ЧЕБЫШЕВСКИЙ СБОРНИК
Том 16 Выпуск 1 (2015)

УДК 519.14

JOINT DISCTRETE UNIVERSALITY OF DIRICHLET L-FUNCTIONS. II

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To the memory of Professor A.A. Karatsuba

Abstract

In 1975, S. M. Voronin obtained the universality of Dirichlet L-functions $L(s,\chi)$, $s=\sigma+it$. This means that, for every compact K of the strip $\{s\in\mathbb{C}: \frac{1}{2}<\sigma<1\}$, every continuous non-vanishing function on K which is analytic in the interior of K can be approximated uniformly on K by shifts $L(s+i\tau,\chi)$, $\tau\in\mathbb{R}$. Also, S. M. Voronin investigating the functional independence of Dirichlet L-functions obtained the joint universality. In this case, a collection of analytic functions is approximated simultaneously by shifts $L(s+i\tau,\chi_1),\dots,L(s+i\tau,\chi_r)$, where $\chi_1,\dots,\chi_r$ are pairwise non-equivalent Dirichlet characters.

The above universality is of continuous type. Also, a joint discrete universality for Dirichlet L-functions is known. In this case, a collection of analytic functions is approximated by discrete shifts $L(s+ikh,\chi_1),\dots,L(s+ikh,\chi_r)$, where $h>0$ is a fixed number and $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and was proposed by B. Bagchi in 1981. For joint discrete universality of Dirichlet L-functions, a more general setting is possible. In [3], the approximation by shifts $L(s+ikh_1,\chi_1),\dots,L(s+ikh_r,\chi_r)$ with different $h_1>0,\dots,h_r>0$ was considered. This paper is devoted to approximation by shifts $L(s+ikh_1,\chi_1),\dots,L(s+ikh_{r_1},\chi_{r_1}),L(s+ikh,\chi_{r_1+1}),\dots,L(s+ikh,\chi_r)$, with different $h_1,\dots,h_{r_1},h$. For this, the linear independence over $\mathbb{Q}$ of the set

$$L(h_1,\dots,h_{r_1},h;\pi)=\big\{(h_1\log p:\ p\in\mathcal{P}),\dots,(h_{r_1}\log p:\ p\in\mathcal{P}),\\ (h\log p:\ p\in\mathcal{P});\pi\big\},$$

where $\mathcal{P}$ denotes the set of all prime numbers, is applied.

Keywords: analytic function, Dirichlet L-function, linear independence, universality.

Bibliography: 10 titles.

СОВМЕСТНАЯ ДИСКРЕТНАЯ УНИВЕРСАЛЬНОСТЬ L-ФУНКЦИЙ ДИРИХЛЕ. II

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Аннотация

В 1975 г. С. М. Воронин доказал универсальность L-функций Дирихле $L(s,\chi)$, $s=\sigma+it$. Это означает, что для всякого компакта K полосы $\{s\in\mathbb{C}: \frac{1}{2}<\sigma<1\}$ любая непрерывная и неимеющая нулей в K, и аналитическая внутри K функция может быть приближена равномерно на K сдвигами $L(s+i\tau,\chi)$, $\tau\in\mathbb{R}$. Изучая функциональную независимость L-функций Дирихле, С. М. Воронин также установил их совместную универсальность. В этом случае набор аналитических функций одновременно приближается сдвигами $L(s+i\tau,\chi_1),\dots,L(s+i\tau,\chi_r)$, где $\chi_1,\dots,\chi_r$ попарно не эквивалентные характеры Дирихле.

Такая универсальность называется непрерывной универсальностью. Также известна дискретная универсальность L-функций Дирихле. В этом случае набор аналитических функций приближается дискретными сдвигами $L(s+ikh,\chi_1),\dots,L(s+ikh,\chi_r)$, где h некоторое фиксированное положительное число, а $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Такая постановка задачи была дана Б. Багчи в 1981 г., однако может рассматриваться более общий случай. В [3] было изучено приближение аналитических функций сдвигами $L(s+ikh_1,\chi_1),\dots,L(s+ikh_r,\chi_r)$ с различными $h_1>0,\dots,h_r>0$. Настоящая статья посвящена приближению сдвигами $L(s+ikh_1,\chi_1)$, $\dots,L(s+ikh_{r_1},\chi_{r_1}), L(s+ikh,\chi_{r_1+1}), \dots, L(s+ikh,\chi_r)$, с различными $h_1,\dots,h_{r_1},h$. При этом требуется линейная независимость над полем рациональных чисел для множества

$$L(h_1,\dots,h_{r_1},h;\pi)=\big\{(h_1\log p:\ p\in\mathcal{P}),\dots,(h_{r_1}\log p:\ p\in\mathcal{P}),\\ (h\log p:\ p\in\mathcal{P});\pi\big\},$$

где $\mathcal{P}$ – множество всех простых чисел.

Ключевые слова: аналитическая функция, L-функция Дирихле, линейная независимость, универсальность.

Библиография: 10 названий.

1. Introduction

Let $s = \sigma + it$ be a complex variable, and χ be a Dirichlet character. The corresponding Dirichlet L-function $L(s, \chi)$ is defined, for $\sigma > 1$, by the series

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

and is analytically continued to an entire function if χ is non-principal character. If χ is the principal character modulo q, then $L(s, \chi)$ has a meromorphic continuation to the whole complex plane with a simple pole at the point $s = 1$ with residue

$$\prod_{p|q} \left(1 - \frac{1}{p}\right),$$

where p denotes a prime number.

In [9], S. M. Voronin discovered the universality property of Dirichlet L-functions. Roughly speaking, this means that any function from a wide class of analytic functions can be approximated by shifts $L(s + i\tau, \chi)$, $\tau \in \mathbb{R}$. A strong statement of the modern version of the Voronin theorem is the following.

Let $\mathcal{K}$ be the class of compact subsets of the strip $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$ with connected complements, and let, for $K \in \mathcal{K}$, $H_0(K)$ denote the class of continuous non-vanishing functions on K which are analytic in the interior of K. Moreover, let meas$A$ be the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Theorem 1. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{\tau \in [0, T] : \sup_{s \in K} |L(s + i\tau, \chi) - f(s)| < \varepsilon\right\} > 0.$$

Dirichlet L-functions also are jointly universal, and this was obtained by S. M. Voronin in [10]. We state modern version of a joint universality theorem for Dirichlet L-functions which can be found in [5], [8].

Theorem 2. *Suppose that $\chi_1, \dots, \chi_r$ are pairwise non-equivalent Dirichlet characters. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{\tau \in [0, T] : \sup_{1 \le j \le r} \sup_{s \in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon\right\} > 0.$$

Theorems 1 and 2 are called continuous universality theorems because the real shift τ in $L(s + i\tau, \chi)$ takes arbitrary real values. Also, discrete universality theorem can be considered where τ takes values from the discrete set $\{hk : k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$, where $h > 0$ is a fixed number. B. Bagchi proved [1] a joint discrete universality theorem for Dirichlet L-functions which we state in a more general form. Denote by $\#A$ the number of elements of the set A.

THEOREM 3. *Suppose that $\chi_1, \dots, \chi_r$ are pairwise non-equivalent Dirichlet characters. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{1\le j\le r} \sup_{s\in K_j} |L(s+ikh, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

In [3], a version of Theorem 3 with different h for each L-function $L(s, \chi_j)$ was obtained. For its proof, a certain additional independence hypothesis is applied. Denote by $\mathcal{P}$ the set of all prime numbers, and define, for $h_1 > 0, \dots, h_r > 0$, the set

$$L(h_1, \dots, h_r; \pi) = \{(h_1 \log p : \ p \in \mathcal{P}), \dots, (h_r \log p : \ p \in \mathcal{P}); \pi\}.$$

THEOREM 4 ([3]). *Suppose that $\chi_1, \dots, \chi_r$ are pairwise non-equivalent Dirichlet characters, and that the set $L(h_1, \dots, h_r; \pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{1\le j\le r} \sup_{s\in K_j} |L(s+ikh_j, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

It is known [3] that the set $L(h_1, \dots, h_r; \pi)$ is linearly independent over $\mathbb{Q}$ for almost all $(h_1, \dots, h_r) \in \mathbb{R}^r_{>0}$ with respect to the Lebesgue measure on $\mathbb{R}^r$. During the memorial conference of A. A. Karatsuba, Professor Yu. V. Nesterenko constructed special examples of h_j. For example, in the case $r = 2$, the set $L(1, \sqrt{2}; \pi)$ is linearly independent over $\mathbb{Q}$.

The aim of this note is to give a modification of Theorem 4 which idea belongs to Professor I. S. Rezvyakova. Let $1 \le r_1 < r$, and, for $h_1 > 0, \dots, h_{r_1} > 0$ and $h > 0$,

$$L(h_1, \dots, h_{r_1}, h; \pi) = \big\{(h_1 \log p : \ p \in \mathcal{P}), \dots, (h_{r_1} \log p : \ p \in \mathcal{P}), (h \log p : \ p \in \mathcal{P}); \pi\big\}.$$

THEOREM 5. *Suppose that $\chi_1, \dots, \chi_r$ are pairwise non-equivalent Dirichlet characters, and that the set $L(h_1, \dots, h_{r_1}, h; \pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{1\le j\le r_1} \sup_{s\in K_j} |L(s+ikh_j, \chi_j) - f_j(s)| < \varepsilon, \sup_{r_1< j\le r} \sup_{s\in K_j} |L(s+ikh, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

For example, in the case $r = 4$, we can take $h_1 = 1$, $h_2 = \sqrt{2}$, $h = \sqrt{3}$.

2. Main lemmas

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane. Define the torus

$$\Omega = \prod_{p \in \mathcal{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathcal{P}$. With the product topology and pointwise multiplication, the torus Ω, by the Tikhonov theorem, is a compact topological Abelian group. Therefore, denoting by $\mathcal{B}(X)$ the Borel σ-field of the space X, we have that, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure can be defined. Moreover, we put

$$\Omega^{r_1+1} = \Omega_1 \times \cdots \times \Omega_{r_1+1},$$

where $\Omega_j = \Omega$ for $j = 1, \dots, r_1+1$. Then, by the Tikhonov theorem again, Ω^{r_1+1} is a compact topological group, and, on $(\Omega^{r_1+1}, \mathcal{B}(\Omega^{r_1+1}))$, the probability Haar measure m_H exists. Moreover, the measure m_H is the product of the Haar measures m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \dots, r_1+1$. Denote by $\omega_j(p)$ the projection of an element $\omega_j \in \Omega_j$ to the coordinate space γ_p, $p \in \mathcal{P}$, $j = 1, \dots, r_1+1$.

Now, for $A \in \mathcal{B}(\Omega^{r_1+1})$, define

$$Q_N(A) = \frac{1}{N+1} \# \big\{0 \le k \le N : \big((p^{-ikh_1} : p \in \mathcal{P}), \dots, (p^{-ikh_{r_1}} : p \in \mathcal{P}), (p^{-ikh} : p \in \mathcal{P})\big) \in A\big\}.$$

Lemma 1. *Suppose that the set $L(h_1, \dots, h_{r_1}, h; \pi)$ is linearly independent over $\mathbb{Q}$. Then Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We consider the Fourier transform $g_N(\underline{k})$ of the measure Q_N, where $\underline{k} = (k_{jp} : p \in \mathcal{P},\ j = 1, \dots, r_1+1)$. We have that

$$g_N(\underline{k}) = \int_{\Omega^{r_1+1}} \prod_{j=1}^{r_1+1} \prod_{p \in \mathcal{P}} \omega_j^{k_{jp}}(p) \mathrm{d}Q_N,$$

where only a finite number of integers k_{jp} are distinct from zero. Thus, the definition of Q_N gives

$$\begin{aligned} g_N(\underline{k}) &= \frac{1}{N+1} \sum_{k=0}^{N} \prod_{j=1}^{r_1} \prod_{p \in \mathcal{P}} p^{-ikk_{jp}h_j} \prod_{p \in \mathcal{P}} p^{-ik\hat{k}_p h} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \exp\left\{-ik\left(\sum_{j=1}^{r_1} \sum_{p \in \mathcal{P}} k_{jp} h_j \log p + \sum_{p \in \mathcal{P}} \hat{k}_p h \log p\right)\right\}, \end{aligned} \tag{1}$$

where, for brevity, $k_{r_1+1,p} = \hat{k}_p$.

Clearly,

$$g_N(\underline{0}) = 1. \tag{2}$$

Moreover, we observe that, for $\underline{k} \neq \underline{0}$,

$$\exp\left\{-i\left(\sum_{j=1}^{r_1}\sum_{p\in\mathcal{P}} k_{jp}h_j \log p + \sum_{p\in\mathcal{P}} \hat{k}_p h \log p\right)\right\} \neq 1. \tag{3}$$

Indeed, if inequality (3) is not true, then the equality

$$\sum_{j=1}^{r_1}\sum_{p\in\mathcal{P}} k_{jp}h_j \log p + h\sum_{p\in\mathcal{P}} \hat{k}_p \log p = 2\pi l$$

holds for some $l \in \mathbb{Z}$ and some finite number of integers k_{jp}, $\hat{k}_p$. However, this contradicts the linear independence of the set $L(h_1, \dots, h_{r_1}, h; \pi)$ over $\mathbb{Q}$. Now, from (1) – (3) we find that

$$g_N(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ \dfrac{1-\exp\left\{-i(N+1)\left(\sum\limits_{j=1}^{r_1}\sum\limits_{p\in\mathcal{P}} k_{jp}h_j \log p + h\sum\limits_{p\in\mathcal{P}} \hat{k}_p \log p\right)\right\}}{(N+1)\left(1-\exp\left\{-i\left(\sum\limits_{j=1}^{r_1}\sum\limits_{p\in\mathcal{P}} k_{jp}h_j \log p + h\sum\limits_{p\in\mathcal{P}} \hat{k}_p \log p\right)\right\}\right)} & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

Therefore,

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

This and a continuity theorem for probability measures on compact groups, see for example, [4], prove the lemma.

□

Now we will give a modification of Lemma 2.2 from [3] on the ergodicity of one transformation of Ω^{r_1+1}. Define

$$a_{h_1,\dots,h_{r_1},h} = \left((p^{-ih_1} : \; p \in \mathcal{P}), \dots, (p^{-ih_{r_1}} : \; p \in \mathcal{P}), (p^{-ih} : \; p \in \mathcal{P})\right),$$

and

$$\varphi_{h_1,\dots,h_{r_1},h}(\omega) = a_{h_1,\dots,h_{r_1},h}\omega, \quad \omega \in \Omega^{r_1+1}.$$

LEMMA 2. *Suppose that the set $L(h_1, \dots, h_{r_1}, h; \pi)$ is linearly independent over $\mathbb{Q}$. Then the transformation $\varphi_{h_1,\dots,h_{r_1},h}$ is ergodic.*

PROOF. The characters $\psi(\omega)$, $\omega = (\omega_1, \dots, \omega_{r_1+1}) \in \Omega^{r_1+1}$ of the group Ω^{r_1} are of the form

$$\psi(\omega) = \prod_{j=1}^{r_1+1}\prod_{p\in\mathcal{P}} \omega^{k_{jp}}(p),$$

where, as in Lemma 1, only a finite number of integers k_{jp} are distinct from zero. Thus, in view of (3),

$$\psi(a_{h_1,\dots,h_{r_1},h}) = \exp\left\{-i\left(\sum_{j=1}^{r_1}\sum_{p\in\mathcal{P}} k_{jp}h_j\log p + h\sum_{p\in\mathcal{P}}\hat{k}_p\log p\right)\right\} \neq 1. \qquad (4)$$

Let $A \in \mathcal{B}(\in \Omega^{r_1+1})$ be an invariant set of the transformation $\varphi_{h_1,\dots,h_{r_1},h}$, i.e., the sets A and $\varphi_{h_1,\dots,h_{r_1},h}(A)$ can differ one from another at most by a set of zero m_H-measure, let I_A be the indicator function, and let $\hat{g}$ denote the Fourier transform of g. Then, for almost all $\omega \in \Omega^{r_1+1}$,

$$I_A(a_{h_1,\dots,h_{r_1},h}\omega) = I_A(\omega). \qquad (5)$$

If ψ is a non-trivial character, then (5) and the invariance of the measure m_H show that

$$\hat{I}_A(\psi) = \int\limits_{\Omega^{r_1+1}} \psi(\omega)I_A(\omega)m_H(\mathrm{d}\omega) = \psi(a_{h_1,\dots,h_{r_1},h})\hat{I}_A(\psi).$$

Therefore, by (4),

$$\hat{I}_A(\psi) = 0. \qquad (6)$$

Now, suppose that ψ_0 is the trivial character of Ω^{r_1+1}, and let $\hat{I}_A(\psi_0) = u$. Then (6) together with orthogonality of characters shows that, for every character ψ,

$$\hat{I}_A(\psi) = u\int\limits_{\Omega^{r_1+1}} \psi(\omega)m_H(\mathrm{d}\omega) = u\hat{1}(\psi) = \hat{u}(\psi).$$

Therefore, $I_A(\omega) = u$ for almost all $\omega \in \Omega^{r_1+1}$. Hence, $m_H(A) = 1$ or $m_H(A) = 0$, in the other words, the transformation $\varphi_{h_1,\dots,h_{r_1},h}$ is ergodic.

□

3. A limit theorem

Let, for brevity, $\underline{h} = (h_1,\dots,h_{r_1},h)$, $\underline{\chi} = (\chi_1,\dots,\chi_r)$ and

$$\underline{L}(s+ik\underline{h},\underline{\chi}) = (L(s+ikh_1,\chi_1),\dots,L(s+ikh_{r_1},\chi_{r_1}), \\ L(s+ikh,\chi_{r_1+1}),\dots L(s+ikh,\chi_r)).$$

Denote by $H(D)$, $D = \left\{s\in\mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$, the space of analytic functions on D endowed with the topology of uniform convergence on compacta, and, on the probability space $(\Omega^{r_1+1}, \mathcal{B}(\Omega^{r_1+1}), m_H)$, define the $H^r(D)$-valued random element

$$\underline{L}(s,\omega,\underline{\chi}) = \left(\prod_p\left(1-\frac{\chi_1(p)\omega_1(p)}{p^s}\right)^{-1},\dots,\prod_p\left(1-\frac{\chi_{r_1}(p)\omega_{r_1}(p)}{p^s}\right)^{-1}, \right. \\ \left. \prod_p\left(1-\frac{\chi_{r_1+1}(p)\omega_{r_1+1}(p)}{p^s}\right)^{-1},\dots,\prod_p\left(1-\frac{\chi_r(p)\omega_{r_1+1}(p)}{p^s}\right)^{-1}\right).$$

Let $P_{\underline{L}}$ be the distribution of $\underline{L}(s,\omega,\underline{\chi})$, i.e.,

$$P_{\underline{L}}(A) = m_H\left(\omega\in\Omega^{r_1+1} : \underline{L}(s,\omega,\underline{\chi})\in A\right), \quad A\in\mathcal{B}(H^r(D)).$$

THEOREM 6. *Suppose that the set* $L(h_1,\dots,h_{r_1},h;\pi)$ *is linearly independent over* $\mathbb{Q}$. *Then*

$$P_N(A) \overset{def}{=} \frac{1}{N+1}\#\left\{0\le k\le N : \left(\underline{L}(s+ik\underline{h},\underline{\chi})\right)\in A\right\}, \quad A\in\mathcal{B}(H^r(D)),$$

converges weakly to $P_{\underline{L}}$ *as* $N\to\infty$.

PROOF. Let, for a fixed $\sigma_1>\frac{1}{2}$,

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}, \quad m,n\in\mathbb{N}.$$

Define auxiliary functions

$$L_n(s,\chi_j) = \sum_{m=1}^{\infty}\frac{\chi_j(m)v_n(m)}{m^s}, \quad j=1,\dots,r,$$

and

$$L_n(s,\omega_j,\chi_j) = \sum_{m=1}^{\infty}\frac{\chi_j(m)\omega_j(m)v_n(m)}{m^s}, \quad j=1,\dots,r_1,$$

$$L_n(s,\omega_{r_1+1},\chi_j) = \sum_{m=1}^{\infty}\frac{\chi_j(m)\omega_{r_1+1}(m)v_n(m)}{m^s}, \quad j=r_1+1,\dots,r,$$

the series being absolutely convergent for $\sigma>\frac{1}{2}$, where, for $m\in\mathbb{N}$,

$$\omega_j(m) = \prod_{p^\alpha\|m}\omega_j^\alpha(p), \quad j=1,\dots,r_1+1.$$

Further, we put

$$\begin{aligned}\underline{L}_n(s+ik\underline{h},\underline{\chi}) = (&L_n(s+ikh_1,\chi_1),\dots,L_n(s+ikh_{r_1},\chi_{r_1}),\\ &L_n(s+ikh,\chi_{r_1+1}),\dots,L_n(s+ikh,\chi_r))\end{aligned}$$

and

$$\begin{aligned}\underline{L}_n(s+ik\underline{h},\omega,\underline{\chi}) = (&L_n(s+ikh_1,\omega_1,\chi_1),\dots,L_n(s+ikh_{r_1},\omega_{r_1},\chi_{r_1}),\\ &L_n(s+ikh,\omega_{r_1+1},\chi_{r_1+1}),\dots,L_n(s+ikh,\omega_{r_1+1},\chi_r)).\end{aligned}$$

Then, using Lemma 1 and Theorem 5.1 of [2], we find, in view of the invariance of the Haar measure m_H, that

$$P_{N,n}(A) \overset{def}{=} \frac{1}{N+1}\#\left\{0\le k\le N : \underline{L}_n(s+ik\underline{h},\underline{\chi})\in A\right\},$$

and

$$P_{N,n,\omega}(A) \stackrel{def}{=} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \underline{L}_n(s+ik\underline{h}, \omega, \underline{\chi}) \in A \right\},$$

where $A \in \mathcal{B}(H^r(D))$, both converge weakly to the same probability measure P_n on $(H^r(D), \mathcal{B}(H^r\ (D)))$ as $N \to \infty$.

It remains to pass from $P_{N,n}$ to P_N. For $g_1, g_2 \in H(D)$, let

$$\rho_0(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup\limits_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup\limits_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : \ l \in \mathbb{N}\}$ is a sequence of compact subsets of the strip D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some $l \in \mathbb{N}$. Then we have that ρ_0 is a metric on $H(D)$ which induces its topology of uniform convergence on compacta. Now let, for $\underline{g}_1 = (g_{11}, \dots, g_{1r}),\ \underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D)$,

$$\rho(\underline{g}_1, \underline{g}_2) = \max_{1 \le j \le r} \rho_0(g_{1j}, g_{2j}).$$

Then ρ is a metric on $H^r(D)$ inducing its topology.

Using the estimate

$$\sum_{k=0}^{N} |L(\sigma + ikh_j, \chi_j)|^2 = O(N),$$

which follows from the bound

$$\int_0^T |L(\sigma + it, \chi_j)|^2 \mathrm{d}t = O(T), \quad \sigma > \frac{1}{2}, \quad j = 1, \dots, r,$$

and the Gallagher lemma [7], we obtain by a standard procedure that

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho \left(\underline{L}(s + ik\underline{h}, \underline{\chi}), \underline{L}_n(s + ik\underline{h}, \underline{\chi}) \right) = 0. \tag{7}$$

Also, standard arguments imply, for almost all $\omega \in \Omega$, the estimate

$$\int_0^T |L(\sigma + it, \omega, \chi_j)|^2 \mathrm{d}t = O(T),$$

and this leads to the bound

$$\sum_{k=0}^{N} |L(\sigma + ikh, \omega, \chi_j)|^2 \mathrm{d}t = O(N), \quad \sigma > \frac{1}{2}, \quad j = 1, \dots, r.$$

Hence, for almost all $\omega \in \Omega^{r_1+1}$, we deduce that

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho\left(\underline{L}(s + ik\underline{h}, \omega, \underline{\chi}), \underline{L}_n(s + ik\underline{h}, \omega, \underline{\chi})\right) = 0. \tag{8}$$

Equality (7) allows to show the weak convergence of the measure P_N. Let θ_N be a discrete random variable on a probability space $(\hat{\Omega}, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{P}(\theta_N = k) = \tfrac{1}{N+1}, \quad k = 0, 1, \dots, N,$$

and

$$\underline{X}_{N,n}(s) = \underline{L}_n(s + i\theta_N \underline{h}, \underline{\chi}).$$

Then the weak convergence of $P_{N,n}$ to P_n can be rewritten in the form

$$\underline{X}_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} \underline{X}_n, \tag{9}$$

where $\underline{X}_n$ is the $H^r(D)$-valued random element with the distribution P_n. Using the latter relation, it is not difficult to prove that the family of probability measures $\{P_n : \ n \in \mathbb{N}\}$ is tight. Thus, it is relatively compact, and there exists a sequence $\{n_k\}$ and a probability measure P on $(H^r(D), \mathcal{B}(H^r(D)))$ such that

$$\underline{X}_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{10}$$

Now putting

$$\underline{X}_N(s) = \underline{L}(s + i\theta_N \underline{h}, \underline{\chi})$$

and using (7), we find that

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mathbb{P}\left(\rho(\underline{X}_N, \underline{X}_{N,n}) \geq \varepsilon\right) = 0. \tag{11}$$

Relations (9) - (10) show that all hypotheses of Theorem 4.2 of [2] are satisfied. Therefore,

$$\underline{X}_N \xrightarrow[N\to\infty]{\mathcal{D}} P,$$

and this implies the weak convergence of P_N to P as $N \to \infty$.

It remains to identify the limit measure P. For this, define

$$\begin{aligned}\underline{L}(s + ik\underline{h}, \omega, \underline{\chi}) = &(L(s + ikh_1, \omega_1, \chi_1), \dots, L(s + ikh_{r_1}, \omega_{r_1}, \chi_{r_1}),\\ &L(s + ikh, \omega_{r_1+1}, \chi_{r_1+1}), \dots, L(s + ikh, \omega_{r_1+1}, \chi_r))\end{aligned}$$

and

$$P_{N,\omega}(A) \stackrel{def}{=} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \underline{L}(s+ik\underline{h}, \omega, \underline{\chi}) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)).$$

Then, using the weak convergence of $P_{N,n,\omega}$ to P_n, equality (9) and repeating the above arguments, we obtain that $P_{N,\omega}$ also converges weakly to P as $N \to \infty$. Thus, if A is a continuity set of the measure P, then we have that

$$\lim_{N\to\infty} P_{N,\omega}(A) = P(A). \tag{12}$$

On the space $(\Omega^{r_1+1}, \mathcal{B}(\Omega^{r_1+1}), m_H)$, define the random variable

$$\xi(\omega) = \begin{cases} 1 & \text{if } \underline{L}(s,\omega,\underline{\chi}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathbb{E}\xi = \int_{\Omega} \xi \mathrm{d} m_H = P_{\underline{L}}(A). \tag{13}$$

In view of Lemma 2, we can apply the ergodic Birkhoff-Khintchine theorem, which, for almost all $\omega \in \Omega^{r_1+1}$, gives

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \xi\left(\varphi^k_{h_1,\dots,h_{r_1},h}(\omega)\right) = \mathbb{E}\xi. \tag{14}$$

Therefore, relations (13) and (14) imply

$$\lim_{N\to\infty} P_{N,\omega}(A) = P_{\underline{L}}(A).$$

Hence, by (12), we obtain that $P(A) = P_{\underline{L}}(A)$ for every continuity set A of P. Therefore, $P = P_{\underline{L}}$, and the theorem is proved.

□

4. Proof of Theorem 5

Theorem 5 is a consequence of Theorem 6 and the Mergelyan theorem on the approximation of analytic functions by polynomials [6]. Let $S = \{g \in H(D) : g(s) \ne 0 \quad \text{or} \quad g(s) \equiv 0\}$. It is known [5] that the support of the random element

$$\left(\prod_{p\in\mathcal{P}} \left(1 - \frac{\chi_{r_1+1}(p)\omega_{r_1+1}(p)}{p^s}\right)^{-1}, \dots, \prod_{p\in\mathcal{P}} \left(1 - \frac{\chi_r(p)\omega_{r_1+1}(p)}{p^s}\right)^{-1} \right)$$

is the set S^{r-r_1}. Moreover, the measure m_H is the product of the Haar measures m_{jH} on Ω_j, $j = 1, \dots, r_1 + 1$. Since the support of

$$\prod_{p\in\mathcal{P}} \left(1 - \frac{\chi_j(p)\omega_j(p)}{p^s}\right)^{-1}, \quad j = 1, \dots, r_1,$$

is the set S, we find that the support the random element

$$\left(\prod_{p\in\mathcal{P}}\left(1-\frac{\chi_1(p)\omega_1(p)}{p^s}\right)^{-1},\dots,\prod_{p\in\mathcal{P}}\left(1-\frac{\chi_{r_1}(p)\omega_{r_1}(p)}{p^s}\right)^{-1}\right)$$

is the set S^{r_1}. These remarks show, in view of Theorem 6, that the support of $P_{\underline{L}}$ is the set S^r.

By the Mergelyan theorem [6], there exist polynomials $p_1(s),\dots,p_r(s)$ such that

$$\sup_{1\le j\le r}\sup_{s\in K_j}\left|f_j(s)-e^{p_j(s)}\right|<\frac{\varepsilon}{2}. \tag{15}$$

Define

$$G=\left\{(g_1,\dots,g_r)\in H^r(D):\ \sup_{1\le j\le r}\sup_{s\in K_j}\left|g_j(s)-e^{p_j(s)}\right|<\frac{\varepsilon}{2}\right\}.$$

Obviously, G is an open set in $H^r(D)$. Moreover, $\left(e^{p_1(s)},\dots,e^{p_r(s)}\right)\in S^r$, i.e., is an element of the support of $P_{\underline{L}}$. Thus, $P_{\underline{L}}(G)>0$. Hence, $\liminf\limits_{N\to\infty}P_N(G)>0$, since, by Theorem 6,

$$\liminf_{N\to\infty}P_N(G)\ge P_{\underline{L}}(G).$$

This, the definition of G and inequality (15) prove the theorem.

5. Conclusions

In the paper, a discrete joint universality theorem for Dirichlet L-functions $L(s,\chi)$ is obtained. In this theorem, a collection of analytic functions $f_1(s),\dots,f_r(s)$ is approximated by shifts $L(s+ikh_1,\chi_1),\dots,L(s+ikh_{r_1},\chi_{r_1}),L(s+ikh,\chi_{r_1+1}),\dots,$ $L(s+ikh,\chi_r)$, where $\chi_1,\dots,\chi_r$ are pairwise non-equivalent Dirichlet characters, and $h_1,\dots,h_{r_1},h$ are such positive numbers that the set

$$\begin{aligned}L(h_1,\dots,h_{r_1},h;\pi)=\big\{&(h_1\log p:\ p\in\mathcal{P}),\dots,(h_{r_1}\log p:\ p\in\mathcal{P}),\\&(h\log p:\ p\in\mathcal{P});\pi\big\}.\end{aligned}$$

is linearly independent over the field of rational numbers.

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Получено 18.02.2015