ЧЕБЫШЕВСКИЙ СБОРНИК
Том 19. Выпуск 3.

УДК 511.3 DOI 10.22405/2226-8383-2018-19-3-74-79

О неполных рациональных тригонометрических суммах

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Аннотация

Приводится версия метода Хуа для оценки неполных рациональных тригонометрических сумм. Эти оценки не являются тривиальными для суммы с длинами, превышающими квадратный корень длины полной суммы.

Ключевые слова: метод Хуа оценки полных рациональных тригонометрических сумм, неполные рациональные тригонометрические суммы, полиномиальные сравнения, цепь показателей и корни сравнений.

Библиография: 10 названий.

Для цитирования:
Х. М. Салиба. О неполных рациональных тригонометрических суммах // Чебышевский сборник, 2018, т. 19, вып. 3, с. 74–79.

CHEBYSHEVSKII SBORNIK
Vol. 19. No. 3.

UDC 511.3 DOI 10.22405/2226-8383-2018-19-3-74-79

On non-complete rational trigonometric sums

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Abstract

We give the version of Hua's method for the estimation of non-complete rational trigonometric sums. These estimates are non-trivial one for sums with lengths exceeding a square root of length the complete sum.

Keywords: the Hua's method of complete rational trigonometric sums estimate, non-complete rational trigonometric sums, polynomial congruencies, the chain of exponents and roots of congruencies.

Bibliography: 10 titles.

For citation:
Saliba H. M., 2018, "On non-complete rational trigonometric sums" , *Chebyshevskii sbornik*, vol. 19, no. 3, pp. 74–79.

To the memory of Academician J. V. Linnik

1. Introduction

The purpose of this article is to give the new demonstration of the estimation of non-complete rational trigonometric sums. Early the deduction of similar estimations are realized with using the Fourier analysis [1],[2]. Here we develop the Hua's method of estimations of complete rational sums ([2], p.101–109). We follow the version of this method, proposed by V. N. Chubarikov [6]-[10].

Let $n \geq 2, p$ is a prime number, $f(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial with integer coefficients, $(a_n, \dots a_1, p) = 1, \quad 0 \leq l < k$ and $e(x) = e^{2\pi i x}$,

$$S = S(p^k; k-l, f) = \sum_{x=1}^{p^{k-l}} e(f(x)/p^k) \tag{1}$$

$$S = \sum_{\xi=1}^{p} S(\xi), \quad S(\xi) = \sum_{\substack{x=1 \\ x \equiv \xi \pmod p}}^{p^{k-l}} e(f(x)/p^k), \tag{2}$$

moreover

$$S(\xi) = \sum_{x=1}^{p^{k-l-1}} e(f(\xi + px)/p^k).$$

Let $w = [\ln n / \ln p], \quad p^\tau \| (na_n, \dots, 2a_2, a_1)$, then $\tau \leq w$.

We define, following Hua L.-K. ([2], p. 217), solutions

$$x = \xi_1 + p\xi_2 + \cdots + p^{s-1}\xi_s + \dots \tag{3}$$

of congruence $f'(x) \equiv 0 \pmod{p^k}$ in the next way

$$p^{-\tau_0} f'(\xi_1) \equiv 0 \pmod p, p^{u_1} g_{\xi_1}(x) = f(\xi_1 + px) - f(\xi), \tag{4}$$

where the coefficients of the polynomial $g_{\xi_1}(x)$ and the number p have no common factor excepted 1, and further by the analogy for $s \geq 1$ we put

$$p^{-\tau_{s-1}} g_{(\xi_1,\dots,\xi_{s-1})}(\xi_s) \equiv 0 \pmod p, \tag{5}$$

$$p^{u_r} g_{(\xi_1,\dots,\xi_r)}(x) = g_{(\xi_1,\dots,\xi_{r-1})}(\xi_r + px) - g_{(\xi_1,\dots,\xi_{r-1})}(\xi_r), \tag{6}$$

$$k_s = k_{s-1} - u_s, l_s = l_{s-1} - u_s + 1. \tag{7}$$

Now we formulate statements of following theorems.

Theorem 1. *Let inequalities $k_{r-1} \geq 2(l_{r-1} + w + 1), k_r < 2(l_r + w + 1)$ be define the number r. Then we have*

$$S(p^k; k-l, f) = \sum_{(\xi_1,\dots,\xi_r)} e\left(\frac{f(\xi_1)}{p^k} + \frac{g_{\xi_1}(\xi_2)}{p^{k_1}} + \cdots + \frac{g_{\xi_1,\dots,\xi_{r-1}}(\xi_r)}{p^{k_{r-1}}}\right) S(p^{k_r}; k_r - l_r, g_{(\xi_1,\dots,\xi_r)}).$$

Theorem 2. *Let r be the smallest number over all solutions $(\xi_1, \xi_2, \dots, \xi_r)$, defining early, and satisfying inequalities $k_{r-1} \geq 2(l_{r-1} + w + 1), k_r < 2(l_r + w + 1)$. Then*

$$|S(p^k; k-l, f)| \leq (n-1)p^{k-l-r}. \tag{8}$$

2. Lemmas

Further we have the following statement.

LEMMA 1. *Let ξ is not a solution of the congruence $p^{-\tau}f'(x)\equiv 0 \pmod p$, and let $0\le l<k$. Then for $k\ge l+2(l+w)$ we get $S(\xi)=0$.*

PROOF. We put $x=y+p^{k-l-\tau-2}z$, where $1\le y\le p^{k-l-\tau-2}, 0\le z\le p^{\tau+1}-1$. It gives

$$S(\xi)=\sum_{y=1}^{p^{k-l-\tau-2}}\sum_{z=0}^{p^{\tau+1}-1} e(f(\xi+py+p^{k-l-\tau-1}z)/p^k)=$$

$$=\sum_{y=1}^{p^{k-l-\tau-2}} e(e(f(\xi+py)/p^k))\sum_{z=0}^{p^{\tau+1}-1} e(f'(\xi+py)z/p^{\tau+1})=0,$$

as $p^{-\tau}f'(\xi)\not\equiv 0 \pmod p$ and $k\ge 2(l+\tau+1)$.

Lemma is proved.

LEMMA 2. *Let ξ be a solution of the congruence $p^{-\tau}f'(\xi)\equiv 0 \pmod p$. Let*

$$p^u g(x)=f(\xi+px)-f(\xi),\quad g(x)=g_\xi(x)=b_nx^n+\cdots+b_1x,\quad (b_n,\dots b_1,p)=1.$$

Then we have

$$S(\xi)=e(f(\xi)/p^k)\sum_{x=1}^{p^{k-l-1}} e(g(x)/p^{k-u}).$$

PROOF. We find

$$e(-f(\xi)/p^k)S(\xi)=\sum_{x=1}^{p^{k-l-1}} e((f(\xi+px)-f(\xi))/p^k)=\sum_{x=1}^{p^{k-l-1}} e(g(x)/p^{k-u}).$$

Lemma is proved.

LEMMA 3. *The number of solutions of the congruence $f'(x)\equiv 0 \pmod{p^k}$ in the sense described (3)–(7) is at most $n-1$.*

PROOF. See ([2], p.217, Lemma 6.1).

LEMMA 4. *We have*

$$n-1\ge u_1\ge u_2\ge\cdots\ge u_r\ge 2.$$

PROOF. See ([2], p.219, Lemma 7.1).

LEMMA 5. *We have*

$$k-l+r-(u_1+\cdots+u_r)\le[\ln n/\ln p].$$

PROOF. See ([2], p.220, Lemma 7.2).

3. Proof of theorems

1. For $k\ge 2(l+w+1)$ we get

$$S(p^k;k-l,f)=\sum_\xi e(f(\xi)/p^k)S(p^{k-u};k-l-1,g_\xi),$$

where $\xi = \xi_1$ runs all solutions of the congruence $p^{-\tau} f'(\xi) \equiv 0 \pmod p$, and $u = u_1 = u_1(\xi)$ is defined in the statement of the Lemma 2.

Putting $k_1 = k - u_1, l_1 = l + 1 - u_1$, we have

$$S(p^k; k-l, f) = \sum_{\xi} e(f(\xi)/p^k) S(p^{k_1}; k_1 - l_1, g_\xi).$$

Thus if $k_1 \geq 2(l_1 + w + 1)$ then we obtain from Lemmas 1 and 2

$$S(p^k; k-l, f) = \sum_{(\xi_1,\xi_2)} e\left(\frac{f(\xi_1)}{p^k} + \frac{g_{\xi_1}(xi_2)}{p^{k_1}}\right) S(p^{k_1-u_2}; k_1 - l_1 - 1, g_{(\xi_1,\xi_2)}),$$

where ξ_2 is a solution of the congruence

$$p^{-\tau_1} g'_{\xi_1}(\xi_2) \equiv 0 \pmod p, \quad p^{\tau_1} \| (nb_n, \ldots, 2b_2, b_1), \quad \tau_1 \leq w,$$

and

$$p^{u_2} g_{(\xi_1,\xi_2)}(x) = g_{\xi_1}(\xi_2 + px) - g_{\xi_1}(\xi_2), \quad g_{(\xi_1,\xi_2)}(x) = c_n x^n + \cdots + c_1 x, \quad (c_n, \ldots c_1, p) = 1.$$

We carry on doing this procedure further. For $r \geq 1$ we put

$$k_r = k_{r-1} - u_r, l_r = l_{r-1} + 1 - u_r.$$

$$p^{u_r} g_{(\xi_1,\ldots,\xi_r)}(x) = g_{(\xi_1,\ldots,\xi_{r-1})}(\xi_r + px) - g_{(\xi_1,\ldots,\xi_{r-1})}(\xi_r),$$

$$g_{(\xi_1,\ldots,\xi_r)}(x) = c_n^{(r)} x^n + \cdots + c_1^{(r)} x, \quad (c_n^{(r)}, \ldots c_1^{(r)}, p) = 1.$$

Then finally by Lemma 4 and 5 for some r from conditions

$$k_{r-1} \geq 2(l_{r-1} + w + 1), k_r < 2(l_r + w + 1),$$

we find

$$S(p^k; k-l, f) = \sum_{(\xi_1,\ldots,\xi_r)} e\left(\frac{f(\xi_1)}{p^k} + \frac{g_{\xi_1}(\xi_2)}{p^{k_1}} + \cdots + \frac{g_{\xi_1,\ldots,\xi_{r-1}}(\xi_r)}{p^{k_{r-1}}}\right) \times$$

$$\times S(p^{k_{r-1}-u_r}; k_{r-1} - l_{r-1} - 1, g_{(\xi_1,\ldots,\xi_r)}) =$$

$$= \sum_{(\xi_1,\ldots,\xi_r)} e\left(\frac{f(\xi_1)}{p^k} + \frac{g_{\xi_1}(\xi_2)}{p^{k_1}} + \cdots + \frac{g_{\xi_1,\ldots,\xi_{r-1}}(\xi_r)}{p^{k_{r-1}}}\right) S(p^{k_r}; k_r - l_r, g_{(\xi_1,\ldots,\xi_r)}),$$

where ξ_r is a solution of the congruence

$$p^{-\tau_{r-1}} g'_{(\xi_1,\ldots,\xi_{r-1})}(\xi_r) \equiv 0 \pmod p, \quad p^{\tau_{r-1}} \| (nb_n^{(r-1)}, \ldots, 2b_2^{(r-1)}, b_1^{(r-1)}), \quad \tau_{r-1} \leq w.$$

The theorem 1 is proved.

2. Further we have

$$k_r = k_{r-1} - u_r = k_{r-2} - u_{r-1} - u_r = \cdots = k - u_1 - u_1 - \cdots - u_r,$$

$$l_r = l_{r-1} + 1 - u_r = l_{r-2} + 1 - u_{r-1} + 1 - u_{r-2} = \cdots = l + r - u_1 - \cdots - u_r.$$

Hence

$$k_r - l_r = k - l - r.$$

From here we get

$$S(p^k;k-l,f)=\sum_{(\xi_1,\dots,\xi_r)} e\left(\frac{f(\xi_1)}{p^k}+\frac{g_{\xi_1}(\xi_2)}{p^{k_1}}+\dots+\frac{g_{\xi_1,\dots,\xi_{r-1}}(\xi_r)}{p^{k_{r-1}}}\right)S(p^{k_r};k-l-r,g_{(\xi_1,\dots,\xi_r)}).$$

Therefore, using the Lemma 3, we find

$$|S(p^k;k-l,f)|\le(n-1)p^{k-l-r}.$$

The theorem 2 is proved.

Conclusive notes. It's interesting to get non-trivial estimations for shorter non-complete rational trigonometric sums.

We continue recent studies of Professor V.N.Chubarikov on complete arithmetical sums. The problem of this paper due to him. I express my gratitude to Professor V.N.Chubarikov for the discussion of this problem.

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Получено 16.09.2018
Принято к печати 10.10.2018