepsilon_2$, $\varepsilon_1, \varepsilon_2 > 0$. Hence, for at most countably many $\varepsilon > 0$, the sets $\partial\hat{G}_\varepsilon$ have a positive $P_{\zeta,F}$ measure. Using Theorem 4 and equivalent of weak convergence of probability measures in terms of continuity sets, we obtain that

$$\lim_{T\to\infty} P_{T,F}(\hat{G}_\varepsilon) = P_{\zeta,F}(\hat{G}_\varepsilon) \tag{13}$$

for all but at most countably many $\varepsilon > 0$. Let G_ε be from the proof of Theorem 2. Then, in view of (11), we obtain that $G_\varepsilon \subset \hat{G}_\varepsilon$, and thus, by (12), $P_{\zeta,F}(\hat{G}_\varepsilon) > 0$. This, the definition of $P_{T,F}$ and (13) prove the theorem. □

4. Conclusions

In the paper, a generalized version of the Laurinčikas-Matsumoto universality theorem for zeta functions of certain cusp forms $\zeta(s, F)$ is proved in two different forms. Namely, it is shown that the shifts $\zeta(s + i\varphi(\tau), F)$, where $\varphi(\tau)$ belongs to a certain class of differentiable functions $U(\tau_0)$ can approximate with a given accuracy all non-vanishing analytic functions defined in the strip $\left\{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\right\}$, where κ is the weight of the form F, and the lower density of the set of such shifts is positive.

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