ЧЕБЫШЕВСКИЙ СБОРНИК

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Новое применение дисперсионного метода Линника

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Аннотация

Пусть α_m и β_n — две последовательности вещественных чисел с носителями на отрезках $[M, 2M]$ и $[N, 2N]$, где $M = X^{1/2-\delta}$ и $N = X^{1/2+\delta}$. Мы доказываем существование такой постоянной δ_0, что мультипликативная свертка α_m и β_n имеет уровень распределения $1/2 + \delta - \varepsilon$ (в слабом смысле), если только $0 \leqslant \delta < \delta_0$, последовательность β_n является последовательностью Зигеля-Вальфиша, и обе последовательности α_m и β_n ограничены сверху функцией делителей. Наш результат, таким образом, представляет собой общую дисперсионную оценку для "коротких" сумм II типа. Доказательство существенно использует дисперсионный метод Линника и недавние оценки трилинейных сумм с дробями Клоостермана, принадлежащие Беттин и Чанди. Также мы остановимся на применении полученного результата к проблеме делителей Титчмарша.

Ключевые слова: равнораспределение в арифметических прогрессиях, метод дисперсии.

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Another application of Linnik dispersion method

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Abstract

Let α_m and β_n be two sequences of real numbers supported on $[M, 2M]$ and $[N, 2N]$ with $M = X^{1/2-\delta}$ and $N = X^{1/2+\delta}$. We show that there exists a $\delta_0 > 0$ such that the multiplicative convolution of α_m and β_n has exponent of distribution $\frac{1}{2} + \delta - \varepsilon$ (in a weak sense) as long as $0 \le \delta < \delta_0$, the sequence β_n is Siegel-Walfisz and both sequences α_m and β_n are bounded above by divisor functions. Our result is thus a general dispersion estimate for "narrow" type-II sums. The proof relies crucially on Linnik's dispersion method and recent bounds for trilinear forms in Kloosterman fractions due to Bettin-Chandee. We highlight an application related to the Titchmarsh divisor problem.

Keywords: equidistribution in arithmetic progressions, dispersion method.

Bibliography: 10 titles.

In memoriam Professor Yu. V. Linnik (1915-1972)

1. Introduction

An important theme in analytic number theory is the study of the distribution of sequences in arithmetic progressions. A representative result in this field is the Bombieri-Vinogradov theorem [2], according to which for any $A > 0$,

$$\sum_{q \le Q} \max_{(a,q)=1} \left| \sum_{\substack{p \le x \\ p \equiv a \pmod q}} 1 - \frac{1}{\varphi(q)} \sum_{p \le x} 1 \right| \ll_A x(\log x)^{-A} \tag{1}$$

provided that $Q \le \sqrt{x}(\log x)^{-B}$ for some constant $B = B(A)$ depending on $A > 0$.

Nothing of the strength of (1) is known in the range $Q > x^{1/2+\varepsilon}$ for any fixed $\varepsilon > 0$ and already establishing for any fixed integer $a \ne 0$ and for all $A > 0$ the weaker estimate,

$$\sum_{q \le Q} \left| \sum_{\substack{p \le x \\ p \equiv a \pmod q}} 1 - \frac{1}{\varphi(q)} \sum_{p \le x} 1 \right| \ll_{a,A} x(\log x)^{-A} \tag{2}$$

with $Q = x^{1/2+\delta}$ and some $\delta > 0$ is a major open problem. If we could show (2) then we would say that *the primes have exponent of distribution $\frac{1}{2} + \delta$ in a weak sense*. However we note that there are results of this type if one allows to restrict the sum over $q \le Q$ in (2) to integers that are x^ε smooth, for a sufficiently small $\varepsilon > 0$ (see [16, 5]).

Any known approach to (2) goes through combinatorial formulas which decompose the sequence of prime numbers as a linear combination of multiplicative convolutions of other sequences (see for example [13, Chapter 13]). If one attempts to establish (2) by using such a combinatorial formula then one is led to the problem of showing that for any $A > 0$,

$$\sum_{q \le Q} \left| \sum_{\substack{M \le m \le 2M \\ N \le n \le 2N \\ mn \equiv a \pmod q}} \alpha_m \beta_n - \sum_{\substack{M \le m \le 2M \\ N \le n \le 2N \\ (mn,q)=1}} \alpha_m \beta_n \right| \ll X(\log X)^{-A} \ , \ X := MN \tag{3}$$

with $Q > X^{1/2+\varepsilon}$ for some $\varepsilon > 0$. In [14] Linnik developed his "dispersion method" to tackle such expressions. The method relies crucially on the bilinearity of the problem, followed by the use of various estimates for Kloosterman sums of analytic or algebraic origins. For a bound such as (3) to hold one needs to impose a "Siegel-Walfisz condition" on at least one of the sequences α_m or β_n.

DEFINITION 1. *We say that a sequence $\boldsymbol{\beta} = (\beta_n)$ satisfies a Siegel-Walfisz condition (alternatively we also say that $\boldsymbol{\beta}$ is Siegel-Walfisz), if there exists an integer $k > 0$ such that for any fixed $A > 0$, uniformly in $x \geq 2$, $q > |a| \geq 1, r \geq 1$ and $(a, q) = 1$, we have,*

$$\sum_{\substack{x < n \leq 2x \\ n \equiv a \pmod{q} \\ (n,r)=1}} \beta_n - \frac{1}{\varphi(q)} \sum_{\substack{x < n \leq 2x \\ (n,qr)=1}} \beta_n = O_A(\tau_k(r) \cdot x(\log x)^{-A}).$$

where $\tau_k(n) := \sum_{n_1 \ldots n_k = n} 1$ is the kth divisor function.

It is widely expected (see e.g [3, Conjecture 1]) that (3) should hold as soon as $\min(M, N) > X^\varepsilon$ provided that at least one of the sequences α_n, β_m is Siegel-Walfisz, and that there exists an integer $k > 0$ such that $|\alpha_m| \leq \tau_k(m)$ and $|\beta_n| \leq \tau_k(n)$ for all integers $m, n \geq 1$. We are however very far from proving a result of this type.

When $Q > X^{1/2+\varepsilon}$ for some $\varepsilon > 0$, there are only a few results establishing (3) unconditionally in specific ranges of M and N (precisely [9, Théorème 1], [3, Theorem 3], [11, Corollaire 1], [12, Corollary 1.1 (i)]). All the results that establish (3) unconditionally place a restriction on one of the variable N or M being much smaller than the other. We call such cases "unbalanced convolutions" and this forms the topic of our previous paper [12].

In applications a recurring range is one where M and N are roughly of the same size. This often corresponds to the case of "type II sums" in which one is permitted to exploit bilinearity but not much else. This is the range to which we contribute in this paper.

THEOREM 1. *Let $k \geq 1$ be an integer and $M, N \geq 1$ be given. Set $X = MN$. Let α_m and β_n be two sequences of real numbers supported respectively on $[M, 2M]$ and $[N, 2N]$. Suppose that $\boldsymbol{\beta} = (\beta_n)$ is Siegel–Walfisz and suppose that $|\alpha_m| \leq \tau_k(m)$ and $|\beta_n| \leq \tau_k(n)$ for all integers $m, n \geq 1$. Then, for every $\varepsilon > 0$ and every $A > 0$,*

$$\sum_{\substack{Q \leq q \leq 2Q \\ (q,a)=1}} \Big| \sum_{mn \equiv a \pmod{q}} \alpha_m \beta_n - \frac{1}{\varphi(q)} \sum_{(mn,q)=1} \alpha_m \beta_n \Big| \ll_A X(\log X)^{-A} \qquad (4)$$

uniformly in $N^{56/23}X^{-17/23+\varepsilon} \leq Q \leq NX^{-\varepsilon}$ and $1 \leq |a| \leq X$.

Setting $N = X^{1/2+\delta}$ and $M = X^{1/2-\delta}$ in Theorem 1 it follows from Theorem 1 and the Bombieri-Vinogradov theorem that (4) holds for all $Q \leq NX^{-\varepsilon}$ with $0 \leq \delta < \delta_0 := \frac{1}{112}$. Previously the existence of such a $\delta_0 > 0$ was established conditionally on Hooley's $R^\star$ conjecture on cancellations in short incomplete Kloosterman sums in [8, Théorème 1] and in that case one can take $\delta_0 = \frac{1}{14}$. Similarly to our previous paper, we use the work of Bettin-Chandee [1] and Duke-Friedlander-Iwaniec [7] as an unconditional substitute for Hooley's $R^\star$ conjecture. In fact the proof of Theorem 1 follows closely the proof of the conditional result in [8, Théorème 1] up to the point where Hooley's $R^\star$ conjecture is applied. Incidentally we notice that the largest Q that Theorem 1 allows to take is $Q = X^{17/33-5\varepsilon}$ provided that one chooses $N = X^{17/33-4\varepsilon}$.

Unfortunately the type-II sums that our Theorem 1 allows to estimate are too narrow to make Theorem 1 widely applicable in many problems (however see [15] for an interesting connection with cancellations in character sums). We record nonetheless below one corollary, which is related to Titchmarsh's divisor problem concerning the estimation of $\sum_{p \leq x} \tau_2(p-1)$ (for the best results on this problem see [10, Corollaire 2], [3, Corollary 1] and [6]). The proof of the Corollary below will be given in §5.

COROLLARY 1. *Let $k \geq 1$ and let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two sequences of real numbers as in Theorem 1. Let δ be a constant satisfying*

$$0 < \delta < \frac{1}{112},$$

and let
$$X \geq 2, \ M = X^{1/2-\delta}, \text{ and } N = X^{1/2+\delta}.$$

Then for every $A > 0$ we have the equality

$$\sum_{m \sim M} \sum_{n \sim N} \alpha_m \beta_n \tau_2(mn - 1) = 2 \sum_{q \geq 1} \frac{1}{\varphi(q)} \sum_{\substack{m \sim M, \, n \sim N, \, mn > q^2 \\ (mn,q)=1}} \alpha_m \beta_n + O\big(X(\log X)^{-A}\big).$$

2. Conventions and lemmas

2.1. Conventions

For M and $N \geq 1$, we put $X = MN$ and $\mathcal{L} = \log 2X$. Whenever it appears in the subscript of a sum the notation $n \sim N$ will means $N \leq n < 2N$. Given an integer $a \neq 0$ and two sequences $\boldsymbol{\alpha} = (\alpha_m)_{M \leq m < 2M}$ and $\boldsymbol{\beta} = (\beta_n)_{N \leq n < 2N}$ supported respectively on $[M, 2M]$ and $[N, 2N]$ we define the discrepancy

$$E(\boldsymbol{\alpha}, \boldsymbol{\beta}, M, N, q, a) := \sum_{\substack{m \sim M \\ mn \equiv a \bmod q}} \sum_{n \sim N} \alpha_m \beta_n - \frac{1}{\varphi(q)} \sum_{\substack{m \sim M \\ (mn,q)=1}} \sum_{n \sim N} \alpha_m \beta_n,$$

and we also define the mean-discrepancy,

$$\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta}, M, N, q, a) := \sum_{\substack{q \sim Q \\ (q,a)=1}} |E(\boldsymbol{\alpha}, \boldsymbol{\beta}, M, N, q, a)|. \tag{5}$$

Throughout η will denote any positive number the value of which may change at each occurence. The dependency on η will not be recalled in the O or $\ll$–symbols. Typical examples are $\tau_k(n) = O(n^\eta)$ or $(\log x)^{10} = O(x^\eta)$, uniformly for $x \geq 1$.

If f is a smooth real function, its Fourier transform is defined by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e(-\xi t) \, \mathrm{d}t,$$

where $e(\cdot) = \exp(2\pi i \cdot)$.

2.2. Lemmas

Our first lemma is a classical finite version of the Poisson summation formula in arithmetic progressions, with a good error term.

LEMMA 1. *There exists a smooth function $\psi : \mathbb{R} \longrightarrow \mathbb{R}^+$, with compact support equal to $[1/2, 5/2]$, larger than the characteristic function of the interval $[1, 2]$, equal to 1 on this interval such that, uniformly for integers a and $q \geq 1$, for $M \geq 1$ and $H \geq (q/M) \log^4 2M$ one has the equality*

$$\sum_{m \equiv a \bmod q} \psi\Big(\frac{m}{M}\Big) = \hat{\psi}(0)\frac{M}{q} + \frac{M}{q} \sum_{0 < |h| \leq H} e\Big(\frac{ah}{q}\Big) \hat{\psi}\Big(\frac{h}{q/M}\Big) + O(M^{-1}). \tag{6}$$

Furthermore, uniformly for $q \geq 1$ and $M \geq 1$ one has the equality

$$\sum_{(m,q)=1} \psi\Big(\frac{m}{M}\Big) = \frac{\varphi(q)}{q}\hat{\psi}(0)M + O\big(\tau_2(q) \log^4 2M\big). \tag{7}$$

Доказательство. See Lemma 2.1 of [12], inspired by [4, Lemma 7]. □ We now recall a classical lemma on the average behavior of the τ_k-function in arithmetic progressions (see [14, Lemma 1.1.5], for instance).

Lemma 2. *For every $k \geq 1$, for every $\varepsilon > 0$, there exists $C(k, \varepsilon)$ such that, for every $x \geq 2$, for every $x^\varepsilon < y < x$, for every $1 \leq q \leq yx^{-\varepsilon}$, for every integer a coprime with q, one has the inequality*

$$\sum_{\substack{x-y < n \leq x \\ n \equiv a \bmod q}} \tau_k(n) \leq C(k, \varepsilon) \frac{y}{\varphi(q)} (\log 2x)^{k-1}.$$

The following lemma is one of the various forms of the so–called Barban–Davenport–Halberstam Theorem (for a proof see for instance [3, Theorem 0 (a)].

Lemma 3. *Let $k > 0$ be an integer. Let $\boldsymbol{\beta} = (\beta_n)$ be a Siegel–Walfisz sequence such that $|\beta_n| \leq \tau_k(n)$ for all integer $n \geq 1$. Then for every $A > 0$ there exists $B = B(A)$ such that, uniformly for $N \geq 1$ one has the equality*

$$\sum_{q \leq N(\log 2N)^{-B}} \sum_{a,\, (a,q)=1} \Big| \sum_{\substack{n \sim N \\ n \equiv a \bmod q}} \beta_n - \frac{1}{\varphi(q)} \sum_{\substack{n \sim N \\ (n,q)=1}} \beta_n \Big|^2 = O_A\big(N(\log 2N)^{-A}\big).$$

We now recall an easy consequence of Weil's bound for Kloosterman sums.

Lemma 4. *Let a and b two integers ≥ 1. Let $\mathcal{I}$ an interval included in $[1, a]$. Then for every integer ℓ for every $\varepsilon > 0$ we have the inequality*

$$\sum_{\substack{n \in \mathcal{I} \\ (n,ab)=1}} \frac{n}{\varphi(n)} e\Big(\ell \frac{\overline{n}}{a}\Big) = O_\varepsilon\Big((\ell, a)^{\frac{1}{2}}(ab)^\varepsilon a^{\frac{1}{2}}\Big).$$

Доказательство. We begin we the case $b = 1$. We write the factor $\frac{n}{\varphi(n)}$ as

$$\frac{n}{\varphi(n)} = \sum_{\nu | n^\infty} \nu^{-1} = \sum_{\kappa(\nu)|n} \nu^{-1},$$

where $\kappa(\nu)$ is the largest squarefree integer dividing ν (sometimes $\kappa(\nu)$ is called the *kernel* of ν). This gives the equality

$$\Big| \sum_{\substack{n \in \mathcal{I} \\ (n,a)=1}} \frac{n}{\varphi(n)} e\Big(\ell \frac{\overline{n}}{a}\Big) \Big| \leq \sum_{\nu \geq 1} \nu^{-1} \Big| \sum_{\substack{n \in \mathcal{I} \\ \kappa(\nu)|n \\ (n,a)=1}} e\Big(\ell \frac{\overline{n}}{a}\Big) \Big| = \sum_{\substack{\nu \geq 1 \\ (\nu,a)=1}} \nu^{-1} \Big| \sum_{\substack{m \in \mathcal{I}/\kappa(\nu) \\ (m,a)=1}} e\Big(\ell \frac{\overline{\kappa(\nu)\,\overline{m}}}{a}\Big) \Big|.$$

In the summation we can restrict to the ν such that $\kappa(\nu) \leq a$. Applying the classical bound for short Kloosterman sums, we deduce that

$$\Big| \sum_{\substack{n \in \mathcal{I} \\ (n,a)=1}} \frac{n}{\varphi(n)} e\Big(\ell \frac{\overline{n}}{a}\Big) \Big| \ll_\varepsilon (\ell, a)^{\frac{1}{2}} a^{\frac{1}{2}+\varepsilon} \prod_{p \leq a} \Big(1 - \frac{1}{p}\Big)^{-1} \ll_\varepsilon (\ell, a)^{\frac{1}{2}} a^{\frac{1}{2}+2\varepsilon}.$$

This proves Lemma 4 in the case where $b = 1$. When $b \neq 1$, we use the Möbius inversion formula to detect the condition $(n, b) = 1$. □

Our central tool is a bound for trilinear forms for Kloosterman fractions, due to Bettin and Chandee [1, Theorem 1]. The result of Bettin-Chandee builds on work of Duke-Friedlander-Iwaniec [7, Theorem 2] who considered the case of bilinear forms. These two papers show cancellations in exponential sums involving Kloosterman fractions $e(a\overline{m}/n)$ with $m \asymp n$. We state below the main theorem of Bettin-Chandee.

Lemma 5. *For every $\epsilon > 0$ there exists $C(\varepsilon)$ such that for every non zero integer ϑ, for every sequences let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\nu}$ be of complex numbers, for every A, M and $N \geq 1$, one has the inequality*

$$\left| \sum_{a \sim A} \sum_{m \sim M} \sum_{n \sim N} \alpha(m)\beta(n)\nu(a)e\left(\vartheta \frac{a\overline{m}}{n}\right) \right| \leq C(\varepsilon)\|\boldsymbol{\alpha}\|_{2,M}\|\boldsymbol{\beta}\|_{2,N}\|\boldsymbol{\nu}\|_{2,A}$$

$$\times \left(1 + \frac{|\vartheta|A}{MN}\right)^{\frac{1}{2}}\left((AMN)^{\frac{7}{20}+\varepsilon}(M+N)^{\frac{1}{4}} + (AMN)^{\frac{3}{8}+\varepsilon}(AM+AN)^{\frac{1}{8}}\right).$$

3. Proof of Theorem 1

All along the proof we will suppose that the inequality $1 \leq |a| \leq X$ holds and that we also have

$$X^{\frac{3}{8}} \leq M \leq X^{\frac{1}{2}} \leq N \text{ and } Q \leq N. \tag{8}$$

3.1. Beginning of the dispersion

Without loss of generality we can suppose that the sequence $\boldsymbol{\beta}$ satisfies the following property

$$n \mid a \Rightarrow \beta_n = 0. \tag{9}$$

Such an assumption is justified because the contribution to $\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta}, M, N, Q, a)$ of the (q, m, n) such that $n \mid a$ is

$$\ll QX^\eta + X^\eta \sum_{n \mid a} \sum_{\substack{m \sim M \\ mn \neq a}} \tau_2(|mn - a|) + MX^\varepsilon \ll (M+Q)X^\eta.$$

By (5), we have the inequality

$$\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta}, M, N, Q, a) \leq \sum_{q \sim Q} \sum_{\substack{m \sim M \\ (m,q)=1}} |\alpha_m| \left| \sum_{\substack{n \sim N \\ n \equiv a\overline{m} \bmod q}} \beta_n - \frac{1}{\varphi(q)} \sum_{\substack{n \sim N \\ (n,q)=1}} \beta_n \right|.$$

Let ψ be the smooth function constructed in Lemma 1. By the Cauchy–Schwarz inequality, the inequality $|\alpha_m| \leq \tau_k(m)$ and by Lemma 2 we deduce

$$\Delta^2(\boldsymbol{\alpha}, \boldsymbol{\beta}, M, N, Q, a) \ll MQ\mathcal{L}^{k^2-1}\left\{W(Q) - 2V(Q) + U(Q)\right\}, \tag{10}$$

with

$$U(Q) = \sum_{(q,a)=1} \frac{\psi(q/Q)}{\varphi^2(q)}\left(\sum_{\substack{n \sim N \\ (n,q)=1}} \beta_n\right)^2 \sum_{(m,q)=1} \psi\left(\frac{m}{M}\right), \tag{11}$$

$$V(Q) = \sum_{(q,a)=1} \frac{\psi(q/Q)}{\varphi(q)}\left(\sum_{\substack{n_1 \sim N \\ (n_1,q)=1}} \beta_{n_1}\right)\left(\sum_{\substack{n_2 \sim N \\ (n_2,q)=1}} \beta_{n_2}\right) \sum_{m \equiv a\overline{n_1} \bmod q} \psi\left(\frac{m}{M}\right),$$

$$W(Q) = \sum_{(q,a)=1} \psi(q/Q)\left(\sum_{\substack{n_1 \sim N \\ (n_1,q)=1}} \beta_{n_1}\right)\left(\sum_{\substack{n_2 \sim N \\ (n_2,q)=1}} \beta_{n_2}\right) \sum_{\substack{m \equiv a\overline{n_1} \bmod q \\ m \equiv a\overline{n_2} \bmod q}} \psi\left(\frac{m}{M}\right). \tag{12}$$

3.2. Study of $U(Q)$

A direct application of (7) of Lemma 1 in the definition (11) gives the equality

$$U(Q) = \hat{\psi}(0)M \sum_{(q,a)=1} \frac{\psi(q/Q)}{q\varphi(q)}\left(\sum_{\substack{n \sim N \\ (n,q)=1}} \beta_n\right)^2 + O\left(N^2Q^{-1}X^\eta\right)$$

$$= U^{\mathrm{MT}}(Q) + O\left(N^2Q^{-1}X^\eta\right), \tag{13}$$

by definition.

3.3. Study of $V(Q)$

Let ε be a fixed positive number. We now apply (6) of Lemma 1 with

$$H = M^{-1}QX^{\varepsilon}. \tag{14}$$

This leads to the equality

$$V(Q) = V^{\mathrm{MT}}(Q) + V^{\mathrm{Err1}}(Q) + V^{\mathrm{Err2}}(Q), \tag{15}$$

where each of the three terms corresponds to the contribution of the three terms on the right hand–side of (6). We directly have the equality

$$V^{\mathrm{Err2}}(Q) = O\bigl(M^{-1}N^2X^{\eta}\bigr). \tag{16}$$

For the main term we get

$$V^{\mathrm{MT}}(Q) = \hat{\psi}(0)M \sum_{(q,a)=1} \frac{\psi(q/Q)}{q\varphi(q)} \Bigl(\sum_{\substack{n \sim N \\ (n,q)=1}} \beta_n \Bigr)^2. \tag{17}$$

By the definition of $V^{\mathrm{Err1}}(Q)$ we have the equality

$$V^{\mathrm{Err1}}(Q) = M \sum_{(q,a)=1} \frac{\psi(q/Q)}{q\varphi(q)} \Bigl(\sum_{\substack{n_2 \sim N \\ (n_2,q)=1}} \beta_{n_2} \Bigr)$$

$$\Bigl(\sum_{\substack{n_1 \sim N \\ (n_1,q)=1}} \beta_{n_1} \sum_{0 < |h| \leq H} \hat{\psi}\Bigl(\frac{h}{q/M}\Bigr) e\Bigl(\frac{ah\overline{n_1}}{q}\Bigr) \Bigr),$$

from which we deduce the inequality

$$\bigl| V^{\mathrm{Err1}}(Q) \bigr| \leq MQ^{-2} \sum_{n_1 \sim N} |\beta_{n_1}| \sum_{n_2 \sim N} |\beta_{n_2}| \sum_{0 < |h| \leq H} \bigl| \mathcal{V}(n_1, n_2, h) \bigr| \tag{18}$$

with

$$\mathcal{V}(n_1, n_2, h) = \sum_{(q,an_1n_2)=1} \psi(q/Q)\frac{Q^2}{q\varphi(q)}\hat{\psi}\Bigl(\frac{h}{q/M}\Bigr)e\Bigl(\frac{ah\overline{n_1}}{q}\Bigr).$$

Since $(q, n_1) = 1$ Bézout's relation gives the equality

$$\frac{ah\overline{n_1}}{q} = -ah\frac{\overline{q}}{n_1} + \frac{ah}{n_1 q} \quad \mod 1.$$

By the inequality $1 \leq |a| \leq X$ and by the definition of H, the derivative of the bounded function

$$t \mapsto \psi(t/Q)\frac{Q^2}{t^2}\hat{\psi}\Bigl(\frac{h}{t/M}\Bigr)e\Bigl(\frac{ah}{n_1 t}\Bigr)$$

is $\ll X^{\varepsilon}t^{-1}$ when $t \asymp Q$. This allows to make a partial summation over the variable q with the loss of a factor X^{ε}. After all these considerations, we see that there exists a subinterval $\mathcal{J} \subset [Q/2, 5Q/2]$ such that we have the inequality

$$\bigl| \mathcal{V}(n_1, n_2, h) \bigr| \ll X^{\varepsilon} \Bigl| \sum_{\substack{q \in \mathcal{J} \\ (q, n_1 n_2)=1}} \frac{q}{\varphi(q)}e\Bigl(ah\frac{\overline{q}}{n_1}\Bigr) \Bigr|.$$

Lemma 4 leads to the bound

$$\big|\, \mathcal{V}(n_1, n_2, h)\,\big| \ll X^\varepsilon (ah, n_1)^{\frac{1}{2}} (n_1 n_2)^\eta n_1^{\frac{1}{2}}.$$

Inserting this into (18), we obtain

$$V^{\mathrm{Err1}}(Q) \ll M N^{\frac{3}{2}} Q^{-2} X^{\varepsilon + \eta} \sum_{n_1 \sim N} |\beta_{n_1}| \sum_{0 < |h| \leq H} (h, n_1)^{\frac{1}{2}},$$

which finally gives

$$V^{\mathrm{Err1}}(Q) \ll N^{\frac{5}{2}} Q^{-1} X^{2\varepsilon + \eta} \tag{19}$$

using the inequality $|\beta_n| \leq \tau_k(n)$ and the definition of H. Combining (15), (16), (17) and (19) we obtain the equality

$$V(Q) = V^{\mathrm{MT}}(Q) + O_\varepsilon\big((M^{-1} N^2 + N^{\frac{5}{2}} Q^{-1}) X^{2\varepsilon + \eta}\big). \tag{20}$$

where $V^{\mathrm{MT}}(Q)$ is defined in (17) and where the constant implicit in the O_ε–symbol is uniform for a satisfying $1 \leq |a| \leq X$.

4. Study of $W(Q)$

4.1. The preparation of the variables

The conditions of the last summation in (12) imply the congruence restriction

$$n_1 \equiv n_2 \bmod q \quad \text{and} \quad (n_1 n_2, q) = 1. \tag{21}$$

In order to control the mutual multiplicative properties of n_1 and n_2 we decompose these variables as

$$\begin{cases} (n_1, n_2) = d, \\ n_1 = d\nu_1, \ n_2 = d\nu_2, \ (\nu_1, \nu_2) = 1, \\ \nu_1 = d_1 \nu_1' \ \text{with} \ d_1 \mid d^\infty \ \text{and} \ (\nu_1', d) = 1. \end{cases} \tag{22}$$

Thanks to $|\beta_n| \leq \tau_k(n)$ and to (9) the contribution of the pairs (n_1, n_2) with $d > X^\varepsilon$ to the right–hand side of (12) is negligible since it is

$$\ll X^\eta \sum_{\substack{X^\varepsilon < d \leq 2N}} \sum_{m \sim M} \sum_{\substack{\nu_1 \sim N/d \\ dm\nu_1 - a \neq 0}} \sum_{\substack{q \sim Q \\ q \mid d\nu_1 m - a}} \sum_{\substack{\nu_2 \sim N/d \\ \nu_2 \equiv \nu_1 \bmod q}} 1$$

$$\ll X^\eta \sum_{\substack{X^\varepsilon < d \leq 2N}} \sum_{m \sim M} \sum_{\substack{\nu_1 \sim N/d \\ dm\nu_1 - a \neq 0}} \tau_2(|d\nu_1 m - a|) \Big(\frac{N}{dQ} + 1 \Big)$$

$$\ll M N^2 Q^{-1} X^{\eta - \varepsilon} + X^{1 + \eta}. \tag{23}$$

Now consider the contribution of the pairs (n_1, n_2) with $d \leq X^\varepsilon$ and $d_1 > X^\varepsilon$ to the right–hand side of (12). It is

$$\ll X^\eta \sum_{\substack{d \leq X^\varepsilon \\ d_1 | d^\infty}} \sum_{X^\varepsilon < d_1 < 2N} \sum_{m \sim M} \sum_{\substack{\nu_1' \sim N/(dd_1) \\ dd_1 m \nu_1' - a \neq 0}} \sum_{\substack{q \sim Q \\ q | dd_1 \nu_1' m - a}} \sum_{\substack{\nu_2 \sim N/d \\ \nu_2 \equiv d_1 \nu_1' \bmod q}} 1$$

$$\ll X^\eta \sum_{\substack{d \leq X^\varepsilon \\ d_1 | d^\infty}} \sum_{X^\varepsilon < d_1 < 2N} \sum_{m \sim M} \sum_{\substack{\nu_1' \sim N/(dd_1) \\ dd_1 m \nu_1' - a \neq 0}} \tau_2(|dd_1 \nu_1' m - a|) \Big(\frac{N}{dQ} + 1 \Big)$$

$$\ll X^\eta M N^2 Q^{-1} \sum_{d \leq X^\varepsilon} \frac{1}{d^2} \sum_{\substack{d_1 > X^\varepsilon \\ d_1 | d^\infty}} \frac{1}{d_1} + X^\eta M N \sum_{d \leq X^\varepsilon} \frac{1}{d} \sum_{\substack{d_1 > X^\varepsilon \\ d_1 | d^\infty}} \frac{1}{d_1}$$

$$\ll M N^2 Q^{-1} X^{\eta - \frac{\varepsilon}{2}} + X^{1 + \eta - \frac{\varepsilon}{2}}. \tag{24}$$

Consider the conditions

$$d < X^\varepsilon \text{ and } d_1 < X^\varepsilon, \tag{25}$$

and the subsum $\widetilde{W}(Q)$ of $W(Q)$ where the variables n_1 and n_2 satisfy the condition (25). By (23) and (24) we have the equality

$$W(Q) = \widetilde{W}(Q) + O\big(MN^2Q^{-1}X^{\eta - \frac{\varepsilon}{2}} + X^{1+\eta}\big). \tag{26}$$

4.2. Expansion in Fourier series

We apply Lemma 1 to the last sum over m in (12) with H defined in (14). This decomposes $\widetilde{W}(Q)$ into the sum

$$\widetilde{W}(Q) = \widetilde{W}^{\mathrm{MT}}(Q) + \widetilde{W}^{\mathrm{Err1}}(Q) + \widetilde{W}^{\mathrm{Err2}}(Q), \tag{27}$$

where each of the three terms corresponds to the contribution of each term on the right–hand side of (6).

The easiest term is $\widetilde{W}^{\mathrm{Err2}}(Q)$ since, by $|\beta_n| \leq \tau_k(n)$ and (8), it satisfies the inequality

$$\widetilde{W}^{\mathrm{Err2}}(Q) \ll M^{-1} \sum_{Q/2 \leq q \leq 5Q/2} \sum_{\substack{n_1,\ n_1 \sim N \\ n_1 \equiv n_2 \bmod q}} \sum \tau_k(n_1) \tau_k(n_2)$$

$$\ll M^{-1} N^2 X^\eta \tag{28}$$

According to the restriction (21), we see that the main term is

$$\widetilde{W}^{\mathrm{MT}}(Q) = \hat{\psi}(0) M \sum_{(q,a)=1} \frac{\psi(q/Q)}{q} \sum_{(\delta,q)=1} \Big(\sum_{\substack{n_1,\ n_2 \sim N \\ n_1 \equiv n_2 \equiv \delta \bmod q}} \beta_{n_1} \beta_{n_2} \Big), \tag{29}$$

where the variables n_1 and n_2 satisfy the conditions (25). By a similar computation leading to (23) and (24) we can drop these conditions at the cost of the same error term. In other words the equality (29) can be written as

$$\widetilde{W}^{\mathrm{MT}}(Q) = W^{\mathrm{MT}}(Q) + O\big(MN^2Q^{-1}X^{\eta - \frac{\varepsilon}{2}} + X^{1+\eta}\big), \tag{30}$$

where $W^{\mathrm{MT}}(Q)$ is the new main term, which is defined by

$$W^{\mathrm{MT}}(Q) = \hat{\psi}(0) M \sum_{(q,a)=1} \frac{\psi(q/Q)}{q} \sum_{(\delta,q)=1} \Big(\sum_{\substack{n \sim N \\ n \equiv \delta \bmod q}} \beta_n \Big)^2. \tag{31}$$

4.3. Dealing with the main terms

We now gather the main terms appearing in (13), (17), (26), (27), (30), and in (31). The main term of $W(Q) - 2V(Q) + U(Q)$ is

$$W^{\mathrm{MT}}(Q) - 2V^{\mathrm{MT}}(Q) + U^{\mathrm{MT}}(Q)$$

$$= \hat{\psi}(0) M \sum_{(q,a)=1} \frac{\psi(q/Q)}{q} \sum_{(\delta,q)=1} \Big(\sum_{\substack{n \sim N \\ n \equiv \delta \bmod q}} \beta_n - \frac{1}{\varphi(q)} \sum_{\substack{n \sim N \\ (n,q)=1}} \beta_n \Big)^2.$$

Appealing to Lemma 3 we deduce that, for any A, we have the equality

$$W^{\mathrm{MT}}(Q) - 2V^{\mathrm{MT}}(Q) + U^{\mathrm{MT}}(Q) = O\Big(M \cdot Q^{-1} \cdot N^2 (\log 2N)^{-A} \Big) \tag{32}$$

provided that

$$Q \le N(\log 2N)^{-B}, \tag{33}$$

for some $B = B(A)$.

4.4. Preparation of the exponential sums

By the definition (27), we have the equality

$$\widetilde{W}^{\mathrm{Err1}}(Q) = M \sum_q \frac{\psi(q/Q)}{q} \sum_{\substack{n_1, \, n_2 \sim N \\ n_1 \equiv n_2 \bmod q}} \beta_{n_1} \beta_{n_2} \sum_{0 < |h| \le H} \hat{\psi}\Big(\frac{h}{q/M}\Big) e\Big(\frac{ah\overline{n_1}}{q}\Big),$$

where the variables (n_1, n_2) are such the associated d and d_1 satisfy (25).

This implies that any pair (n_1, n_2) satisfies $n_1 - n_2 \ne 0$ and since we have $n_1 \equiv n_2 \bmod q$ (see (21)) these integers cannot be near to each other, indeed they satisfy the inequality

$$|n_1 - n_2| \ge Q/2.$$

Since we have $(n_1 n_2, q) = 1$, we can equivalently write the congruence $n_1 - n_2 \equiv 0 \bmod q$ as

$$\nu_1 - \nu_2 = d_1 \nu_1' - \nu_2 = qr, \tag{34}$$

and instead of summing over q, we will sum over r. Note that $1 \le |r| \le R/d$, where

$$R = 2NQ^{-1}. \tag{35}$$

In the summations, the pair of variables (n_1, n_2) is replaced by the quadruple (d, d_1, ν_1', ν_2) (see (22)). The variables d and d_1 are small, so we expect no substantial cancellations when summing over them. Hence for some

$$d, d_1 \le X^\varepsilon, \ d_1 \mid d^\infty,$$

we have the inequality

$$\widetilde{W}^{\mathrm{Err1}}(Q) \ll X^{2\varepsilon} M Q^{-1} \big| \mathcal{W} \big|, \tag{36}$$

where $\mathcal{W} = \mathcal{W}(d, d_1)$ is the quadrilinear form in the four variables r, ν_1', ν_2 and h defined by

$$\mathcal{W} = \sum_{1 \le |r| \le R/d} \sum_{\substack{dd_1 \nu_1', \, d\nu_2 \sim N \\ d_1 \nu_1' \equiv \nu_2 \bmod r}} \beta_{dd_1 \nu_1'} \beta_{d\nu_2} \frac{\psi\big((d_1 \nu_1' - \nu_2)/(rQ)\big)}{(d_1 \nu_1' - \nu_2)/(rQ)}$$

$$\sum_{0 < |h| \le H} \hat{\psi}\Big(\frac{h}{(d_1 \nu_1' - \nu_2)/(rM)}\Big) e(\cdot), \tag{37}$$

where $e(\cdot)$ is the oscillating factor

$$e(\cdot) = e\Big(\frac{ah\,\overline{dd_1\nu_1'}}{(d_1\nu_1' - \nu_2)/r}\Big),$$

and where the variables satisfy the following divisibility conditions:

$$(d_1\nu_1', \nu_2) = 1, \ (\nu_1', d) = 1 \text{ and } (dd_1\nu_1'r, d_1\nu_1' - \nu_2) = r.$$

Using Bézout's reciprocity formula we transform the factor $e(\cdot)$ as follows:

$$\frac{ah\,\overline{dd_1\nu_1'}}{(d_1\nu_1' - \nu_2)/r} = -ah\frac{\overline{(d_1\nu_1' - \nu_2)/r}}{dd_1\nu_1'} + \frac{ahr}{dd_1\nu_1'(d_1\nu_1' - \nu_2)} \text{ mod } 1.$$

Since $(dd_1, \nu_1') = (r, \nu_1') = 1$ we can apply Bézout formula again, giving the equalities

$$ah\frac{\overline{(d_1\nu_1' - \nu_2)/r}}{dd_1\nu_1'} = ah\frac{\overline{\nu_1'}\,\overline{(d_1\nu_1' - \nu_2)/r}}{dd_1} + ah\frac{\overline{dd_1}\,\overline{(d_1\nu_1' - \nu_2)/r}}{\nu_1'} \text{ mod } 1$$

$$= ah\frac{\overline{\nu_1'}\,\overline{(d_1\nu_1' - \nu_2)/r}}{dd_1} - ah\frac{r\overline{dd_1}\nu_2}{\nu_1'} \text{ mod } 1$$

The first term on the right–hand side of the above equality depends only on the congruences classes of a, h, r, ν_1' and ν_2 modulo dd_1. As a consequence of the above discussion, we see that there exists a coefficient $\xi = \xi(a, h, r, \nu_1', \nu_2)$ of modulus 1, depending only on the congruence classes of a, h, r, ν_1' and ν_2 modulo dd_1 such that we have the equality

$$e(\cdot) = \xi \cdot e\Big(\frac{ahr}{dd_1\nu_1'(d_1\nu_1' - \nu_2)}\Big) \cdot e\Big(\frac{ahr\overline{dd_1}\nu_2}{\nu_1'}\Big).$$

Returning to (37), and fixing the congruences classes modulo dd_1 of the variables h, r, ν_1' and ν_2, we see that there exists

$$0 \le a_1, a_2, a_3, a_4 < dd_1$$

such that $\mathcal{W}$ satisfies the inequality,

$$|\mathcal{W}| \le X^{6\varepsilon}$$

$$\sum_{\substack{1 \le |r| \le R/d \\ r \equiv a_1 \bmod dd_1}} \Big| \sum_{\substack{dd_1\nu_1', d\nu_2 \sim N \\ d_1\nu_1' \equiv \nu_2 \bmod r \\ \nu_1' \equiv a_2 \bmod dd_1 \\ \nu_2 \equiv a_3 \bmod dd_1}} \beta_{dd_1\nu_1'}\beta_{d\nu_2} \sum_{\substack{1 \le |h| \le H \\ h \equiv a_4 \bmod dd_1}} \Psi_r(h, \nu_1', \nu_2)e\Big(\frac{ahr\overline{dd_1}\nu_2}{\nu_1'}\Big)\Big|, \quad (38)$$

where Ψ_r is the differentiable function

$$\Psi_r(h, \nu_1', \nu_2) = \frac{\psi\big((d_1\nu_1' - \nu_2)/(rQ)\big)}{(d_1\nu_1' - \nu_2)/(rQ)}\,\hat{\psi}\Big(\frac{h}{(d_1\nu_1' - \nu_2)/(rM)}\Big)e\Big(\frac{ahr}{dd_1\nu_1'(d_1\nu_1' - \nu_2)}\Big),$$

In order to perform the Abel summation over the variables ν_1', ν_2 and h (see for instance [9, Lemme 5]) we must have information on the partial derivatives of the Ψ_r–function. Indeed for $0 \le \epsilon_0, \epsilon_1, \epsilon_2 \le 1$, we have the inequality

$$\frac{\partial^{\epsilon_0 + \epsilon_1 + \epsilon_2}}{\partial h^{\epsilon_0} \partial \nu_1'^{\epsilon_1} \partial \nu_2^{\epsilon_2}}\Psi_r(h, \nu_1', \nu_2) \ll X^{50\varepsilon}|h|^{-\epsilon_0}\,\nu_1'^{-\epsilon_1}\,\nu_2^{-\epsilon_2}\big(N/(rQ)\big)^{\epsilon_1 + \epsilon_2}, \quad (39)$$

as a consequence of the inequality $|d_1\nu_1' - \nu_2| \geq rQ/2$ (see(34)), of the definition of H (see (14)) and of the inequality $1 \leq |a| \leq X$.

Since $(d_1\nu_1'\nu_2, r) = 1$ we detect the congruence $d_1\nu_1' \equiv \nu_2 \bmod r$ by the $\varphi(r)$ Dirichlet characters χ modulo r. By (39) we eliminate the function Ψ_r in the inequality (38) which becomes

$$|\mathcal{W}| \leq X^{60\varepsilon} N^2 Q^{-2} \sum_{\substack{1 \leq |r| \leq R/d \\ r \equiv a_1 \bmod dd_1}} \frac{1}{\varphi(r)\, r^2} \sum_{\chi \bmod r}$$

$$\Big| \sum_{\substack{dd_1\nu_1' \in \mathcal{N}_1 \\ \nu_1' \equiv a_2 \bmod dd_1}} \sum_{\substack{d\nu_2 \in \mathcal{N}_2 \\ \nu_2 \equiv a_3 \bmod dd_1}} \chi(d\nu_1')\overline{\chi}(\nu_2)\beta_{dd_1\nu_1'}\beta_{d\nu_2} \sum_{\substack{h \in \mathcal{H} \\ h \equiv a_4 \bmod dd_1}} e\Big(\frac{ahr\overline{dd_1\nu_2}}{\nu_1'}\Big)\Big|, \quad (40)$$

- where $\mathcal{N}_1$ and $\mathcal{N}_2$ are two intervals included in $[N, 2N]$,
- and where $\mathcal{H}$ is the union of two intervals included in $[-H, -1]$ and $[1, H]$ respectively.

Denote by $\mathcal{W}_1(r, \chi)$ the inner sum over ν_1', ν_2 and h in (40). Remark that the trivial bound for $\mathcal{W}_1(r, \chi)$ is $O(X^\eta HN^2/(d^2d_1))$. We now can apply Lemma 5 to the sum $\mathcal{W}_1(r, \chi)$, with the choice of parameters

$$\vartheta \to ar, \ A \to H, M \to N \text{ and } N \to N.$$

We obtain the bound

$$\mathcal{W}_1(r, \chi) \ll H^{\frac{1}{2}} N^{\frac{1}{2}} N^{\frac{1}{2}} X^{\varepsilon+\eta}\Big(1 + \frac{|a|\,|r|H}{N^2}\Big)\Big((HN^2)^{\frac{7}{20}+\varepsilon} N^{\frac{1}{4}} + (HN^2)^{\frac{3}{8}+\varepsilon}(HN)^{\frac{1}{8}}\Big).$$

By the definition (35), (14) and the inequality $1 \leq |a| \leq X$ we deduce the inequality

$$\mathcal{W}_1(r, \chi) \ll X^{4\varepsilon+\eta}\big(H^{\frac{17}{20}} N^{\frac{39}{20}} + HN^{\frac{15}{8}}\big),$$

and using (14) we finally deduce

$$\mathcal{W}_1(r, \chi) \ll X^{5\varepsilon+\eta}\big(M^{-\frac{17}{20}} N^{\frac{39}{20}} Q^{\frac{17}{20}} + M^{-1} N^{\frac{15}{8}} Q\big).$$

Returning to (40), summing over χ and r and inserting into (36) we obtain the bound

$$\widetilde{W}^{\mathrm{Err1}}(Q) \ll X^{67\varepsilon+\eta}\big(M^{\frac{3}{20}} N^{\frac{79}{20}} Q^{-\frac{43}{20}} + N^{\frac{31}{8}} Q^{-2}\big). \tag{41}$$

4.5. Conclusion

We have now all the elements to bound $\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta}, M, N, Q, a)$. By (10), (13), (20), (26), (27), (28), (30) and (41) we have the inequality

$$\Delta^2 \ll MQ\mathcal{L}^{k^2-1}\Big\{\Big(W^{\mathrm{MT}}(Q) - 2V^{\mathrm{MT}}(Q) + U^{\mathrm{MT}}(Q)\Big) + N^2 Q^{-1} X^\eta$$
$$+ (M^{-1}N^2 + N^{\frac{5}{2}} Q^{-1})X^{2\varepsilon+\eta}$$
$$+ \big(MN^2 Q^{-1} X^{\eta-\frac{\varepsilon}{2}} + X^{1+\eta}\big) + X^{67\varepsilon+\eta}\big(M^{\frac{3}{20}} N^{\frac{79}{20}} Q^{-\frac{43}{20}} + N^{\frac{31}{8}} Q^{-2}\big)\Big\},$$

which is shortened in (recall (8))

$$\Delta^2 \ll MQ\mathcal{L}^{k^2-1}\Big\{MN^2 Q^{-1}(\log 2N)^{-A}+$$
$$+ MN^2 Q^{-1} X^{\eta-\frac{\varepsilon}{2}} + X^{67\varepsilon+\eta}\big(M^{\frac{3}{20}} N^{\frac{79}{20}} Q^{-\frac{43}{20}} + N^{\frac{31}{8}} Q^{-2}\big)\Big\},$$

by (32) and (33) if one assumes

$$Q \le NX^{-\varepsilon}. \tag{42}$$

To finish the proof of Theorem 1, it remains to find sufficient conditions over M, N and Q to ensure the bound $\Delta^2 \ll M^2 N^2 \mathcal{L}^{-A}$. Choosing $\eta = \varepsilon/5$, we have to study the following three inequalities hold

$$\begin{cases} MQ \cdot MN^2 Q^{-1} X^{-\frac{\varepsilon}{4}} & \ll M^2 N^2 X^{-\frac{\varepsilon}{4}}, \\ MQ \cdot M^{\frac{3}{20}} N^{\frac{79}{20}} Q^{-\frac{43}{20}} X^{68\varepsilon} & \ll M^2 N^2 X^{-\frac{\varepsilon}{4}}, \\ MQ \cdot N^{\frac{31}{8}} Q^{-2} X^{68\varepsilon} & \ll M^2 N^2 X^{-\frac{\varepsilon}{4}}. \end{cases} \tag{43}$$

The first inequality is trivially satisfied. The second inequality of (43) is satisfied as soon as

$$Q > N^{\frac{56}{23}} X^{-\frac{17}{23}+65\varepsilon}. \tag{44}$$

This inequality combined with (42) implies that $N < X^{\frac{17}{33}}$. The last condition of (43) is satisfied as soon as

$$Q > N^{\frac{23}{8}} X^{-1+69\varepsilon}.$$

We can drop this condition since it is a consequence of (44) and of the inequality $N < X^{\frac{17}{33}}$. The proof of Theorem 1 is now complete.

5. Proof of Corollary 1

Let $S(M, N)$ be the sum we are studying in this corollary. We use Dirichlet's hyperbola argument to write

$$mn - 1 = qr, \tag{45}$$

and by symmetry we can impose the condition $q < r$. This symmetry creates a factor 2 unless $mn - 1$ is a perfect square. The contribution to $S(M, N)$ of the (m, n) such that $mn - 1$ is a square is bounded by $O(X^{\frac{1}{2}+\eta})$ with $\eta > 0$ arbitrary. This is a consequence of $|\beta_n| \le \tau_k(n)$.

The decomposition (45), the constraint $q < r$ and the inequalities $X - 1 \le mn - 1 < 4X$ imply that $q \le 2X^{\frac{1}{2}}$. In counterpart, if $q < X^{\frac{1}{2}}$ we are sure that $q < r$. Thus we have the equality

$$S(M, N) = 2 \sum_{\substack{q \le X^{1/2}}} \sum_{\substack{m \sim M \\ mn \equiv 1 \bmod q}} \sum_{\substack{n \sim N}} \alpha_m \beta_n + 2 \sum_{\substack{mn-1=qr, q<r \\ m \sim M, \ n \sim N, X^{1/2} < q \le 2X^{1/2}}} \sum \sum \sum \alpha_m \beta_n + O(X^{\frac{1}{2}+\eta})$$

$$= 2S_0(M, N) + 2S_1(M, N) + O(X^{\frac{1}{2}+\eta}), \tag{46}$$

by definition. A direct application of Theorem 1 with $Q = X^{\frac{1}{2}}$ gives the equality

$$S_0(M, N) = \sum_{\substack{q \le X^{1/2}}} \frac{1}{\varphi(q)} \sum_{\substack{m \sim M}} \sum_{\substack{n \sim N \\ (mn,q)=1}} \alpha_m \beta_n + O(X\mathcal{L}^{-C}), \tag{47}$$

for any C.

For the second term $S_1(M, N)$, we must get rid of the constraint $q < r$. A technique among others is to precisely control the size of the variables m, n and q. If it is so, then $r = (mn-1)/q$ is also controlled and one can check if it satisfies $r > q$. We introduce the following factor of dissection:

$$\Delta = 2^{\frac{1}{[\mathcal{L}^B]}},$$

where $B = B(A)$ is a parameter to be fixed later, and where $[y]$ is the largest integer $\leq y$. If we denote by $L_0 = [\mathcal{L}^B]$ we see that $\Delta^{L_0} = 2$ and that $\Delta = 1 + O(\mathcal{L}^{-B})$. We denote by M_0, N_0 and Q_0 any numbers in the sets

$$\mathcal{M}_0 := \{M, \Delta M, \Delta^2 M, \Delta^3 M, \cdots, \Delta^{L_0-1} M\}$$
$$\mathcal{N}_0 := \{N, \Delta N, \Delta^2 N, \Delta^3 N, \cdots, \Delta^{L_0-1} N\}$$
$$\mathcal{Q}_0 := \{X^{\frac{1}{2}}, \Delta X^{\frac{1}{2}}, \Delta^2 X^{\frac{1}{2}}, \Delta^3 X^{\frac{1}{2}}, \cdots, \Delta^{L_0-1} X^{\frac{1}{2}}\},$$

respectively. We split $S_1(M, N)$ into

$$S_1(M, N) = \sum_{M_0 \in \mathcal{M}_0} \sum_{N_0 \in \mathcal{N}_0} \sum_{Q_0 \in \mathcal{Q}_0} S_1(M_0, N_0, Q_0), \tag{48}$$

where $S_1(M_0, N_0, Q_0)$ is defined by

$$S_1(M_0, N_0, Q_0) = \sum_{q \simeq Q_0} \sum_{\substack{m \simeq M_0, \ n \simeq N_0 \\ mn \equiv 1 \bmod q}} \alpha_m \beta_n.$$

- where the notation $y \simeq Y_0$ means that the integer y satisfies the inequalities $Y_0 \leq y < \Delta Y_0$,

- where the variables m, n and q satisfy the extra condition

$$mn - 1 > q^2. \tag{49}$$

Note that the decomposition (48) contains

$$O(\mathcal{L}^{3B}), \tag{50}$$

terms.

Since $mn - 1 \geq M_0 N_0 - 1$ and $q^2 < Q_0^2 \Delta^2$ in each sum $S_1(M_0, N_0, Q_0)$, we can drop the condition (49) in the definition of this sum as soon as we have

$$M_0 N_0 - 1 > Q_0^2 \Delta^2. \tag{51}$$

When (51) is satisfied, the variables m, n and q are independent and a direct application of Theorem 1 gives for each sum $S_1(M_0, N_0, Q_0)$, the equality

$$S_1(M_0, N_0, Q_0) = \sum_{q \simeq Q_0} \frac{1}{\varphi(q)} \sum_{\substack{m \simeq M_0, \ n \simeq N_0 \\ (mn, q) = 1}} \alpha_m \beta_n + O_C(X \mathcal{L}^{-C}), \tag{52}$$

where C is arbitrary.

It remains to consider the case where (51) is not satisfied, which means that $(M_0, N_0, Q_0) \in \mathcal{E}_0$ where

$$\mathcal{E}_0 := \big\{ (M_0, N_0, Q_0); \ M_0 N_0 - 1 \leq Q_0^2 \Delta^2 \big\}. \tag{53}$$

We now show that the variable n considered in such a $S_1(M_0, N_0, Q_0)$ varies in a rather short interval. More precisely, since $M_0 \Delta > m$, $N_0 \Delta > n$ and $Q_0 < q$ we deduce from the definition (53) that $q^2 \geq mn\Delta^{-4} - \Delta^{-2}$ which implies the inequality $q \geq (mn)^{\frac{1}{2}} \Delta^{-2} - 1$. Combining with (49), we get the inequality

$$(mn)^{\frac{1}{2}} \Delta^{-2} - 1 < q < (mn)^{\frac{1}{2}}$$

which implies

$$(q^2/m) < n < ((q+1)^2/m)\Delta^4.$$

Using the inequality

$$X^{1/2} \le q \le 2X^{1/2} \ll (Q^2/M)(\Delta^4 - 1)X^{-\frac{\delta}{2}},$$

and $|\beta_n| \le \tau_k(n)$ we apply Lemma 2 to see that

$$
\sum_{(M_0,N_0,Q_0)\in\mathcal{E}_0} \sum \sum S_1(M_0, N_0, Q_0)
$$

$$
\ll \sum_{m\sim M} \tau_k(m) \sum_{\substack{q\sim X^{1/2}\\(q,m)=1}} \sum_{(q^2/m)<n<((q+1)^2/m)\Delta^4} \tau_k(n)
$$

$$
\ll (\Delta^4 - 1)\mathcal{L}^{k-1} \sum_{m\sim M} \tau_k(m) \sum_{q\sim X^{1/2}} \frac{1}{\varphi(q)} \cdot \frac{q^2}{m}
$$

$$
\ll \mathcal{L}^{2k-2-B}X.
$$

Actually, by introducing a main term back, which is less than the error term, we can also write this bound as an equality

$$
\sum_{(M_0,N_0,Q_0)\in\mathcal{E}_0} \sum \sum S_1(M_0, N_0, Q_0)
$$

$$
= \sum_{(M_0,N_0,Q_0)\in\mathcal{E}_0} \sum \sum \sum_{q\simeq Q_0} \frac{1}{\varphi(q)} \sum_{\substack{m\simeq M_0,\\(mn,q)=1}} \sum_{n\simeq N_0} \alpha_m\beta_n + O(\mathcal{L}^{2k-2-B}), \quad (54)
$$

where the variables (m, n, q) continue to satisfy (49).

Gathering (46), (47), (48), (50), (52), (54) we obtain

$$
S(M, N) = 2 \sum_{q\le X^{1/2}} \frac{1}{\varphi(q)} \sum_{\substack{m\sim M\\(mn,q)=1}} \sum_{n\sim N} \alpha_m\beta_n
$$

$$
+ 2 \sum_{M_0\in\mathcal{M}_0} \sum_{N_0\in\mathcal{N}_0} \sum_{Q_0\in\mathcal{Q}_0} \sum_{q\simeq Q_0} \frac{1}{\varphi(q)} \sum_{\substack{m\simeq M_0,\\(mn,q)=1}} \sum_{n\simeq N_0} \alpha_m\beta_n
$$

$$
+ O(\mathcal{L}^{3B-C}X) + O(\mathcal{L}^{2k-2-B}X) + O(X^{\frac{1}{2}+\eta}),
$$

where the variables (m, n, q) continue to satisfy (49). Putting the different summations back together, we complete the proof of Corollary 1 by choosing B and C in order to satisfy the equalities $-A = 3B - C = 2k - 2 - B$.

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