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Совместная дискретная универсальность дзета-функций Лерха

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Аннотация

После 1975 г. работы Воронина известно, что некоторые дзета и L-функции универсальны в том смысле, что их сдвигами приближается широкий класс аналитических функций. Рассматриваются два типа сдвигов: непрерывный и дискретный.

В работе изучается универсальность дзета-функций Лерха $L(\lambda,\alpha,s)$, $s=\sigma+it$, которые в полуплоскости $\sigma>1$ определяются рядами Дирихле с членами $e^{2\pi i\lambda m}(m+\alpha)^{-s}$ с фиксированными параметрами $\lambda\in\mathbb{R}$ и α, $0<\alpha\leqslant 1$, и мероморфно продолжаются на всю комплексную плоскость. Получены совместные дискретные теоремы универсальности для дзета-функций Лерха. Именно, набор аналитических функций $f_1(s),\dots,f_r(s)$ одновременно приближаются сдвигами $L(\lambda_1,\alpha_1,s+ikh),\dots,L(\lambda_r,\alpha_r,s+ikh)$, $k=0,1,2,\dots$, где $h>0$ - фиксированное число. При этом требуется линейная независимость над полем рациональных чисел множества $\left\{(\log(m+\alpha_j):m\in\mathbb{N}_0,\ j=1,\dots,r),\frac{2\pi}{h}\right\}$. Доказательство теорем универсальности использует вероятностные предельные теоремы о слабой сходимости вероятностных мер в пространстве аналитических функций.

Ключевые слова: дзета-функция Лерха, пространство аналитических функций, слабая сходимость, теорема Мергеляна, универсальность.

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Joint discrete universality for Lerch zeta-functions[1]

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Abstract

After Voronin's work of 1975, it is known that some of zeta and L-functions are universal in the sense that their shifts approximate a wide class of analytic functions. Two cases of shifts, continuous and discrete, are considered.

The present paper is devoted to the universality of Lerch zeta-functions $L(\lambda, \alpha, s)$, $s = \sigma + it$, which are defined, for $\sigma > 1$, by the Dirichlet series with terms $e^{2\pi i\lambda m}(m+\alpha)^{-s}$ with parameters $\lambda \in \mathbb{R}$ and α, $0 < \alpha \leqslant 1$, and by analytic continuation elsewhere. We obtain joint discrete universality theorems for Lerch zeta-functions. More precisely, a collection of analytic functions $f_1(s), \dots, f_r(s)$ simultaneously is approximated by shifts $L(\lambda_1, \alpha_1, s+ikh), \dots, L(\lambda_r, \alpha_r, s+ikh)$, $k = 0, 1, 2, \dots$, where $h > 0$ is a fixed number. For this, the linear independence over the field of rational numbers for the set $\left\{(\log(m+\alpha_j) : m \in \mathbb{N}_0,\ j = 1, \dots, r), \frac{2\pi}{h}\right\}$ is required. For the proof, probabilistic limit theorems on the weak convergence of probability measures in the space of analytic function are applied.

Keywords: Lerch zeta-function, Mergelyan theorem, space of analytic functions, universality, weak convergence.

Bibliography: 18 titles.

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Dedicated to the 100th birthday of Nikolai Mikhailovich Korobov

1. Introduction

In [18], see also [4], S.M. Voronin discovered the universality of the Riemann zeta-function $\zeta(s)$, $s=\sigma+it$, that a wide class of analytic functions can be approximated by shifts $\zeta(s+i\tau)$, $\tau\in\mathbb{R}$. After Voronin's work, various authors extended his universality theorem for some other zeta- and L-functions, and classes of Dirichlet series. One of universal zeta-functions is the Lerch zeta-function $L(\lambda,\alpha,s)$ with parameters $\lambda\in\mathbb{R}$ and α, $0<\alpha\leqslant 1$, which is defined, for $\sigma>1$, by the Dirichlet series

$$L(\lambda,\alpha,s)=\sum_{m=0}^{\infty}\frac{e^{2\pi i\lambda m}}{(m+\alpha)^s}.$$

The function $L(\lambda,\alpha,s)$ was introduced and studied independently by R. Lipschitz [14] and M. Lerch [13]. The analytic properties of $L(\lambda,\alpha,s)$ depend on the parameters λ and α, and in particular case, this is true for the analytic continuation to the whole complex plane. If $\lambda\notin\mathbb{Z}$, then $L(\lambda,\alpha,s)$ is an entire function, while, for $\lambda\in\mathbb{Z}$, $L(\lambda,\alpha,s)$ reduces to the Hurwitz zeta-function

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},\quad \sigma>1,$$

which is analytically continued to the whole complex plane, except for a simple pole at the point $s=1$ with residue 1. In virtue of the periodicity of $e^{2\pi i\lambda m}$, it suffices to suppose that $0<\lambda\leqslant 1$. The theory of the Lerch zeta-function is given in [7].

The first universality result for the function $L(\lambda,\alpha,s)$ was obtained in [5]. Let

$$D=\left\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\right\},$$

$\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, and let $H(K)$ with $K\in\mathcal{K}$ denote the class of continuous functions on K that are analytic in the interior of K. Let $\operatorname{meas}A$ denote the Lebesgue measure of a measurable set $A\subset\mathbb{R}$. Then it was obtained in [5] that if α is transcendental, then for $K\in\mathcal{K}$, $f(s)\in H(K)$, $0<\lambda\leqslant 1$ and every $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|L(\lambda,\alpha,s+i\tau)-f(s)|<\varepsilon\right\}>0.$$

The case of rational α is more complicated. Some conditional result in this direction has been obtained in [7]. If both α and λ are rational, then the function $L(\alpha,\lambda,s)$ becomes the periodic Hurwitz zeta-function, and, for it, an universality theorem of type of [9] is true. In this case, a certain condition connecting α and λ is involved.

The universality of $L(\alpha,\lambda,s)$ with algebraic irrational α is an open problem. The case of α with linearly independent set $L(\alpha)=\{\log(m+\alpha):m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\}$ over the field of rational numbers $\mathbb{Q}$ can be viewed as a certain approximation to that problem, see [17] and [6].

For the function $L(\alpha,\lambda,s)$, also a discrete universality when τ in $L(\alpha,\lambda,s+i\tau)$ takes values from a certain discrete set is considered. One of the simplest discrete sets is the arithmetic progression $\{kh:k\in\mathbb{N}_0\}$ with $h>0$. Denote by $\# A$ the cardinality of the set A. If α is transcendental and the number $\exp\{\frac{2\pi}{k}\}$ is rational, then it is known [3], [8] that, for $K\in\mathcal{K}$, $f(s)\in H(K)$, $0<\lambda\leqslant 1$ and every $\varepsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}|L(\lambda,\alpha,s+ikh)-f(s)|<\varepsilon\right\}>0.$$

Let, for $h > 0$,

$$L(\alpha, h, \pi) = \left\{ (\log(m+\alpha) : m \in \mathbb{N}_0), \frac{2\pi}{h} \right\}.$$

Then, in [12], the transcendence of α and rationality of $\exp\{\frac{2\pi}{h}\}$ were replaced by the linear independence over $\mathbb{Q}$ of the set $L(\alpha, h, \pi)$.

The aim of this paper is joint discrete universality theorems for Lerch zeta-functions. We note that the joint universality for Lerch zeta-functions is an interesting problem connecting algebraic properties of the parameters $\alpha_1, \dots, \alpha_r$ and $\lambda_1, \dots, \lambda_r$ with analytic properties of a collection $L(\lambda_1, \alpha_1, s), \dots, L(\lambda_r, \alpha_r, s)$, therefore, there are many results of such a kind. The first joint universality theorem for Lerch zeta-functions was proved in [10], [11].

THEOREM 1. *Suppose that the parameters $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, $\lambda_1 = \frac{a_1}{q_1}, \dots, \lambda_r = \frac{a_r}{q_r}$, $(a_1, q_1) = 1, \dots, (a_r, q_r) = 1$, are rational numbers, k is the least common multiple of $q_1, \dots, q_r$, and that the rank of the matrix*

$$\begin{pmatrix} e^{2\pi i\lambda_1} & e^{2\pi i\lambda_2} & \dots & e^{2\pi i\lambda_r} \\ e^{4\pi i\lambda_1} & e^{4\pi i\lambda_2} & \dots & e^{4\pi i\lambda_r} \\ \dots & \dots & \dots & \dots \\ e^{2k\pi i\lambda_1} & e^{2k\pi i\lambda_2} & \dots & e^{2k\pi i\lambda_r} \end{pmatrix}$$

is equal to r. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j \in H(K_j)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(\lambda_j, \alpha_j, s+i\tau) - f_j(s)| < \varepsilon \right\} > 0.$$

Let

$$L(\alpha_1, \dots, \alpha_r) = \{ (\log(m+\alpha_1) : m \in \mathbb{N}_0), \dots, (\log(m+\alpha_r) : m \in \mathbb{N}_0) \}.$$

Then in [16], under the hypothesis that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, it was obtained that the inequality of Theorem 1 is true for all $0 < \lambda \leqslant 1$, $j = 1, \dots, r$.

We will focus on joint discrete analogues of the above results. For $h > 0$, define the set

$$L(\alpha_1, \dots, \alpha_r; h, \pi) = \left\{ (\log(m+\alpha_1) : m \in \mathbb{N}_0), \dots, (\log(m+\alpha_r) : m \in \mathbb{N}_0), \frac{2\pi}{h} \right\}.$$

Then we have

THEOREM 2. *Suppose that the set $L(\alpha_1, \dots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$, $f_j \in H(K_j)$ and $0 < \lambda_j \leqslant 1$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(\lambda_j, \alpha_j, s+ikh) - f_j(s)| < \varepsilon \right\} > 0.$$

Theorem 2 has the following modification.

THEOREM 3. *Suppose that the set $L(\alpha_1, \dots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$, $f_j \in H(K_j)$ and $0 < \lambda_j \leqslant 1$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(\lambda_j, \alpha_j, s+ikh) - f_j(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The proofs of Theorems 2 and 3 are based on statistical properties of Lerch zeta-functions, more precisely, on limit theorems of weakly convergent probability measures in the space of analytic functions.

2. Discrete limit theorems

Denote by $\mathcal{B}(X)$ the Borel σ-field of the space X. We recall that $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. In this section, we consider the weak convergence of probability measures defined on $(H(D), \mathcal{B}(H(D)))$.

We use the notation $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and define

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. Then, by the famous Tikhonov theorem, the torus Ω with the product topology and pointwise multiplication is a compact topological Abelian group. Putting

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \ldots, r$, by the Tikhonov theorem again, we have that Ω^r is a compact topological Abelian group. Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H can be defined. This gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$. Denote by m_{jH} the probability Haar measure on $(\Omega^j, \mathcal{B}(\Omega^j))$, $j = 1, \ldots, r$. Then we have that

$$m_H = m_{1H} \times \cdots \times m_{rH}.$$

Let ω_j be the elements of Ω_j, $j = 1, \ldots, r$, and $\omega = (\omega_1, \ldots, \omega_r$ denote the elements of Ω^r. Moreover, denote by $\omega_j(m)$ the projection of an element $\omega_j \in \Omega_j$ to the circle γ_m, $m \in \mathbb{N}_0$, $j = 1, \ldots, r$. Now, on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$, define the $H^r(D)$-valued random element $L(\underline{\lambda}, \underline{\alpha}, s, \omega)$, where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r)$ and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$, by

$$L(\underline{\lambda}, \underline{\alpha}, s, \omega) = (L_1(\lambda_1, \alpha_1, s, \omega_1), \ldots, L_r(\lambda_r, \alpha_r, s, \omega_r)),$$

where

$$L_j(\lambda_j, \alpha_j, s, \omega_j) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m} \omega_j(m)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r.$$

We note that the latter series are uniformly convergent on compact subsets of the strip D [7], thus, they define the $H(D)$-valued random elements.

Having the above definitions, we state a joint discrete limit theorem for Lerch zeta-functions.

THEOREM 4. *Suppose that the set $L(\alpha_1, \ldots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. Then*

$$P_N(A) \overset{def}{=} \frac{1}{N+1} \# \{0 \leqslant k \leqslant N : L(\underline{\lambda}, \underline{\alpha}, s + ikh) \in A\}, \quad A \in \mathcal{B}(H^r(D)),$$

converges weakly to the distribution P_L of the random element $L(\underline{\lambda}, \underline{\alpha}, s, \omega)$ as $N \to \infty$.

We remind that, for $A \in \mathcal{B}(H^r(D))$,

$$P_L(A) = m_H \{\omega \in \Omega^r : L(\underline{\lambda}, \underline{\alpha}, s, \omega) \in A\}.$$

We divide the proof of Theorem 4 into lemmas. The first of them deals with the weak convergence of probability measures on $(\Omega^r, \mathcal{B}(\Omega^r))$, and for that the linear independence of the set $L(\alpha_1, \ldots, \alpha_r; h, \pi)$ is essentially applied.

Let, for $A \in \mathcal{B}(\Omega^r)$,

$$Q_N(A) = \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : ((m + \alpha_1)^{-ikh} : m \in \mathbb{N}_0), \ldots, ((m + \alpha_r)^{-ikh} : m \in \mathbb{N}_0)) \in A\right\}.$$

LEMMA 1. *Suppose that the set $L(\alpha_1, \dots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. Then Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof.
We consider the Fourier transform of Q_N. Since characters of the group Ω^r are of the form

$$\prod_{j=1}^{r} \prod_{m=0}^{\infty} \omega_j^{k_{jm}}(m),$$

where only a finite number of integers k_{jm} are distinct from zero, we have that the Fourier transform $g_N(\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = (k_{jm} : k_{jm} \in \mathbb{Z},\ m \in \mathbb{N}_0)$, $j = 1, \dots, r$, of Q_N is

$$\begin{aligned} g_N(\underline{k}_1, \dots, \underline{k}_r) &= \int_{\Omega^r} \prod_{j=1}^{r} \prod_{m=0}^{\infty} \omega_j^{k_{jm}}(m) \mathrm{d}\, Q_N = \frac{1}{N+1} \sum_{k=0}^{N} \prod_{j=1}^{r} \prod_{m=0}^{\infty} (m+\alpha_j)^{-ikhk_{jm}} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \exp\left\{ -ikh \sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m+\alpha_j) \right\}, \end{aligned} \tag{1}$$

where $\sum'$ means that only a finite number of integers k_{jm} are distinct from zero. Clearly,

$$g_N(\underline{0}, \dots, \underline{0}) = 1. \tag{2}$$

Since the set $L(\alpha_1, \dots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$,

$$\exp\left\{ -ih \sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m+\alpha_j) \right\} \neq 1$$

for $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$. Actually, if this inequality is not true, the

$$h \sum_{j=1}^{r} {\sum_{m=0}^{\infty}}' k_{jm} \log(m+\alpha_j) - \frac{2\pi l}{h} = 0$$

with $l \in \mathbb{Z}$, and this contradicts the linear independence of the set $L(\alpha_1, \dots, \alpha_r; h, \pi)$. Thus, in this case, we find by (1) that

$$g_N(\underline{k}_1, \dots, \underline{k}_r) = \frac{1 - \exp\left\{ -(N+1)ih \sum_{j=1}^{r} \sum_{m=0}^{\prime\infty} k_{jm} \log(m+\alpha_j) \right\}}{(N+1)\left(1 - \exp\left\{ -ih \sum_{j=1}^{r} \sum_{m=0}^{\prime\infty} k_{jm} \log(m+\alpha_j) \right\}\right)}.$$

This and (2) show that

$$\lim_{N\to\infty} g_N(\underline{k}_1, \dots, \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, \dots, \underline{k}_r) = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{if } (\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H, the lemma is proved. □

Now, we will apply Lemma 1 to obtain a joint limit theorem in the space of analytic functions for functions given by absolutely convergent Dirichlet series connected to Lerch zeta-functions. Let $\hat{\sigma} > \frac{1}{2}$ be a fixed number, and, for $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$,

$$v_n(m, \alpha_j) = \exp\left\{ -\left(\frac{m+\alpha_j}{n+\alpha_j} \right)^{\hat{\sigma}} \right\}, \quad j = 1, \dots, r.$$

Define

$$L_n(\underline{\lambda}, \underline{\alpha}, s) = (L_n(\lambda_1, \alpha_1, s), \ldots, L_n(\lambda_r, \alpha_r, s))$$

and

$$L_n(\underline{\lambda}, \underline{\alpha}, s, \omega) = (L_n(\lambda_1, \alpha_1, s, \omega_1), \ldots, L_n(\lambda_r, \alpha_r, s, \omega_r)),$$

where

$$L_n(\lambda_j, \alpha_j, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m} v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \ldots, r,$$

and

$$L_n(\lambda_j, \alpha_j, s, \omega) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m} \omega_j(m) v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \ldots, r,$$

It is known [7] that the series for $L_n(\lambda_j, \alpha_j, s)$ and $L_n(\lambda_j, \alpha_j, s, \omega_j)$ are absolutely convergent for $\sigma > \frac{1}{2}$.

The next lemma deals with weak convergence for

$$P_{N,n}(A) \stackrel{\text{def}}{=} \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : L_n(\underline{\lambda}, \underline{\alpha}, s + ikh) \in A\right\}, \quad A \in \mathcal{B}(H^r(D)).$$

Define the function $u_n : \Omega^r \to H^r(D)$ by the formula

$$u_n(\omega) = L_n(\underline{\lambda}, \underline{\alpha}, s, \omega), \quad \omega \in \Omega.$$

Since the series for $L_n(\lambda_j, \alpha_j, s, \omega_j), \quad j = 1, \ldots, r$, are absolutely convergent for $\sigma > \frac{1}{2}$, the function u_n is continuous, hence it is $(\mathcal{B}(\Omega^r), \mathcal{B}(H^r(D)))$-measurable. Therefore, the measure m_H induces [1] on $(H^r(D), \mathcal{B}(H^r(D)))$ the unique probability measure $\hat{P}_n \stackrel{\text{def}}{=} m_H u_n^{-1}$, where, for $A \in \mathcal{B}(H^r(D))$,

$$\hat{P}_n(A) = m_H u_n^{-1}(A) = m_H(u_n^{-1} A).$$

Lemma 2. *Suppose that the set $L(\alpha_1, \ldots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. Then $P_{N,n}$ converges weakly to $\hat{P}_n$ as $N \to \infty$.*

Proof.

Let Q_N be defined in Lemma 1. Then the definitions of $P_{N,n}$, Q_N and u_n show that,for every $A \in \mathcal{B}(H^r(D))$,

$$P_{N,n}(A) = \frac{1}{N+1} \# \Big\{0 \leqslant k \leqslant N : \Big(((m+\alpha_1)^{-ikh} : m \in \mathbb{N}_0), \ldots, \\ ((m+\alpha_r)^{-ikh} : m \in \mathbb{N}_0)\Big) \in u_n^{-1} A\Big\} = Q_N(u_n^{-1} A),$$

i.e., $P_{N,n} = Q_N u_n^{-1}$. This, Lemma 1, the continuity of u_n and Theorem 5.1 from [1] show that $P_{N,n}$ converges weakly to the measure $m_H u_n^{-1}$ as $N \to \infty$.

Now, we will approximate $L(\underline{\lambda}, \underline{\alpha}, s)$ by $L_n(\underline{\lambda}, \underline{\alpha}, s)$. For $g_1, g_2 \in H(D)$, let

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\}$ is a sequence of compact subsets of the strip D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact subset, then $K \subset K_l$ for some l. The proof of the existence of the sequence $\{K_l : l \in \mathbb{N}\}$ can be found, for example, in [2]. The metric ρ induces the topology of the space $H(D)$ of uniform convergence on compacta. The metric $\underline{\rho}$ in $H^r(D)$ inducing the product topology is defined by

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1\leqslant j\leqslant r} \rho(\underline{g}_{1j}, \underline{g}_{2,j}),$$

where $\underline{g}_1 = (g_{11}, \dots g_{1r})$, $\quad \underline{g}_2 = (g_{21}, \dots g_{2r}) \in H^r(D)$. □

Lemma 3. *For all $\underline{\lambda}$, $\underline{\alpha}$ and $h > 0$,*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \underline{\rho}\left(L(\underline{\lambda}, \underline{\alpha}, s+ikh), L_n(\underline{\lambda}, \underline{\alpha}, s+ikh)\right) = 0.$$

Proof.

The definition of the metric $\underline{\rho}$ shows that the equality of the lemma follows from the equalities

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho\left(L_j(\lambda_j, \alpha_j, s+ikh), L_n(\lambda_j, \alpha_j, s+ikh)\right) = 0,$$

$j = 1, \dots, r$, that were obtained in Lemma 3 of [12]. □

We recall that the measure $\hat{P}_n$ was defined in Lemma 2.

Lemma 4. *Suppose that the set $L(\alpha_1, \dots, \alpha_r; h, \pi)$ is linearly independent over $\mathbb{Q}$. Then the sequence $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact subset $K = K(\varepsilon) \subset H^r(D)$ such that*

$$\hat{P}_n(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$.

Proof.

Consider the marginal measures of $\hat{P}_n$, i.e., the measures

$$\hat{P}_{n,j}(A) = \hat{P}_n\left(\underbrace{H(D) \times \dots \times H(D)}_{j-1} \times A \times H(D) \times \dots \times H(D)\right), \quad A \in \mathcal{B}(H(D)),$$

where $j = 1, \dots, r$. The linear independence of the set $L(\alpha_1, \dots, \alpha_r; h, \pi)$ implies that for $L(\alpha_j, h, \pi)$, $j = 1, \dots, r$. Therefore, in view of the proof of Lemma 5 from [12], we have that $\hat{P}_{n,j}$ converges weakly to the distribution P_{L_j} of the random element $L_j(\lambda_j, \alpha_j, s, \omega_j)$ as $n \to \infty$, $j = 1, \dots, r$. Hence, the sequence $\{\hat{P}_{n,j} : n \in \mathbb{N}\}$ is relatively compact, $j = 1, \dots, r$. Since the set $H(D)$ is complete and separable, by the inverse Prokhorov Theorem [1, Theorem 6.2], the sequence $\{\hat{P}_{n,j} : n \in \mathbb{N}\}$ is tight, $j = 1, \dots, r$. Thus, for every $\varepsilon > 0$, there exists a compact subset $K_j \subset H(D)$ such that

$$\hat{P}_n(K_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, \dots, r,$$

for all $n \in \mathbb{N}$. The set $K = K_1 \times \dots \times K_r$ is compact in $H^r(D)$. Moreover,

$$\hat{P}_n(H^r(D) \setminus K) = \hat{P}_n\left(\bigcup_{j=1}^{r} (H(D) \setminus K_j)\right) \leqslant \sum_{j=1}^{r} \hat{P}_{n,j}(H(D) \setminus K_j) < \varepsilon$$

for all $n \in \mathbb{N}$, i.e.,the sequence $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight. □

For convenience, we recall one result from [1]. Suppose that (S, ϱ)-valued random elements $Y_n, X_{1n}, X_{2n}, \dots$ are defined on the same probability space with measure $\mathbb{P}$, and that the space S is separable.

Lemma 5. *Suppose that, for every k,*

$$X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k$$

and

$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X.$$

Moreover, for every $\varepsilon > 0$, let

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}\{\rho(X_{kn}, Y_n) \geqslant \varepsilon\} = 0.$$

Then $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

The lemma is Theorem 4.2 from [1].

Proof of Theorem 3. By Lemma 4 and the Prokhorov theorem [1, Theorem 6.1], the sequence $\{\hat{P}_n : n \in \mathbb{N}\}$ is relatively compact. Hence, every subsequence of $\hat{P}_n$ contains a subsequence $\{\hat{P}_{n_k}\}$ such that $\hat{P}_{n_k}$ converges weakly to a certain probability measure P on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \to \infty$. Therefore, denoting by $\hat{X}_n = \hat{X}_n(s)$ the $H^r(D)$-valued random element having the distribution $\hat{P}_n$, we have that

$$\hat{X}_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{3}$$

Moreover, by Lemma 2,

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} \hat{X}_n, \tag{4}$$

where the $H^r(D)$-valued random element $X_{N,n} = X_{N,n}(s)$ is defined by

$$X_{N,n}(s) = L_n(\underline{\lambda}, \underline{\alpha}, s + i\theta_N),$$

and θ_N is a random variable defined on a certain probability space $(\hat{\Omega}, \mathcal{F}, \mathbb{P})$ by the formula

$$\mathbb{P}(\theta_N = kh) = \frac{1}{N+1}, \quad k = 0, 1, \dots, N.$$

Define one more $H^r(D)$-valued random element

$$Y_N = Y_N(s) = L(\underline{\lambda}, \underline{\alpha}, s + i\theta_N).$$

Then, in view of Lemma 3, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \mathbb{P}(\underline{\varrho}(X_{N,n}, Y_N) \geqslant \varepsilon) \\
&\quad = \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : \underline{\rho}\left(L(\underline{\lambda}, \underline{\alpha}, s + ikh), L_n(\underline{\lambda}, \underline{\alpha}, s + ikh)\right) \geqslant \varepsilon\right\} \\
&\quad \leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N+1)\varepsilon} \sum_{k=0}^{N} \underline{\varrho}(L(\underline{\lambda}, \underline{\alpha}, s + ikh), L_n(\underline{\lambda}, \underline{\alpha}, s + ikh)) = 0.
\end{aligned}$$

This equality together with relations (3) and (4) shows that all hypotheses of Lemma 5 are satisfied. Therefore, we obtain the relation

$$Y_N \xrightarrow[N\to\infty]{\mathcal{D}} P. \tag{5}$$

Thus, we have that P_N converges weakly to P as $N \to \infty$. Moreover, the relation (5) shows that the measure P is independent of the choice of the subsequence $\hat{P}_{n_k}$. Since the sequence $\hat{P}_n$ is relatively compact, hence we obtain that

$$\hat{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P.$$

This means that $\hat{X}_n$ converges weakly to P as $n \to \infty$. The latter remark allows easily to identify the measure P. Actually, in [16], it was obtained that, under hypothesis that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$,

$$\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : L(\underline{\lambda}, \underline{\alpha}, s+i\tau) \in A\right\}, \quad A \in \mathcal{B}(H^r(D)), \tag{6}$$

also converges weakly to the limit measure P of $\hat{P}_n$ as $n \to \infty$, and that P coincides with P_L. Obviously,the linear independence of the set $L(\alpha_1, \dots, \alpha_r; h, \pi)$ implies that of the set $L(\alpha_1, \dots, \alpha_r)$. Therefore, P_N also converges weakly to P_L which is the limit measure of $\hat{P}_n$. The theorem is proved. □

3. Proofs of universality

We remind the Mergelyan theorem on approximation of analytic functions by polynomials [15].

LEMMA 6. *Let K be a compact subset on the complex plane with connected complement, and let $f(s)$ be a function continuous on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

We also need the explicit form of the support of the measure P_L. We recall that the support of P_L is a closed minimal set S_L such that $P_L(S_L) = 1$. The set S_L consists of all $\underline{g} \in H^r(D)$ such that, for every open neighbourhood G of $\underline{g}$, the inequality $P_L(G) > 0$ is true.

LEMMA 7. *The support of the measure P_L is the whole of $H^r(D)$.*

Proof.
It was observed above that P_L is the limit measure of (6). Thus, the lemma follows from [16], see the proof of Theorem 2.1. □

We also recall two equivalents of the weak convergence of probability measures. Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(X, \mathcal{B}(X))$. The set $A \in \mathcal{B}(X)$ is called a continuity set of P if $P(\partial A) = 0$, where ∂A is the boundary of A.

LEMMA 8. *The following statements are equivalent:*
1° *P_n converges weakly to P;*
2° *for every open set $G \subset X$,*

$$\liminf_{n \to \infty} P_n(G) \geqslant P(G),$$

3° *for every continuity set A of the measure P,*

$$\lim_{n \to \infty} P_n(A) = P(A).$$

The lemma is a part of Theorem 2.1 from [1].

Proof of Theorem 2.
In view of Lemma 6, there exist polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{2}. \tag{7}$$

Consider the set

$$G_\varepsilon = \left\{(g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |g_j(s) - p_j(s)| < \frac{\varepsilon}{2}\right\}.$$

Then the set G_ε is open, and,by Lemma 7, is a neighborhood of the collection $(p_1(s),\dots,p_r(s))$ which is an element of the support of the measure P_L. Therefore, the inequality

$$P_L(G_\varepsilon) > 0 \tag{8}$$

is satisfied. Hence, by Theorem 4 and 2° of Lemma 8,

$$\liminf_{N\to\infty} P_N(G_\varepsilon) \geqslant P_L(G_\varepsilon) > 0. \tag{9}$$

This, and the definitions of P_N and G_ε show that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(\lambda_j,\alpha_j,s+ikh) - p_j(s)| < \frac{\varepsilon}{2} \right\} > 0. \tag{10}$$

Let $k \in \mathbb{N}$ satisfy the inequality

$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(\lambda_j,\alpha_j,s+ikh) - p_j(s)| < \frac{\varepsilon}{2}.$$

Then, for such k, (7) implies the inequality

$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(\lambda_j,\alpha_j,s+ikh) - f_j(s)| < \varepsilon.$$

Therefore, (10) gives the assertion of the theorem. □

Proof of Theorem 3.

Consider the set

$$\hat{G}_\varepsilon = \left\{ (g_1,\dots,g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$

Then the set $\hat{G}_\varepsilon$ is open. Moreover, the boundary ∂G_ε lies in the set

$$\left\{ (g_1,\dots,g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| = \varepsilon \right\}.$$

Therefore, $\partial\hat{G}_{\varepsilon_1} \cap \partial\hat{G}_{\varepsilon_2} = \varnothing$ for positive $\varepsilon_1 \neq \varepsilon_2$. From this, it follows that $P_L(\hat{G}_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$, i.e., the set $\hat{G}_\varepsilon$ is a continuity set of P_L for all but at most countably many $\varepsilon > 0$. Hence, by Theorem 4, and 1° and 3° of Lemma 8, the limit

$$\lim_{N\to\infty} P_N(\hat{G}_\varepsilon) = P_L(\hat{G}_\varepsilon) \tag{11}$$

exists for all but at most countably many $\varepsilon > 0$. Moreover, it is not difficult to see that if $(g_1,\dots,g_r) \in G_\varepsilon$, where G_ε is defined in the proof of Theorem 2, then, taking into account (7), we find that

$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| \leqslant \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - p_j(s)| + \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |f_j(s) - p_j(s)| < \varepsilon.$$

This shows that $G_\varepsilon \subset \hat{G}_\varepsilon$. Since, by (9), $P_L(G_\varepsilon) > 0$, the monotonicity of the measure gives the inequality $P_L(\hat{G}_\varepsilon) > 0$. This inequality and (11) prove the theorem. □

4. Conclusions

The Lerch zeta-function $L(\lambda, \alpha, s)$, $s = \sigma + it$, with parameters $\lambda \in \mathbb{R}$ and $0 < \alpha \leqslant 1$ is defined, for $\sigma > 1$, by the series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. In the paper, it is obtained that a collection of Lerch zeta-functions $(L(\lambda_1, \alpha_1, s), \ldots, L(\lambda_r, \alpha_r, s))$ has a discrete universality property, i.e., a wide class of analytic functions can be approximated by shifts $L(\lambda_1, \alpha_1, s + ikh), \ldots, L(\lambda_r, \alpha_r, s + ikh)$, $h > 0$, $k = 0, 1, 2, \ldots$. For this, the linear independence over $\mathbb{Q}$ of the set

$$\left\{(\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \ldots, r), \frac{2\pi}{h}\right\}$$

is required. More precisely, if $K_1, \ldots, K_r$ are compact subsets of the strip $\left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$ with connected complements, and $f_1(s), \ldots, f_r(s)$ are functions continuous on $K_1, \ldots, K_r$ and analytic in the interior of $K_1, \ldots, K_r$, respectively, then, for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(\lambda_j, \alpha_j, s + ikh) - f_j(s)| < \varepsilon \right\} > 0.$$

It is possible to consider a more general situation, i.e., to consider the approximation of $f_1(s), \ldots, f_r(s)$ by different shifts $L(\lambda_1, \alpha_1, s+ikh_1), \ldots, L(\lambda_r, \alpha_r, s+ikh_r)$ with $h_1 > 0, \ldots, h_r > 0$. For this case, a new more general method than that of the paper is required, and it will be developed in a subsequent paper.

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