ЧЕБЫШЕВСКИЙ СБОРНИК
Том 16 Выпуск 1 (2015)

УДК 519.14

MIXED JOINT UNIVERSALITY FOR L-FUNCTIONS FROM SELBERG'S CLASS AND PERIODIC HURWITZ ZETA-FUNCTIONS

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To the memory of Professor A.A. Karatsuba

Abstract

In 1975, a Russian mathematician S. M. Voronin discovered the universality property of the Riemann zeta-function $\zeta(s)$, $s=\sigma+it$. Roughly speaking, this means that analytic functions from a wide class can be approximated uniformly on compact subsets of the strip $\{s\in\mathbb{C} : 1/2<\sigma<1\}$ by shifts $\zeta(s+i\tau)$, $\tau\in\mathbb{R}$. Later, it turned out that other classical zeta and L-functions are also universal in the Voronin sense. Moreover, some zeta and L-functions have a joint universality property. In this case, a given collection of analytic functions is approximated simultaneously by shifts of zeta and L-functions.

In the paper, we present our extended report given at the Conference dedicated to the memory of the famous number theorist Professor A. A. Karacuba. The paper contains the basic universality results on the so-called mixed joint universality initiated by H. Mishou who in 2007 obtained the joint universality for the Riemann zeta and Hurwitz zeta-functions. In a wide sense the mixed joint universality is understood as a joint universality for zeta and L-functions having and having no Euler product.

In 1989, A. Selber introduced a famous class $\mathcal{S}$ of Dirichlet series satisfying certain natural hypotheses including the Euler product. Periodic Hurwitz zeta-functions are a generalization of classical Hurwitz zeta-functions, and have no Euler product. In the paper, a new result on mixed joint universality for L-functions from the Selberg clas and periodic Hurwitz zeta-functions is presented. For the proof a probabilistic method can be applied.

Keywords: Riemann zeta-function, Hurwitz zeta-function, periodic Hurwitz zeta-function, Selberg class, universality, joint universality.

Bibliography: 24 titles.

СМЕШАННАЯ СОВМЕСТНАЯ УНИВЕРСАЛЬНОСТЬ ДЛЯ

L-ФУНКЦИЙ КЛАССА СЕЛЬБЕРГА И ПЕРИОДИЧЕСКИХ ДЗЕТА-ФУНКЦИЙ ГУРВИЦА

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Посвящается памяти профессора А.А. Карацубы

Аннотация

В 1975 г. российский математик С. М. Воронин открыл свойство универсальности дзета-функции Римана $\zeta(s)$, $s = \sigma + it$. Грубо говоря, это означает, что широкого класса аналитические функции могут быть приближены равномерно на компактных подмножествах полоса $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ сдвигами $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. Позже окозалось, что и многие другие классические дзета и L-функции также обладают универсальностью в смысле Воронина. Кроме того, некоторые дзета и L-функции имеют совместное свойство универсальности. В этом случае, данный набор аналитических функций одновременно приближается сдвигами дзета или L-функций.

В статье мы даем расширенный текст нашего доклада, прочитанного на конференции, посвященной памяти известного числовика профессора А. А. Карацубы. Статья содержит обзор основных результатов о так называемой смешанной совместной универсальности, начало которой было было дано японским математиком Г. Мишу в 2007, доказавшим совместную универсальность дзета-функций Римана и Гурвица. В широком смысле смешанная совмесная универсальность понимается как совмесная универсальность дзета и L-функций, имеющих эйлеровское произведение по простым числам и неимеющих такого произведения.

В 1989 г. А. Сельберг ввел замечательный класс $\mathcal{S}$ рядов Дирихле, удовлетворяющих некоторым натуральным условиям, включая эйлеровское прозведение. Периодические дзета-функции Гурвица являются обобщением классических дзета-функций Гурвица и не имеют эйлеровного произведения. В статье формулируется новая теорема о смешанной совместной универсальности для L-функций из класса Сельберга и периодических дзета-функций Гурвица. Для доказатеьства может быть применен вероятностный метод.

Ключевые слова: дзета-функция Римана, дзета-функция Гурвица, периодическая дзета-функция Гурвица, класс Сельберга, универсальность, совместная универсальность.

Библиография: 24 названия.

1. Introduction

The start point of the universality theory for zeta and L-functions is a remarkable work of S. M. Voronin [21], a student of Professor A. A. Karatsuba, on the universality of the Riemann zeta function $\zeta(s)$, $s = \sigma + it$, which is defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product is taken over all primes p. The initial form of the Voronin theorem is the following [21].

THEOREM 1. *Let $0 < r < \frac{1}{4}$. Suppose that $f(s)$ is continuous non-vanishing function on the disc $\{s \in \mathbb{C} : |s| \leq r\}$ which is analytic in the interior of this disc. Then, for every $\epsilon > 0$, there exists a real number $\tau = \tau(\epsilon)$ such that*

$$\max_{|s| \leq r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \epsilon.$$

Various authors found a more general form of Theorem 1, and observed that analogues of this theorem are also true for other classical zeta and L-functions. To state a modern version of the Voronin theorem, we need some notations. Let $\Delta = \{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2\}$ be a vertical strip on the complex plane. Denote by $\mathcal{K}(\Delta)$ the class of compact subsets of the strip Δ with connected complements, by $H(K)$, $K \in \mathcal{K}(\Delta)$, the class of continuous functions on K which are analytic in the interior of K, and by $H_0(K)$, $K \in \mathcal{K}(\Delta)$, the class of continuous non-vanishing functions on K which are analytic in the interior of K. Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$, and let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then we have the following statement, see, for example [7], [20].

THEOREM 2. *Suppose that $K \in \mathcal{K}(D)$ and $f(s) \in H_0(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \epsilon \right\} > 0.$$

Different methods are known for the proof of Theorems 1 and 2. The original Voronin method is based on a version of Riemann's theorem on rearrangement of series in Hilbert space, on the approximation of $\zeta(s)$ in the mean by finite Euler's product as well as on the Kronecker approximation theorem. There exists also a probabilistic proof proposed by B. Bagchi in his thesis [1]. This proof uses a limit theorem on the weak convergence of the probability measure

$$P_T(A) \stackrel{def}{=} \frac{1}{T} \{\tau \in [0, T] : \zeta(s + i\tau) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

as $T \to \infty$, where $\mathcal{B}(X)$ denotes the Borel σ-field of the space X, and $H(D)$ is the space of analytic functions on D endowed with the topology of uniform convergence on compacta. It is proved that P_T, as $T \to \infty$, converges weakly to a probability measure on $(H(D), \mathcal{B}(H(D)))$ which support is the set

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Moreover, for the proof of Theorem 2, the Mergelyan theorem on the approximation of analytic functions by polynomials [14], see also [24], is applied.

The universality problems for zeta and L-functions were investigated by a large group of number theorists, among them, already mentioned above B. Bagchi, H. Bauer, A. Dubickas, R. Garunkštis, J. Genys, S. M. Gonek, A. Good, R. Kačinskaitė, J. Kaczorowski, A. Laurinčikas, J.-L. Mauclaire, Kohji Matsumoto, H. Mishou, Y. Nagoshi, T. Nakamura, Ł. Pańkowski, A. Reich, J. Sander, W. Schwarz, J. Steuding, R. Steuding, the author and others.

2. Selberg class $\mathcal{S}$

We will briefly discuss the universality of L-functions from the famous Selberg class $\mathcal{S}$ [18]. The class $\mathcal{S}$ consists from Dirichlet series

$$\mathcal{L}(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}, \quad \sigma > 1,$$

satisfying the following hypotheses:

1^o Ramanujan hypothesis: for every $\epsilon > 0$, the estimate $a(m) \ll m^\epsilon$ is true;

2^o Analytic continuation: there exists an integer $r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that $(s-1)^r \mathcal{L}(s)$ is an entire function of a finite order;

3^o Functional equation: there exist Q, $\lambda_j \in \mathbb{R}_+$, $\mu_j, w \in \mathbb{C}$ with $\Re\mu_j \geq 0$ and $|w| = 1$, $j = 1, ..., l$, such that the function

$$\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s) Q^s \prod_{j=1}^{l} \Gamma(\lambda_j s + \mu_j),$$

where $\Gamma(s)$ is the Euler gamma-function, satisfies the functional equation

$$\Lambda_{\mathcal{L}}(s) = w \overline{\Lambda_{\mathcal{L}}(1 - \overline{s})};$$

4^o Euler product: there exist the numbers $b(p^\alpha)$, $b(p^\alpha) \ll p^{\alpha\theta}$ with a certain $\theta < \frac{1}{2}$, such that

$$\mathcal{L}(s) = \prod_p \exp\left\{\sum_{\alpha=1}^{\infty} \frac{b(p^\alpha)}{p^{\alpha s}}\right\}.$$

Examples of the functions from the class $\mathcal{S}$ are the following:

1. The Riemann zeta function $\zeta(s)$;

2. Dirichlet L-functions

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s},\quad \sigma>1,$$

where χ is a Dirichlet character;

3. L-functions of algebraic number fields

$$L_k(s,\chi)=\sum_{I}\frac{\chi(I)}{N(I)^s},\quad \sigma>1,$$

where k is an algebraic number field, I runs over the non-zero ideals of the ring of integers of k, $N(I)$ denotes the norm of I, and χ is a Hecke character;

4. L-functions of holomorphic modular forms $f(z)$

$$L_f(s)=\sum_{m=1}^{\infty}\frac{a(m)}{m^s},\quad \sigma>1,$$

where $a(m)$ are normalized Fourier coefficients of $f(z)$.

The theory on the structure of the Selberg class $\mathcal{S}$ can be found in [5].

In [19], J. Steuding introduced a new class $\widetilde{\mathcal{S}}$. The functions from $\widetilde{\mathcal{S}}$ satisfy hypotheses 1^o and 2^o of the class $\mathcal{S}$, hypothesis 4^o of $\mathcal{S}$ is replaced by polynomial Euler product, i.e., there exists $m\in\mathbb{N}$ and, for every prime p, there are complex numbers $c_j(p)$, $1\le j\le m$, such that

$$\mathcal{L}(s)=\prod_{p}\prod_{j=1}^{m}\left(1-\frac{c_j(p)}{p^s}\right)^{-1}.$$

Moreover, an additional prime mean-square hypothesis that there exists a constant $\kappa>0$ such that

$$\lim_{x\to\infty}\frac{1}{\pi(x)}\sum_{p\le x}|a(p)|^2=\kappa,$$

where

$$\pi(x)=\sum_{p\le x}1,$$

is required.

The universality of functions from the class $\widetilde{\mathcal{S}}\cap\mathcal{S}$ was considered in [18], [20] and [16]. For $\mathcal{L}\in\mathcal{S}$, the degree $d_{\mathcal{L}}$ of $\mathcal{L}$ is defined by

$$d_{\mathcal{L}}=2\sum_{j=1}^{l}\lambda_j.$$

Let

$$D_{\mathcal{L}}=\left\{s\in\mathbb{C}:\max\left(\frac{1}{2},1-\frac{1}{d_{\mathcal{L}}}\right)<\sigma<1\right\}.$$

We state the universality theorem from [16]. Let $H_{0\mathcal{L}}(K)$, $K \in \mathcal{K}(D_{\mathcal{L}})$, be the class of continuous non-vanishing functions on K which are analytic in the interior of K.

THEOREM 3. *Suppose that $\mathcal{L} \in \mathcal{S}$ and that the prime mean-square hypothesis is satisfied. Let $K \in \mathcal{K}(D_{\mathcal{L}})$ and $f(s) \in H_{0\mathcal{L}}(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\mathcal{L}(s+i\tau) - f(s)| < \epsilon\right\} > 0.$$

3. Hurwitz zeta-function $\zeta(s,\alpha)$

Theorems 2 and 3 are universality theorems for zeta and L-functions having the Euler product over primes. Also, zeta-functions without Euler product are universal in a similar sense. The simplest example of zeta-functions without Euler product is the Hurwitz zeta-function $\zeta(s,\alpha)$ with fixed parameter α, $0 < \alpha \le 1$. The function $\zeta(s,\alpha)$ is defined, for $\sigma > 1$, by the series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and is meromorphically continued to the whole complex plane with unique simple pole at the point $s = 1$ with residue 1. Obviously, $\zeta(s,1) = \zeta(s)$ and

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\,\zeta(s).$$

If $\alpha \ne 1, \frac{1}{2}$, then the function $\zeta(s,\alpha)$ has no Euler product over primes. Universality of $\zeta(s,\alpha)$ is given in the following theorem.

THEOREM 4. *Suppose that the parameter α is transcendental or rational $\ne 1, \frac{1}{2}$. Let $K \in \mathcal{K}(D)$ and $f(s) \in H(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \epsilon\right\} > 0.$$

The case of rational α was obtained in [1], [3] and [22]. The case of transcendental α is given [1]. Universality of $\zeta(s,\alpha)$ with algebraic irrational α till now is an open problem. For $\alpha = 1$ and $\alpha = \frac{1}{2}$, the function $\zeta(s,\alpha)$ remains universal, however, in this cases $f(s) \in H_0(K)$.

A natural generalization of the function $\zeta(s,\alpha)$ is the periodic Hurwitz zeta-function. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. The periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$ is given, for $\sigma > 1$, by the series

$$\zeta(s,\alpha;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$

The periodicity of the sequence $\mathfrak{a}$ implies, for $\sigma > 1$, the equality

$$\zeta(s, \alpha; \mathfrak{a}) = \frac{1}{q^s} \sum_{m=0}^{q-1} a_m \zeta\left(s, \frac{m+\alpha}{q}\right),$$

and this gives meromorphic continuation to the whole complex plane for $\zeta(s, \alpha; \mathfrak{a})$ with a possible simple pole at the point $s = 1$. If

$$\sum_{m=0}^{q-1} a_m = 0,$$

then the function $\zeta(s, \alpha; \mathfrak{a})$ is entire one.

For the function $\zeta(s, \alpha; \mathfrak{a})$ with transcendental α, an analogue of Theorem 4 was obtained in [4].

4. Joint universality

For zeta and L-functions, also the joint universality is known. In this case, a collection of given analytic functions is approximated simultaneously by shifts of zeta or L-functions. The first result in this direction belongs to S. M. Voronin who obtained [22] a joint universality theorem for Dirichlet L-functions. We state a modern version of his theorem [8].

THEOREM 5. *Suppose that* $\chi_1, ..., \chi_r$ *are pairwise non-equivalent Dirichlet characters. For* $j = 1, ..., r$, *let* $K_j \in \mathcal{K}(D)$ *and* $f(s) \in H_0(K_j)$. *Then, for every* $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{1\le j\le r} \sup_{s\in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \epsilon\right\} > 0.$$

Joint universality of periodic zeta-functions was considered in a several works, and we will recall only the most general result obtained in [10]. For $j = 1, ..., r$, let $l_j \in \mathbb{N}$, and, for every $l = 1, \dots, l_j$, let $\mathfrak{a}_{jl} = \{a_{mjl} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $q_{jl} \in \mathbb{N}$. Moreover, for $j = 1, ..., r$, let $0 < \alpha_j \le 1$, q_j be the least common multiple of the periods $q_{j1}, ..., q_{jl_j}$, and

$$A_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \dots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \dots & a_{2jl_j} \\ \dots & \dots & \dots & \dots \\ a_{q_j j1} & a_{q_j j2} & \dots & a_{q_j jl_j} \end{pmatrix}.$$

Then the following statement is true [10].

Theorem 6. *Suppose that the set*

$$\mathrm{L}(\alpha_1,...,\alpha_r) \stackrel{def}{=} \{\log(m+\alpha_j) :\ m \in \mathbb{N}_0,\ j = 1,\dots,r\}$$

is linearly independent over the field of rational numbers $\mathbb{Q}$, *and that* $\operatorname{rank}(A_j) = l_j$, $j = 1,...,r$. *For every* $j = 1,...,r$ *and* $l = 1,...,l_j$, *let* $K_{jl} \in \mathcal{K}(D)$ *and* $f_{jl}(s) \in H(K_{jl})$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] :\ \sup_{1\le j\le r}\ \sup_{1\le l\le l_j}\ \sup_{s\in K_{jl}} |\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\right\} > 0.$$

The aim of this paper is the so-called mixed joint universality when analytic functions from classes $H_0(K)$ and $H(K)$, $K \in \mathcal{K}(D)$, are simultaneously approximated by shifts of zeta functions having and having no Euler product. The first result of such a kind was obtained by H. Mishou [15] for the functions $\zeta(s)$ and $\zeta(s,\alpha)$.

Theorem 7. *Suppose that* α *is a transcendental number,* $K_1, K_2 \in \mathcal{K}(D)$, $f_1 \in H_0(K_1)$ *and* $f_2 \in H(K_2)$. *Then, for every* $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\Big\{\tau \in [0;T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \epsilon,$$
$$\sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \epsilon\Big\} > 0.$$

The Mishou theorem was generalized in [6] for a pair consisting from the periodic zeta-function and periodic Hurwitz zeta-function. In [2], a mixed joint universality theorem was obtained for the zeta-functions $\zeta(s)$, $\zeta(s,\alpha_j;\mathfrak{a}_{jl})$, $j = 1,...,r$, $l = 1,...,l_j$. In [11], [12] and [17], the function $\zeta(s)$ was replaced by zeta-functions of normalized Hecke cusp forms, zeta-functions of newforms and zeta-functions of newforms with a Dirichlet character, respectively.

A very good survey on universality of zeta-functions is given in [13].

Now we state our new mixed joint universality theorem for the functions $\mathcal{L}(s)$ and $\zeta(s,\alpha_j;\mathfrak{a}_{jl})$, $j = 1,...,r$, $l = 1,...,l_j$, where $\mathcal{L} \in \mathcal{S}$.

Theorem 8. *Suppose that* $\mathcal{L} \in \mathcal{S}$, *the prime mean-square hypothesis is satisfied, the numbers* $\alpha_1,..,\alpha_r$ *are algebraically independent over* $\mathbb{Q}$, *and that* $\operatorname{rank}(A_j) = l_j$, $j = 1,...,r$. *Let* $K \in \mathcal{K}(D_{\mathcal{L}})$, $f(s) \in H_{0\mathcal{L}}(K)$, *and, for every* $j = 1,...,r$ *and* $l = 1,...,l_j$, *let* $K_{jl} \in \mathcal{K}(D)$ *and* $f_{jl}(s) \in H(K_{jl})$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\Big\{\tau \in [0,T] :\ \sup_{s\in K} |\mathcal{L}(s+i\tau) - f(s)| < \varepsilon,$$
$$\sup_{1\le j\le r}\ \sup_{1\le l\le l_j}\ \sup_{s\in K_{jl}} |\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\Big\} \ > \ 0.$$

Let

$$u = \sum_{j=1}^{r} l_j, \quad v = u + 1,$$

and

$$H^v = H^v(D_{\mathcal{L}}, D) = H(D_{\mathcal{L}}) \times H^u(D).$$

where $H(D_{\mathcal{L}})$ is the space of analytic on $D_{\mathcal{L}}$ functions. The proof of Theorem 8 is based on a limit theorem for the vector

$$Z(s_1,s,\underline{\alpha};\underline{\mathfrak{a}},\mathcal{L})=(\mathcal{L}(s_1),\zeta(s,\alpha_1;\mathfrak{a}_{11}),\dots,\zeta(s,\alpha_1;\mathfrak{a}_{1l_1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{rl_r})),$$

where $\underline{\alpha} = (\alpha_1, ..., \alpha_r)$, $\underline{\mathfrak{a}} = (\mathfrak{a}_{11}, ..., \mathfrak{a}_{1l_1}, ..., \mathfrak{a}_{r1}, ..., \mathfrak{a}_{rl_r})$. More precisely, the weak convergence for

$$\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \ Z(s_1 + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) \in A\right\}, \quad A \in \mathcal{B}(H^v),$$

is considered, as $T \to \infty$, and it is obtained that the latter measure converges weakly to the explicitly given probability measure on $(H^v, \mathcal{B}(H^v))$ with the support

$$\{g \in H(D_{\mathcal{L}}) : g(s) \neq 0 \text{ or } g(s) \equiv 0\} \times H^v(D).$$

This result together with the Mergelyan theorem leads to the assertion of Theorem 8.

5. Conclusion

In the paper, a survey of results on mixed joint universality of zeta and L-functions is given. Also, a new mixed joint universality theorem for L-functions from the Selberg class and and periodic Hurwitz zeta functions is stated.

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Получено 25.02.2015