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Характеризация чисел Фибоначчи

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Аннотация

В согласии с философско-математической мыслью ранних пифагорейцев, для заданных отрезков s и t мог быть найден отрезок u, содержащийся ровно n раз в s и m раз в t при некоторых подходящих числах n и m. Справедливость этого положения была подвергнута самими же пифагорейцами при обнаружении ими несоизмеримости стороны и диагонали правильного пятиугольника. Это фундаментальное историческое открытие, прославившие Пифагорейскую школу, оставило «забытым» предшествующий ему этап исследований. Именно фаза поиска u, начатая многочисленными неудачными попытками и завершившаяся разработкой известной техники доказательства «чётное-нечётное», является объектом нашей «творческой интерпритации» исследований Пифагора, которую мы приводим в этой статье. В частности, будет выявлена сильная связь между пифагорейским тождеством $b(b + a) - a^2 = 0$ относительно стороны b и диагонали a правильного пятиугольника и тождеством Кассини $F_iF_{i+2} - F_{i+1}^2 = (-1)^i$ для трех последовательных чисел Фибоначчи. Более того, эти два тождества были обнаружены Пифагорейской школой «почти одновременно», и, следовательно, числа Фибоначчи и тождество Кассини имеют пифагорейское происхождение. Нам не известны архивные документы (уже столь редкие для изучаемого периода!), касающиеся этого утверждения, но в статье приводятся ряд математических заключений в его подтверждение. Приведенный в работе анализ дает новую (и естественную) характеризацию чисел Фибоначчи, до сих пор отсутствующую в литературе.

Ключевые слова: несоизмеримость, золотое сечение, числа Фибоначчи.

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A characterization of Fibonacci numbers

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Abstract

For the early *Pythagoreans*, in perfect agreement with their philosophical-mathematical thought, given segments s and t there was a segment u contained exactly n times in s and m times in t, for some suitable integers n and m. In the sequel, the *Pythagorean* system is been put in crisis by their own discovery of the incommensurability of the *side* and *diagonal* of a *regular pentagon*. This fundamental historical discovery, glory of the *Pythagorean School*, did however “*forget*” the research phase that preceded their achievement. This phase, started with numerous attempts, all failed, to find the desired common measure and culminated with the very famous odd even argument, is precisely the object of our “*creative interpretation*” of the *Pythagorean* research that we present in this paper: the link between the *Pythagorean identity* $b(b+a)-a^2=0$ concerning the *side* b and the *diagonal* a of a *regular pentagon* and the *Cassini identity* $F_iF_{i+2}-F_{i+1}^2=(-1)^i$, concerning three consecutive *Fibonacci numbers*, is very strong. Moreover, the two just mentioned equations were “*almost simultaneously*” discovered by the *Pythagorean School* and consequently *Fibonacci numbers* and *Cassini identity* are of *Pythagorean origin*. There are no historical documents (so rare for that period!) concerning our audacious thesis, but we present solid mathematical arguments that support it. These arguments provide in any case a new (and natural!) characterization of the Fibonacci numbers, until now absent in literature

Keywords: incommensurability, golden ratio, Fibonacci numbers.

Bibliography: 24 titles.

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1. Introduction

Let $F_0 = 1$, $F_1 = 1$ and, for $n \geqslant 2$, $F_n = F_{n-2} + F_{n-1}$ be the *Fibonacci numbers*. It is well known that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi = \frac{1+\sqrt{5}}{2}$ and that in theoretical computer science the *Fibonacci word* $f = 1011010110110\ldots$ is a *cutting sequence* representing the *golden ratio* Φ (also called *Divina Proportione* by Luca Pacioli). Concerning the Fibonacci numbers, the Fibonacci word and the golden ratio, see [3], [7], [14], [9], [10], [11], [12], [13], [15], [16], [17], [18], [19], [20] and [24].

It is also well known that, given three consecutive Fibonacci numbers $F_i \leqslant F_{i+1} < F_{i+2}$, the following *Cassini identity* $F_iF_{i+2} - F_{i+1}^2 = (-1)^i$ holds. In this paper we support our thesis that the discovery of incommensurability and of the previous equalities came "almost simultaneously", most likely first the Pythagorean identity and immediately after the Cassini identity.

Indeed the Cassini identity is strictly related to the studies and the fundamental results of the Pythagorean School (hereafter simply School) on the incommensurability: *side and diagonal of the regular pentagon are incommensurable* (see Figure 1). The result: *if b is the side and a is the diagonal of a regular pentagon, then $b : a = a : (b + a)$ and $b(b + a) - a^2 = 0$* precedes of a very short period of time the discoveries of Fibonacci numbers and Cassini identity $F_iF_{i+2} - F_{i+1}^2 = (-1)^i$ see [19].

2. The irrational number Φ

The School tried for a long time to find a common measure between the diagonal and the side of the regular pentagon. In the proof of these fundamental results (that we shortly recall hereafter) the following *Pythagorean Proposition* 1 (see [18]) plays a crucial role (and the same will happen in the first proof of the main result of this paper, Proposition 7).

Proposition 1. *(***Pythagorean Proposition.***) A strictly decreasing sequence of positive integers is necessarily finite.*

A common measure of diagonal and side of a regular pentagon implies the existence of a segment U and two positive integers β and α such that U is contained β times in b, the side, and α time in a, the diagonal. Using elementary results on similar triangles, we easily reach the equalities $\beta : \alpha = \alpha : (\beta + \alpha)$ and $\beta(\beta + \alpha) = \alpha^2$.

But, two such integers β and α do not exist by an old well-known odd-even argument: i) β and α both odd implies $\beta(\beta + \alpha)$ even and α^2 odd (contradiction), ii) β odd and α even implies $\beta(\beta + \alpha)$ odd and α^2 even (contradiction), iii) β even and α odd implies $\beta(\beta + \alpha)$ even and α^2 odd (contradiction), iv) β and α both even then, using the Pythagorean Proposition 1, we retrieve one of the three previous cases i), ii) and iii) (contradiction). So β and α cannot be both integers. So *side* and *diagonal* of the *regular pentagon* cannot have a common measure and the following theorem is proved.

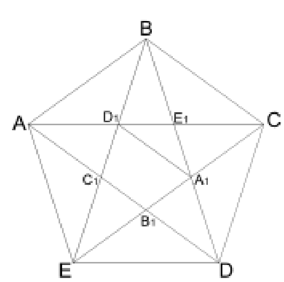

Рис. 1: Regular pentagon

Теорема 1. *Side and diagonal of the regular pentagon are incommensurable.*

3. Fibonacci numbers and their relation with incommensurability

We will present hereafter an argument that shows how the Fibonacci numbers and the Cassini identity appeared naturally during the development of the argument of the incommensurability. Several attempts to find a common measure of side and diagonal of the regular pentagon were not successful and will hereafter be examined in depth. Consider two Propositions on the triangle well known today and also well known to the School:

Proposition 2. *The greatest side of a triangle is that opposite to the greatest angle.*

Proposition 3. *The sum of two sides is greater than the third side.*

Considering the isosceles triangle formed by two consecutive sides and by a diagonal of a regular pentagon, the School would have noticed, by Proposition 2, the inequality $\beta < \alpha$ and, by Proposition 3, the inequality $\alpha < 2\beta$. This is enough to immediately eliminate the side as a common measure ($\beta = 1$).

Now, let $\beta \geqslant 2$. Being β and α integers, from

$$\beta < \alpha < 2\beta,$$

we have

$$\beta + 1 \leqslant \alpha \leqslant 2\beta - 1.$$

Considering the necessary equality $\beta(\beta + \alpha) = \alpha^2$ and using the above lower bound and upper bound, the School easily eliminated the following segments as common measure: the half of the side ($2(2 + 3) - 3^2 \neq 0$), the third of the side ($3(3 + 4) - 4^2 \neq 0$ and $3(3 + 5) - 5^2 \neq 0$), the fourth part of the side ($4(4 + 5) - 5^2 \neq 0$, $4(4 + 6) - 6^2 \neq 0$ e $4(4 + 7) - 7^2 \neq 0$) and so on.

On the other hand, continuing in this way the calculation is increasingly long and difficult as, for each $\beta > 1$, one must consider $\beta - 1$ candidates for α. The departing geometric problem (find a common measure U) is now an arithmetic problem: *given an integer β does there exist an integer $\alpha \geqslant \beta$ such that $\beta(\beta + \alpha) - \alpha^2 = 0$?*

When the recalled argument of incommensurability was completed and consequently it was clear that the answer to this question would be "NO" for each β, we believe that the School has considered the just obtained result as a motivation for a new research and has been argumented as follows: as $\beta(\beta+\alpha)-\alpha^2$ is never 0, we wish to see for what values of β and α the difference between the greatest and the smallest of the numbers $\beta(\beta + \alpha)$ and α^2 assumes the value 1, which is the minimum possible one. This is a typical curiosity of mathematicians: when they solve a problem, their attention is immediately attracted by the new and often numerous problems that the solution always carries with it. So, we simply believe that, after the discovery of the incommensurability, the School has focused on this new problem.

Today, to find the above recalled values of β and α is very easy using a computer. It is possible to write a program that searches, finds and puts all these values in the following table. Our brother Mario wrote the program and this is what happens:

Рис. 2: Pythagoreans at work

β	α	$\alpha+\beta$	$\beta(\alpha+\beta)$	α^2
1	1	2	1^2+1	1^2
1	2	3	2^2-1	2^2
2	3	5	3^2+1	3^2
3	5	8	5^2-1	5^2
5	8	13	8^2+1	8^2
8	13	21	13^2-1	13^2
13	21	34	21^2+1	21^2
21	34	55	34^2-1	34^2
34	55	89	55^2+1	55^2
55	89	144	89^2-1	89^2
89	144	233	144^2+1	144^2
144	233	377	233^2-1	233^2
233	377	610	377^2+1	377^2
377	610	987	610^2-1	610^2
610	987	1597	987^2+1	987^2
987	1597	2584	1597^2-1	1597^2

If, as we think, the School has really tried to find these values of β and α then they have all noticed the peculiarity of the numbers in the table. The Fibonacci numbers are in the first, second and third column and, in addition, the square of the Fibonacci numbers are in the fifth column while the fourth column contains alternately the predecessor and the successor of these squares, see [19].

Now, let $i \geqslant 0$ and F_i the i^{th} Fibonacci number. *Does there exist an integer $\alpha \geqslant F_i$ such that the difference between the greatest and the smallest of the numbers $F_i(F_i+\alpha)$ and α^2 assumes the value 1?* Sure, it exists. The table shows that, for each i, $1 \leqslant F_i \leqslant 1000$, the required number α is exactly F_{i+1} and $F_i(F_i+F_{i+1}) - F_{i+1}^2 = (-1)^i$. Being $F_i + F_{i+1} = F_{i+2}$, this equality becomes $F_iF_{i+2} - F_{i+1}^2 = (-1)^i$ and, as it is well-known, the following lemma holds (see for instance [14]).

Lemma 1. **Cassini identitity.** *For each non negative integer i and for each Fibonacci number F_i the following equality holds*

$$F_iF_{i+2} - F_{i+1}^2 = (-1)^i.$$

As we have seen before, step by step the School has picked up new Fibonacci numbers. Each new one discovered corresponded to a more accurate (but not exact!) measurement of the side and diagonal of the regular pentagon. In this sense, the School has discovered and proved the equality $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi$, certainly not in the very precise form of the current modern epsilon-delta definition that it has today, but surely in the sense that the difference $\Phi - \frac{F_{n+1}}{F_n}$ became ever smaller and smaller.

4. Cassini identity and characterization of Fibonacci numbers

We introduce a definition which will be crucial in the rest of the paper.

Определение 1. *Let β a positive integer. When there exists a positive integer α such that, for some non-negative integer γ, the equality*

$$\beta(\beta+\alpha) - \alpha^2 = (-1)^\gamma$$

holds, then we say that β is a Hippasus number *and that α is a* Hippasus successor *of β.*

For the aims of this paper, using the previous definition [1], we can obtain a more suitable reformulation of the *Cassini identitity*.

PROPOSITION 4. *If an integer $\beta \geqslant 2$ is a Hippasus number then any its Hippasus successor α satisfies $\beta+1 \leqslant \alpha \leqslant 2\beta-1$.*

ДОКАЗАТЕЛЬСТВО. . By way of contradiction, suppose $\alpha < \beta$. There exist a positive integer ζ such that $\beta = \alpha+\zeta$. Then $\beta(\beta+\alpha)-\alpha^2 = (\alpha+\zeta)(2\alpha+\zeta)-\alpha^2 = \alpha^2+3\alpha\zeta+\zeta^2 \geqslant 5$. Contradiction. By way of contradiction, suppose $\alpha 2\beta$. There exist a positive integer $\eta > 0$, such that $\alpha = 2\beta+\eta$. Then $\beta(\beta+2\beta+\eta)-(2\beta+\eta)^2 = -\beta^2-3\beta\eta-\eta^2 \leqslant -11$. Contradiction. Moreover, the equalities

$$\beta(\beta+\beta)-(\beta)^2 = (-1)^\gamma$$
$$\beta(\beta+2\beta)-(2\beta)^2 = (-1)^\gamma$$

are impossible for each integer γ.

So if α exist we must have

$$\beta+1 \leqslant \alpha \leqslant 2\beta-1.$$

□

Proposition 4 underlines a clear relation with the geometric origin of the Hippasus number definition.

The proof of the following proposition is very easy.

PROPOSITION 5. *For each $i \geqslant 0$ the Fibonacci number F_i is a Hippasus number and F_{i+i} is a Hippasus successor of it.*

The following Proposition 6 offers an even more precise reformulation of the *Cassini identitity*. In order to prove Proposition 7 we need several lemmas.

LEMMA 2. *The number 1 is a Hippasus number and 1 itself is one of its Hippasus successor.*

ДОКАЗАТЕЛЬСТВО. The equality $1(1+1)-1^2 = 1$ holds. □

LEMMA 3. *The number 1 has also 2 as a Hippasus successor.*

ДОКАЗАТЕЛЬСТВО. The equality $1(1+2)-2^2 = -1$ holds. □

LEMMA 4. *No positive integer different from 1 and 2 is a Hippasus successor of 1.*

ДОКАЗАТЕЛЬСТВО. For $n > 2$, we have $1(1+n)-n^2 \leqslant -5$. □

LEMMA 5. *A Hippasus number greater than 1 has a unique Hippasus successor.*

ДОКАЗАТЕЛЬСТВО. Let $\beta > 1$ a Hippasus number and α and α', $\alpha \neq \alpha'$, both Hippasus successors of β. By the previous Lemma, we have $\alpha > \beta$ and $\alpha' > \beta$.

Without loss of generality, suppose $\alpha < \alpha'$. There exists $\delta > 0$ and γ, γ' non negative integers such that $\alpha' = \alpha+\delta$, $\beta(\beta+\alpha)-\alpha^2 = (-1)^\gamma$ and $\beta(\beta+\alpha+\delta)-(\alpha+\delta)^2 = (-1)^{\gamma'}$. Now,

$$\beta(\beta+\alpha+\delta)-(\alpha+\delta)^2 =$$
$$\beta(\beta+\alpha)+\beta\delta-(\alpha^2+2\alpha\delta+\delta^2) =$$
$$(\beta(\beta+\alpha)-\alpha^2)+(\beta\delta-2\alpha\delta-\delta^2) =$$
$$(-1)^\gamma-\delta(-\beta+2\alpha+\delta) =$$
$$(-1)^\gamma-\delta((\alpha-\beta)+\alpha+\delta).$$

Being $\alpha \geqslant 3$ (as $\alpha > \beta \geqslant 2$), $\alpha-\beta \geqslant 1$ (as $\alpha > \beta$) and $\delta \geqslant 1$ (as $\alpha' > \alpha$), we have $(\alpha-\beta)+\alpha+\delta \geqslant 5$ and $-\delta((\alpha-\beta)+\alpha+\delta) \leqslant -5$. So

[1]This terminology seems suitable. Tradition, see [23], attributes to Hippasus the discovery of incommensurability and our thesis is the following: the discoveries of incommensurability and of a particular class of numbers came simultaneously, see [19]. So these numbers that we show here to be Fibonacci numbers can provisionally be called *Hippasus numbers*.

$$\beta(\beta+\alpha+\delta)-(\alpha+\delta)^2 \leqslant (-1)^\gamma - 5 < (-1)^{\gamma'}$$

and $\alpha' = \alpha + \delta$ cannot be a Hippasus successor of β. Contradiction. Then two different integers α, α' cannot be both Hippasus successors of the same β. $\square$

So, with the exception of 1 (that is, in a sense, *ambiguous*) any other Hippasus number β has a *unique* Hippasus successor α that is strictly greater than β.

Now, we can precise Proposition 5

Proposition 6. *For the Fibonacci numbers the following statements hold:*
i) $F_0 = 1$ is an Hippasus number and $F_1 = 1$ is an Hippasus successor of it,
ii) $F_1 = 1$ is an Hippasus number and $F_2 = 2$ is an Hippasus successor of it,
iii) for each $i > 1$, F_i is an Hippasus number and F_{i+1} is its **unique** *Hippasus successor.*

Доказательство. i) follows by Lemma 2, ii) follows by Lemma 3 and finally, as for $i > 1$ we have $F_i \geqslant 2$, iii) follows by Proposition 5. $\square$

Lemma 6. *Let β be a Hippasus number and α be a Hippasus successor of β. Then $\alpha - \beta \leqslant \beta$.*

Доказательство. By a trivial verification if $\beta = 1$ and $\alpha = 1$ and if $\beta = 1$ and $\alpha = 2$ and by Proposition 4 for $\beta > 1$. $\square$

In some sense 0 is a “Hippasus number” having 1 as one of its Hippasus successors (indeed we have $0(0+1) - 1 = -1$) but by our choice, a Hippasus number must be positive, see Definition 1. For this reason in the next lemma we add the condition $\alpha > \beta$ with which we exclude the case $\beta = 1$ and $\alpha = 1$.

Lemma 7. *Let β be a Hippasus number and α be a Hippasus successor of β with $\alpha > \beta$. Then $\alpha - \beta$ is a Hippasus number and β is a Hippasus successor of $\alpha - \beta$.*

Доказательство. By Lemma 6 we have $0 < \alpha - \beta \leqslant \beta$. Moreover, we know that for some γ we have $\beta(\beta+\alpha) - \alpha^2 = (-1)^\gamma$. So, $(\alpha-\beta)((\alpha-\beta)+\beta) - \beta^2 = (\alpha-\beta)\alpha - \beta^2 = (-1)[\beta(\beta+\alpha) - \alpha^2] = (-1)^{\gamma+1}$ that exactly says that $\alpha - \beta$ is a Hippasus number and β is a Hippasus successor of $\alpha - \beta$. $\square$

Lemma 8. *Let $\beta \geqslant 1$ be a Hippasus number and α a Hippasus successor of β. If $\alpha - \beta = \beta$ then $\alpha - \beta = 1$, $\beta = 1$ and $\alpha = 2$.*

Доказательство. Consider three cases:

$$\text{a) } \beta = 1, \alpha = 1; \quad \text{b) } \beta = 1, \alpha = 2 \quad \text{and} \quad \text{c) } \beta > 1.$$

We have $\alpha = 2\beta$. Case a): $\alpha = 2\beta$ is not true. Case c): $\alpha = 2\beta$ is not true by Lemma 4. So, it remains only case b) in which the statement trivially holds. $\square$

Now, we are ready to prove the following proposition of which we present two proofs.

Proposition 7. *Any Hippasus number is a Fibonacci number.*

Доказательство. Let β be a Hippasus number and let α be a Hippasus successor of it. If $\beta = 1$ and $\alpha = 1$ then β is a Fibonacci number. If $\beta = 1$ and $\alpha = 2$ then β is a Fibonacci number too. (The set of Hippasus numbers contain two times the value 1, see Lemma 2 and 3, as well as the sequence of Fibonacci numbers.)

So, we have to prove that a Hippasus number greater than 1 is a Fibonacci number. Let β be such a number. We know, by Definition 1, that β has a Hippasus successor α and, being $\beta > 1$, we also know that $\alpha > \beta$ (by Lemma 4) and that α is unique (by Lemma 5).

We know, by Lemma 7, that $\alpha - \beta$ is a Hippasus number and that β is a Hippasus successor of $\alpha - \beta$. By Lemma 6 we have that $\alpha - \beta \leqslant \beta$, i.e., there are two possibilities

$$\alpha - \beta = \beta \quad \text{or} \quad \alpha - \beta < \beta.$$

If $\alpha - \beta = \beta$, then by Lemma 8, $\beta = 1$. Contradiction.

So we must have $\alpha - \beta < \beta$. Put $\beta = \beta_1$ and $\alpha - \beta = \beta_2$.

It may happen that $\beta_1 - \beta_2 < \beta_2$. Put $\beta_3 = \beta_1 - \beta_2$.

It may similarly happen that $\beta_2 - \beta_3 < \beta_3$. Put $\beta_4 = \beta_2 - \beta_3$.

And so on indefinitely.

In principle, we thus have two possibilities:

-either, for each positive integer k, after the selection of the integer β_k we select β_{k+1} with $\beta_{k+1} < \beta_k$;

-either the process of selection of β_{k+1} strictly smaller of β_k will fail at a certain stage.

Let us take these two possibilities in turn [2].

By Pythagorean Proposition 1 (*an infinite strictly decreasing sequence of positive integers cannot exist*) the first possibility cannot happen. So, the process of selection of β_{k+1} strictly smaller of β_k will fail at a certain stage when, for a given integer, say i, $\beta_{i+1} = \beta_i$.

So, we suppose that we have selected $\beta_1, \beta_2, \ldots, \beta_{i-2}, \beta_{i-1}, \beta_i, \beta_{i+1}$ with $\alpha - \beta = \alpha - \beta_1 < \beta_1$, $\beta_1 - \beta_2 = \beta_3 < \beta_2$, $\beta_2 - \beta_3 = \beta_4 < \beta_3, \ldots, \beta_{i-2} - \beta_{i-1} = \beta_i < \beta_{i-1}$ and $\beta_{i-1} - \beta_i = \beta_{i+1} = \beta_i$.

By hypothesis $\beta = \beta_1$ is a Hippasus number and $\beta_2, \ldots, \beta_{i-2}, \beta_{i-1}, \beta_i, \beta_{i+1}$ are all Hippasus numbers by Lemma 7. Moreover, again by Lemma 7, β_i is a successor of β_{i+1}, β_{i-1} is a successor of $\beta_i, \ldots, \beta_1$ is a successor of β_2, α is a successor of $\beta = \beta_1$.

Considering $\beta_{i-1} - \beta_i = \beta_{i+1} = \beta_i$, by Lemma 8, we have:

$$\beta_{i+1} = 1 = F_0,$$
$$\beta_i = 1 = F_1,$$
$$\beta_{i-1} = 2 = F_2.$$

By construction $\beta_{i-1} = 2 = F_2$ has a unique Hippasus successor that is β_{i-2} but, as the Fibonacci number F_2 has a unique Hippasus successor that is F_3 (see Lemma 6), we have that

$$\beta_{i-2} = 3 = F_3.$$

Similarly,

$$\beta_{i-3} = 5 = F_4,$$
$$\beta_{i-4} = 8 = F_5,$$
$$\ldots\ldots\ldots,$$

and, continuing in this way,

$$\ldots\ldots\ldots,$$
$$\beta_3 = F_{i-2},$$
$$\beta_2 = F_{i-1},$$
$$\beta_1 = F_i.$$

□

A second proof could be the following. By way of contradiction, suppose that the set of Hippasus numbers which are not Fibonacci numbers is non empty. By the *minimum principle* this set admits a minimum element, say β. Necessarily, β is strictly greater than 2 and has a unique Hippasus successor, say α. Consider $\alpha - \beta$ that, by Lemma 7, is a Hippasus number. If $\alpha - \beta = \beta$ then, by Lemma 8, $\beta = 1$ that is a Fibonacci number. Contradiction. If $\alpha - \beta < \beta$ then, by Lemma 7, $\alpha - \beta$ is a Hippasus number and strictly smaller than β. Contradiction too.

The second proof, that uses the *minimum principle*, is shorter than the first one, which we prefer as it uses explicitly the *Pythagorean Proposition* 1.

Proposition 5 and Proposition 7 imply the following

PROPOSITION 8. *A positive integer is a Hippasus number if, and only if, it is a Fibonacci number.*

[2] Here we try to imitate a clear, elegant and powerful model of exposition that *Ramsey* presented in [22].

By our previous results we are convinced that the relations between the *Pythagorean identity* $b(b+a)-a^2=0$ and the *Cassini Identity* $\beta(\beta+\alpha)-\alpha^2=(-1)^\gamma$ are really very strict. At least in our thesis, the School, that discovered the first equality, hardly could have ignored the second one. In other terms, when the School found a Hippasus number then the same School simultaneusly found a Fibonacci number, because no other number could have be found. In order to add another argument to our previous ones (in particular Proposition 4), we prove directly the following proposition.

Proposition 9. *Let β be a Hippasus number and α be a Hippasus successor of β. Then α is a Hippasus number and $\alpha+\beta$ is a Hippasus successor of α.*

Доказательство. For some γ we have $(-1)^\gamma=\beta(\beta+\alpha)-\alpha^2$. So $\alpha(\alpha+\alpha+\beta)-(\alpha+\beta)^2=$ $=\alpha^2+\alpha(\alpha+\beta)-(\alpha+\beta)^2=\alpha^2-\alpha\beta-\beta^2=(-1)(-\alpha^2+\alpha\beta+\beta^2)=(-1)(\beta(\beta+\alpha)-\alpha^2)=(-1)^{\gamma+1}$ i.e. α is a Hippasus number and $\alpha+\beta$ is a Hippasus successor of it. □

Следствие 1. *If α is a Hippasus number and β is its Hippasus successor then $\alpha+\beta$ is a Hippasus number.*

Corollary 1 certifies that the laws of formation of Fibonacci numbers and of Hippasus numbers are the same! Much better, the Fibonacci law $F_n+F_{n+1}=F_{n+2}$ rediscovers the Pythagorean law given in the previous Corollary 1. Moreover, the Definition 1 of Hippasus numbers is operational and allows us to find Hippasus numbers one after the other.

The Wasteel result of next section is just a criterion to decide if two integers are consecutive Fibonacci numbers.

5. With Fibonacci numbers the surprises never end

Dickson recalls in [8] the following result of Wasteels, proved in [24].

Proposition 10. *Two positive integers x and y for which y^2-xy-x^2 equals $+1$ or -1 are consecutive terms of the series of Fibonacci.*

Matiyasevich in [13] with reference to the result of Wasteels says: *The fact that successive Fibonacci numbers give the solution of Eq. (25) was presented by Jean-Dominique Cassini to the Academie Royale des Sciences as long ago as 1680. It can be proved by a trivial induction. At the same time the stronger fact that Eq. (25) is characteristic of the Fibonacci numbers is somehow not given in standard textbooks. The induction required to prove the converse is less obvious, and that fact seems to be the reason for the inclusion of the problem of inverting Cassini's identity as Exercise 6.44 in Concrete Mathematics by Ronald Graham, Donald Knuth, and Oren Patashnik [13]. As the original source of this problem the authors cite my paper [21], but I have always suspected that such a simple and fundamental fact must have been discovered long before me. This suspicion turned out to be justified: I have recently found a paper of M. Wasteels [41] published in 1902 in the obscure journal Mathesis.* [3]

A *pentagon* on a *portale* of "*Duomo di Prato*" refers to *Fibonacci numbers* [4] and a *octagon* on the same *portale* seems to have a reference to a singular construction of an octagon that uses Fibonacci numbers! This octagon is *not* regular but very impressively similar to a regular octagon: we design two concentric circles having diameters F_n and F_{n+2}, the two horizontal straight line tangent to the inner circle and the two vertical straight line tangent to the same inner circle. These four lines

[3] In this citation Eq. (25) is the Cassini identity $F_iF_{i+2}-F_{i+1}^2=(-1)^i$. Paper [13] corresponds to [11] here, paper [21] is the fundamental and historical paper of Matiyasevich (here [12]) and paper of Wasteels [41] is [24] here.

[4] Recently, the Fibonacci numbers have been rediscovered in a tarsia of the Church of San Nicola in Pisa (see Armienti [2] and Albano [1]).

cut the larger circle into 8 points. We denote by P_n and Q_n the two of them having the following coordinates and lying in the first quadrant:

$$P_n = \left(\tfrac{F_n}{2}, \sqrt{\left(\tfrac{F_{n+2}}{2}\right)^2 - \left(\tfrac{F_n}{2}\right)^2}\right) \quad , \quad Q_n = \left(\sqrt{\left(\tfrac{F_{n+2}}{2}\right)^2 - \left(\tfrac{F_n}{2}\right)^2}, \tfrac{F_n}{2}\right) .$$

They are the extremes of one of the eigth sides of our octagon. We note that their distance d_n is $\sqrt{2}\left[\sqrt{\left(\frac{F_{n+2}}{2}\right)^2 - \left(\frac{F_n}{2}\right)^2} - \frac{F_n}{2}\right]$. We also denote by e_n the side of the regular octagon inscribed in the circle of diameter F_{n+2}. We have that:

-the value $\frac{d_n}{F_n}$ tends to the limit $\frac{\sqrt{2}}{2}\left[\sqrt{\Phi^4 - 1} - 1\right]$, i.e. about 1.00375,

-the value $\frac{d_n}{e_n}$ tends to the limit $\frac{\sqrt{2}}{\sqrt{2-\sqrt{2}}}\left[\sqrt{1 - \Phi^{-4}} - \Phi^{-2}\right]$, i.e. about 1.001874,

-the value $\frac{e_n}{F_n}$ tends to the limit $\frac{\sqrt{2-\sqrt{2}}}{2}\Phi^2$, i.e. about 1.001878.

It seems that the architech of the "*Duomo di Prato*" was Carboncettus marmorarius see [5] and [6]. For these reasons one can speak about Carboncettus octagon!

Рис. 3: The portal of the Duomo of Prato

6. Conclusions

In this paper we reconsider two of our old questions: when, for the first time, the Fibonacci Numbers were mathematically well defined and who defined them? Conventional wisdom suggests that the Fibonacci Numbers were first introduced in 1202 by Leonardo of Pisa, better known today as Fibonacci, in his book *Liber abbaci*. The intent of this article is to offer a plausible conjecture on the origin of the Fibonacci Numbers. Indeed, our paper contains comments on the relationship between golden ratio and the Fibonacci Numbers. We try to imagine the work of the Pythagorean School and the first steps that led to their discovery of the irrational number Φ, the golden ratio. We suppose that before discovering that no common measure was possible for the side and diagonal of a regular pentagon, in particular they verified that: i) the side was not a common measure, ii) the half of the side was not a common measure, iii) the third of the side was not a common measure and so on. We analyze these "unsuccessful" attempts, during this analysis we realize that the Fibonacci Numbers appear and we conclude that probably the Pythagorean School also noticed ... these same Fibonacci Numbers! Finally we would like to point out that this paper is solely based on some remarks about the arguments used by the Pythagorean School and not on historical documents.

In conclusion, during our personal investigation on the discovery of the irrational numbers (made in the absolute absence of documents), we find enough traces of the Fibonacci numbers and of their properties to convince us that these numbers were born in Crotone in the VI-IV century B.C. On the other hand, *stricto sensu* we present no historical discovery but we present a new characterization of the Fibonacci numbers and, perhaps, we provide maths teachers some useful educational suggestion.

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