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ON A. V. MALYSHEV'S APPROACH TO MINKOWSKI'S CONJECTURE CONCERNING THE CRITICAL DETERMINANT OF THE REGION $|x|^p + |y|^p < 1$ for $p > 1$

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Аннотация

Целью статьи является представление подхода А. В. Малышева к исследованию и доказательству гипотезы Минковского (с уточнениями С. Дэвиса (C. Davis)) о критическом определителе области $|x|^p + |y|^p < 1$ для $p > 1$ и краткое изложение метода Малышева и полученных на его основе результатов.

Ключевые слова: критическая решетка; критический определитель области; Диофантово неравенство; Диофантовы приближения; лучевая функция; звездное тело; многообразие модулей;

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Abstract

We present A. V. Malyshev's approach to Minkowski's conjecture (in Davis's amendment) concerning the critical determinant of the region $|x|^p + |y|^p < 1$ for $p > 1$ and Malyshev's method. In the sequel of this article we use these approach and method to obtain the main result.

Keywords: critical lattice; critical determinant; Diophantine inequality; Diophantine approximation; distance function; star body; moduli space;
Bibliography: 21 titles.

1. Introduction

Let

$$|\alpha x + \beta y|^p + |\gamma x + \delta y|^p \leq c|\det(\alpha\delta - \beta\gamma)|^{p/2},$$

be a diophantine inequality defined for a given real $p > 1$; hear $\alpha, \beta, \gamma, \delta$ are real numbers with $\alpha\delta - \beta\gamma \neq 0$.

H. Minkowski in his monograph [2] raise the question about minimum constant c such that the inequality has integer solution other than origin. Minkowski with the help of his theorem on convex body has found a sufficient condition for the solvability of Diophantine inequalities in integers not both zero:

$$c = {\kappa_p}^p, \kappa_p = \frac{\Gamma(1 + \frac{2}{p})^{1/2}}{\Gamma(1 + \frac{1}{p})}.$$

But this result is not optimal, and Minkowski also raised the issue of not improving constant c. For this purpose Minkowski has proposed to use the critical determinant.

Given any set $\mathcal{R} \subset \mathbb{R}^n$, a lattice Λ is *admissible* for $\mathcal{R}$ (or is $\mathcal{R}$-*admissible*) if $\mathcal{R} \bigcap \Lambda = \emptyset$ or $\{0\}$. The infimum $\Delta(\mathcal{R})$ of the determinants (the determinant of a lattice Λ is written $d(\Lambda)$) of all lattices admissible for $\mathcal{R}$ is called the *critical determinant* of $\mathcal{R}$. A lattice Λ is *critical* for $\mathcal{R}$ if $d(\Lambda) = \Delta(\mathcal{R})$.

Critical determinant is one of the main notion of the geometry of numbers [2, 3, 6]. It has been investigated in the framework of problem of Minkowski in papers by Mordell [4], by Davis [5], by Cohn [7], by Watson [8], by Malyshev [9] and by Malyshev with colleagues.

2. Minkowski's conjecture as a problem of Diophantine approximation theory

Diophantine approximations connect with critical determinants and with solutions in integer numbers $x_1, \dots x_n$ (with some restrictions, for instance not all $x_1, \dots x_n$ are equal to zero) of inequalities

$$F(x_1, \dots x_n) < c,$$

or more generally

$$F(x) < c, \ x \in \Lambda, x \neq 0.$$

Recall the definitions [6].

Let $\mathcal{R}$ be a set and Λ be a lattice with base $\{a_1, \dots, a_n\}$ in $\mathbf{R}^n$. A lattice Λ is *admissible* for body $\mathcal{R}$ ($\mathcal{R}-$*admissible*) if $\mathcal{D} \bigcap \Lambda = \emptyset$ or 0. Let $d(\Lambda)$ be the determinant of Λ. The infimum $\Delta(\mathcal{R})$ of determinants of all lattices admissible for $\mathcal{R}$ is called *the critical determinant* of $\mathcal{R}$; if there is no $\mathcal{R}-$admissible lattices then puts $\Delta(\mathcal{R}) = \infty$. A lattice Λ is *critical* if $d(\Lambda) = \Delta(\mathcal{R})$.

Usually in the geometry of numbers the function $F(x)$ is a distance function. A real function $F(x)$ defined on $\mathbf{R}^n$ is *distance function* if

(i) $F(x) \geq 0, x \in \mathbf{R}^n, F(0) = 0$;

(ii) $F(x)$ is continuous;

(iii) $F(x)$ is homogenous: $F(\lambda x) = \lambda F(x), \lambda > 0, \lambda \in \mathbf{R}$.

The problem of solving of diophantine inequality $F(x) < c$, with a distance function F has the next framework.

Let $\overline{M}$ be the closure of a set M and $\#P$ be the number of elements of a finite set P. An open set $S \subset \mathbf{R}^n$ is a *star body* if S includes the origin of $\mathbf{R}^n$ and for any ray r beginning in the origin $\#(r \cap (\overline{M} \setminus M)) \leq 1$. If $F(x)$ is a distance function then the set

$$M_F = \{x : F(x) < 1\}$$

is a star body.

One of the main particular case of a distance function is the case of convex symmetrical function $F(x)$ which with conditions (i) - (iii) satisfies the additional conditions

(iv)$F(x+y) \leq F(x) + F(y)$;

(v) $F(-x) = F(x)$.

The Minkowski's problem can be reformulated as a conjecture concerning the critical determinant of the region $\mid x \mid^p + \mid y \mid^p \leq 1, \ p > 1$. Recall once more that mentioned mathematical problems are closely connected with Diophantine Approximation.

For the given 2-dimension region $D_p \subset \mathbf{R}^2 = (x, y), \ p > 1 :$

$$|x|^p + |y|^p < 1,$$

let $\Delta(D_p)$ be the critical determinant of the region.

Let $a \in \Lambda, a \neq 0$ and let

$$m(F, \Lambda) = \inf_a F(a).$$

The Hermite constant of the function F is defined as

$$\gamma(F) = \sup_\Lambda \frac{m(F, \Lambda)}{d(\Lambda)^{1/n}}.$$

3. Moduli Spaces

What is moduli? Classically Riemann claimed that $6g-6$ (real) parameters could be for Riemann surface of genus $g > 1$ which would determine its conformal structure (for elliptic curves, when $g = 1$, it is needs one parameter). From algebraic point of view we have the following problem: given some kind of variety, classify the set of all varieties having something in common with the given one (same numerical invariants of some kind, belonging to a common algebraic family). For instance, for an elliptic curve the invariant is the modular invariant of the elliptic curve.

Let $\mathbf{B}$ be a class of objects. Let S be a scheme. A family of objects parametrized by the S is the set of objects $X_s : s \in S, X_s \in \mathbf{B}$ equipped with an additional structure compatible with the structure of the base S. Algebraic moduli spaces are defined in the papers by Mumford, Harris and Morrison [15, 16].

A possibility of the parameterization of all admissible lattices of regions $D_p = \{|x|^p + |y|^p < 1\}$, under varying $p > 1$, by some analitical manifolds was mentioned in the book by Minkowski in 1907 [2]. In 1950 H. Cohn published the paper on the Minkowski's conjecture [7]. The parameterization and the corresponding moduli space were one of the main tools of his approach to the investigation of the conjecture.

Recall some definitions. Let M be an arbitrary set in $\mathbf{R}^n$, $O = (0, \dots, 0) \in \mathbf{R}^n$. A lattice Λ is called admissible for M, or $M-$*admissible*, if it has no points $\neq O$ in the interior of M. It is called *strictly admissible* for M if it does not contain a point $\neq O$ of M.

The *critical determinant* of a set M is the quantity $\Delta(M)$ given by

$$\Delta(M) = inf\{d(\Lambda) : \Lambda \; strictly\, admissible\, for M\}$$

with the understanding that $\Delta(M) = \infty$ if there are no strictly admissible lattices. The set M is said to be of the finite or the infinity type according to whether $\Delta(M)$ is finite or infinite.

The moduli space is defined by the equation

$$\Delta(p, \sigma) = (\tau + \sigma)(1 + \tau^p)^{-\frac{1}{p}}(1 + \sigma^p)^{-\frac{1}{p}}, \quad (1)$$

in the domain

$$\mathcal{M} : \; \infty > p > 1, \; 1 \leq \sigma \leq \sigma_p = (2^p - 1)^{\frac{1}{p}},$$

of the $\{p, \sigma\}$ plane, where σ is some real parameter; here $\tau = \tau(p, \sigma)$ is the function uniquely determined by the conditions

$$A^p + B^p = 1, \; 0 \leq \tau \leq \tau_p,$$

where

$$A = A(p, \sigma) = (1 + \tau^p)^{-\frac{1}{p}} - (1 + \sigma^p)^{-\frac{1}{p}}, \; B = B(p, \sigma) = \sigma(1 + \sigma^p)^{-\frac{1}{p}} + \tau(1 + \tau^p)^{-\frac{1}{p}},$$

τ_p is defined by the equation $2(1 - \tau_p)^p = 1 + \tau_p^p, \; 0 \leq \tau_p < 1$.

Definition 1. *In the notation above, the surface*

$$\Delta - (\tau + \sigma)(1 + \tau^p)^{-1/p}(1 + \sigma^p)^{-1/p} = 0,$$

in $\mathbf{R}^3$ *with coordinates* (σ, p, Δ) *we will called the Minkowski-Cohn moduli space.*

4. Minkowski's analytic conjecture

In considering the question of the minimum value taken by the expression $|x|^p+|y|^p$, with $p \geq 1$, at points, other that the origin, of a lattice Λ of determinant $d(\Lambda)$, Minkowski [2] shows that the problem of determining the maximum value of the minimum for different lattices may be reduced to that of finding the minimum possible area of a parallelogram with one vertex at the origin and the three remaining vertices on the curve $|x|^p+|y|^p = 1$. The problem with $p = 1, 2$ and ∞ is trivial: in these cases the minimum areas are $1/2$, $\sqrt{3}/2$ and 1 respectively. Let $D_p \subset \mathbf{R}^2 = (x, y),\ p > 1$ be the 2-dimension region:

$$|x|^p + |y|^p < 1.$$

Let $\Delta(D_p)$ be the critical determinant of the region. Recall considerations of the previous section. For $p > 1$, let

$$D_p = \{(x, y) \in \mathbb{R}^2 \mid |x|^p + |y|^p < 1\}.$$

Minkowski [2] raised a question about critical determinants and critical lattices of regions D_p for varying $p > 1$. Let $\Lambda_p^{(0)}$ and $\Lambda_p^{(1)}$ be two D_p-admissible lattices each of which contains three pairs of points on the boundary of D_p and with the property that $(1, 0) \in \Lambda_p^{(0)}$, $(-2^{-1/p}, 2^{-1/p}) \in \Lambda_p^{(1)}$, (under these conditions the lattices are uniquely defined). Using analytic parameterization Cohn [7] gives analytic formulation of Minkowski's conjecture.

Let

$$\Delta(p, \sigma) = (\tau + \sigma)(1 + \tau^p)^{-\frac{1}{p}}(1 + \sigma^p)^{-\frac{1}{p}}, \quad (1)$$

be the function defined in the domain

$$\mathcal{M}:\ \infty > p > 1,\ 1 \leq \sigma \leq \sigma_p = (2^p - 1)^{\frac{1}{p}},$$

of the $\{p, \sigma\}$ plane, where σ is some real parameter; here $\tau = \tau(p, \sigma)$ is the function uniquely determined by the conditions

$$A^p + B^p = 1,\ 0 \leq \tau \leq \tau_p,$$

where

$$A = A(p, \sigma) = (1 + \tau^p)^{-\frac{1}{p}} - (1 + \sigma^p)^{-\frac{1}{p}}$$

$$B = B(p, \sigma) = \sigma(1 + \sigma^p)^{-\frac{1}{p}} + \tau(1 + \tau^p)^{-\frac{1}{p}},$$

τ_p is defined by the equation

$$2(1 - \tau_p)^p = 1 + \tau_p^p,\ 0 \leq \tau_p \leq 1.$$

In this case needs to extend the notion of parameter variety to parameter manifold. The function $\Delta(p, \sigma)$ in region $\mathcal{M}$ determines the parameter manifold.

Minkowski's analytic conjecture:

For any real p with conditions $p > 1,\ p \neq 2,\ 1 < \sigma < \sigma_p$

$$\Delta(p, \sigma) > min(\Delta(p, 1), \Delta(p, \sigma_p)).$$

In the vicinity of the point $p = 1$ and in the vicinity of the point $(2, \sigma_2)$ the (p, τ) variant of the Minkowski's analytic conjecture is used.

Minkowski's analytic $(p, \tau)-$**conjecture:**

For any real p and τ *with conditions* $p > 1,\ p \neq 2,\ 0 < \tau < \tau_p$

$$\Delta(p,\tau) > min(\Delta(p,1),\Delta(p,\tau_p)).$$

For investigation of properties of function $\Delta(p,\sigma)$ which are need for proof of Minkowski's conjecture [2,7] we considered the value of $\Delta = \Delta(p,\sigma)$ and its derivatives

$$\Delta^{'}_{\sigma}\ ,\ \Delta^{''}_{\sigma^2}\ ,\ \Delta^{'}_{p}\ ,\ \Delta^{''}_{\sigma p}\ ,\ \Delta^{'''}_{\sigma^2 p}$$

on some subdomains of the domain $\mathcal{M}$ [14].

5. Validated numerics

Validated numerics (sometimes called as interval computations) allow [17–21]
(1) rigorous enclosure for roundoff error, truncation error, and error of data;
(2) computation of rigorous bounds of the ranges of functions and maps.

A compact closed interval $I = [a,b]$ is the set of real numbers x such that (s.t.) $a \leqslant x \leqslant b$. Interval analysis with this type of intervals uses usually two sorts of intervals. Wide intervals are used for representing uncertainty of the real world or lack of information. Narrow intervals are used for rounding error bounds. In any of these two cases on each step of an interval computation we compute the interval I which contains an (ideal) solution of our problem. Some examples of implementations of the intervals are given in papers [21,22].

There are many numerical algorithms for solving mathematical problems. The majority of these algorithms are iterative, so, since stopping the algorithms after a certain number of steps, we only get an approximation $\tilde{x}$ to the desired solution x. A perfect solution would if we could estimate the errors of the result not after the iteration process, but simultaneously with the iteration process. This is one of the main ideas of interval analysis [17–20].

6. Malyshev‘s method

At first expressing $\Delta^{'}_{\sigma}$, $\Delta^{''}_{\sigma^2}$, $\Delta^{'}_{p}$, $\Delta^{''}_{\sigma p}$, $\Delta^{'''}_{\sigma^2 p}$ in terms of a sum of derivatives of "atoms"$s_i = \sigma^{p-i}$, $t_i = \tau^{p-i}$, $a_i = (1+\sigma^p)^{-i-\frac{1}{p}}$, $b_i = (1+\tau^p)^{-i-\frac{1}{p}}$, $A = b_0 - a_0$, $B = \tau b_0 + \sigma a_0$, $\alpha_i = A^{p-i}$, $\beta_i = B^{p-i}$ $(i = 0,1,2,\ldots)$.

Then by the implicit function theorem computing $\tau = \tau(p,\sigma)$ by means of the following iteration process:

$$\tau_{i+1} = (1+\tau_i^p)^{\frac{1}{p}}((1-((1+\tau_i^p)^{-\frac{1}{p}} - (1+\sigma^p)^{-\frac{1}{p}})^p)^{\frac{1}{p}} - \sigma(1+\sigma^p)^{-\frac{1}{p}}),$$

For computation of the expression for τ_p we apply the following iteration:

$$(\tau_p)_{i+1} = 1 - (2^{-\frac{1}{p}})(1+(\tau_p)_i^p)^{\frac{1}{p}},\ p > 1,\ (\tau_p)_0 \in [0, 0.36].$$

So we really have represented the function Δ as the function $\Delta(p,\sigma)$ of two variables. The same fact is true for it's derivatives. A.V. Malyshev and the author have constructed algebraic expressions for $\Delta, \Delta^{'}_{\sigma}$, $\Delta^{''}_{\sigma^2}$, $\Delta^{'}_{p}$, $\Delta^{''}_{\sigma p}$, $\Delta^{'''}_{\sigma^2 p}$ and at first compute their by fixed point and float point computations.

Let $\mathbf{X} = (\mathbf{x}_1,\cdots,\mathbf{x}_n) = ([\underline{x}_1,\overline{x}_1],\cdots,[\underline{x}_n,\overline{x}_n]$ be the n-dimensional real interval vector with $\underline{x}_i \leq x_i \leq \overline{x}_i$ ("rectangle"or "box"). The *interval evaluation* of a function $G(x_1,\cdots,x_n)$ on an interval $\mathbf{X}$ is the interval $[\underline{G},\overline{G}]$ such that for any $x \in \mathbf{X}$, $G(x) \in [\underline{G},\overline{G}]$. The interval evaluation is called *optimal* if $\underline{G} = \min G$, and $\overline{G} = \max G$ on the interval $\mathbf{X}$.

Let D be a subdomain of $\mathcal{M}$. Under evaluation in D a mentioned function the domain is covered by rectangles of the form

$$[\underline{p}, \overline{p};\ \underline{\sigma}, \overline{\sigma}].$$

In the case of the formula that expressing $\Delta^{'}_{\sigma}$, $\Delta^{''}_{\sigma^2}$, $\Delta^{'}_{p}$, $\Delta^{''}_{\sigma p}$, $\Delta^{'''}_{\sigma^2 p}$

in terms of a sum of derivatives of "atoms"$s_i = \sigma^{p-i}$, $t_i = \tau^{p-i}$, $a_i = (1+\sigma^p)^{-i-\frac{1}{p}}$, $b_i = (1+\tau^p)^{-i-\frac{1}{p}}$, $A = b_0 - a_0$, $B = \tau b_0 + \sigma a_0$, $\alpha_i = A^{p-i}$, $\beta_i = B^{p-i}$ $(i = 0, 1, 2, \ldots)$ one applies the rational interval evaluation to construct formulas for lower bounds and upper bounds of the functions, which in the end can be expressed in terms of $\underline{p}$, $\overline{p}$, $\underline{\sigma}$, $\overline{\sigma}$, $\underline{\tau}$, $\overline{\tau}$, ; here the bounds $\underline{\tau}$, $\overline{\tau}$, are obtained with the help of the iteration process:

$$\underline{t}_{i+1} = (1+\underline{t}_i^{\overline{p}})^{\frac{1}{\overline{p}}}((1-((1+\underline{t}_i^{\overline{p}})^{-\frac{1}{\overline{p}}} - (1+\overline{\sigma}^{\underline{p}})^{-\frac{1}{\underline{p}}})^{\underline{p}})^{\frac{1}{\underline{p}}} - \overline{\sigma}(1+\overline{\sigma}^{\underline{p}})^{-\frac{1}{\underline{p}}}),$$

$$\overline{t}_{i+1} = (1+\overline{t}_i^{\underline{p}})^{\frac{1}{\underline{p}}}((1-((1+\overline{t}_i^{\underline{p}})^{-\frac{1}{\underline{p}}} - (1+\underline{\sigma}^{\overline{p}})^{-\frac{1}{\overline{p}}})^{\overline{p}})^{\frac{1}{\overline{p}}} - \underline{\sigma}(1+\underline{\sigma}^{\overline{p}})^{-\frac{1}{\overline{p}}}).$$

$$i = 0, 1, \cdots$$

As interval computation is the enclosure method, we have to put:

$$[\underline{\tau},\ \overline{\tau}] = [\underline{t}_N,\ \overline{t}_N] \bigcap [\underline{\tau}_0,\ \overline{\tau}_0]\ .$$

N is computed on the last step of the iteration.

For initial values we may take : $[\underline{t}_0,\ \overline{t}_0] = [\underline{\tau}_0,\ \overline{\tau}_0] = [0,\ 0.36]$.

6.1. Algorithms

Here we give names, input and output of algorithms for interval evaluation only. All these algorithms are implemented, tested and applied under the computer-assisted proof of Minkowski‘s conjecture [9,10,12–14] .

Algorithm *TPV*
Input: An implicitly defined function τ_p from Section 5.
Interval $[\underline{p}, \overline{p};\ \underline{\sigma}, \overline{\sigma}]$.
Method: Iterative interval computation.
Output: The interval evaluation of τ_p.
Algorithm *TAUV*
Input: Implicitly defined function τ from this Section.
Interval $[\underline{p}, \overline{p};\ \underline{\sigma}, \overline{\sigma}]$.
Method: Described in this Section.
Output: The interval evaluation of τ.
Algorithm *L0V*
Input: Function $l^0 = \Delta(p, \sigma) - \Delta_p^{(0)}$.
Interval $[\underline{p}, \overline{p};\ \underline{\sigma}, \overline{\sigma}]$.
Method: Interval computations.
Output: The interval evaluation of l^0.
Algorithm *L1V*
Input: Function $l^1 = \Delta(p, \sigma) - \Delta_p^{(1)}$.
Interval $[\underline{p}, \overline{p};\ \underline{\sigma}, \overline{\sigma}]$.
Method: Interval computations.
Output: The interval evaluation of l^1.
Algorithm *GV*

Input: A function $g(p,\sigma)$ which has the same sign as function Δ'_{σ}.
Interval $[\underline{p},\overline{p};\ \underline{\sigma},\overline{\sigma}]$.
Method: Interval computations.
Output: The interval evaluation of $g(p,\sigma)$.
Algorithm *HV*
Input: A function $h(p,\sigma)$ which is the partial derivative by σ the function $g(p,\sigma)$.
$[\underline{p},\overline{p};\ \underline{\sigma},\overline{\sigma}]$.
Method: Interval computations.
Output: The interval evaluation of $h(p,\sigma)$.
Next two algorithms are described in [22].
Algorithm *MonotoneFunction*
Input: A real function $F(x,y)$ monotonous by x and by y.
Interval $[\underline{x},\overline{x};\underline{y},\overline{y}]$.
Output: The interval evaluation of F.
Algorithm *RationalFunction*
Input: A rational function $R(x,y)$. Interval $[\underline{x},\overline{x};\underline{y},\overline{y}]$.
Output: The interval evaluation of R.

7. Results

It is important to note that our method gives possibility to prove that a value of the target minimum is an analytic function but is not a point. Ordinary numerical methods do not allow to obtain results of the kind.

In notations [14] next result have proved:

THEOREM 1. *[14]*

$$\Delta(D_p)=\begin{cases}\Delta(p,1),\ 1<p\leqslant 2,\ p\geqslant p_0,\\ \Delta(p,\sigma_p),\ 2\leqslant p\leqslant p_0;\end{cases}$$

here p_0 is a real number that is defined unique by conditions $\Delta(p_0,\sigma_p)=\Delta(p_0,1),\ 2,57\leqslant p_0\leqslant 2,58$.

COROLLARY 1.

$$\kappa_p=\Delta(D_p)^{-\frac{p}{2}}.$$

8. Strong Minkowski's analytic conjecture

A.V. Malishev and the author on the base of some theoretical evidences and results of mentioned computation have proposed the strong Minkowski's analytic conjecture (MAS)

Strong Minkowski's analytic (MAS) conjecture:
For given $p>1$ and increasing σ from 0 to σ_p the function $\Delta(p,\sigma)$
1) increase strictly monotonous if $1<p<2$ and $p\geq p^{(1)}$,
2) decrease strictly monotonous if $2\leq p\leq p^{(2)}$,
3) has a unique maximum on the segment $(1,\sigma_p)$; until the maximum $\Delta(p,\sigma)$ increase strictly monotonous and then decrease strictly monotonous if $p^{(2)}<p<p^{(1)}$;
4) constant, if $p=2$;
here
$p^{(1)}>2$ is a root of the equation $\Delta''_{\sigma^2}|_{\sigma=\sigma_p}=0$;
$p^{(2)}>2$ is a root of the equation $\Delta''_{\sigma^2}|_{\sigma=1}=0$.

It seems that the conjecture (MAS) has not been proved for any parameter except the trivial $p=2$.

9. Conclusion

A.V. Malyshev‘s approach to Minkowski‘s conjecture (in Davis‘s amendment) concerning the critical determinant of the region $|x|^p + |y|^p < 1$ for $p > 1$ is proposed and A.V. Malyshev‘s method of its prove is given. Applications of the approach and of the method are presented.

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