Abstract
In this article we prove that, if integer polynomial $P$ satisfies $|P(\omega)|_p < H^{-w}$, then for $w > 2n - 2$ and sufficiently large $H$ the root $\gamma$ belongs to the field of $p$-adic numbers.

Keywords: integer polynomials, discriminants of polynomials.

Bibliography: 16 titles.

1. Introduction

Throughout this paper, $p$ is a prime number, $\mathbb{Q}_p$ is the field of $p$-adic numbers,

$$P(x) = a_n x^n + \ldots + a_1 x + a_0$$

is an integer polynomial with degree $\deg P(x) = n$ and height $H(P) = \max_{0 \leq j \leq n} |a_j|$. We denote by $\mathcal{P}_n$ the set of integer polynomials of degree $n$. Let $\mathcal{P}_n(H) = \{ P \in \mathcal{P}_n : H(P) = H \}$.

In this paper, a result originally considered by Y. V. Nesterenko is examined. In [1] Y.V. Nesterenko discussed the solvability of the equation $P(x) = 0$ in the ring of $p$-adic integers $\mathbb{Z}_p$ and proved the following result.

**Theorem 1.** Let $x$ be an integer and $P \in \mathcal{P}_n(H)$. If

$$|P(x)|_p \leq e^{-8n^2} H^{-4n},$$

then there exists a $p$-adic number $\gamma$ such that

$$P(\gamma) = 0, \ |x - \gamma|_p < 1.$$ 

Note that a similar problem was considered in [2] and there was given a criteria for when the closest root of a polynomial to a real point belongs to the field of real numbers. Knowledge of the nature of the roots is very important in the problems of Diophantine approximations for construction of regular systems [3, 4]. Numerous applications of this concept arose when obtaining estimates for the Hausdorff measure and Hausdorff dimension of Diophantine sets [5] and proving analogues of the Khintchine theorem [6, 7]. Using the regular systems, the exact theorems on approximation of real numbers by real algebraic [6], by algebraic integers [8], of complex numbers by complex algebraic [9] were obtained, and similar problems in the field of $p$-adic numbers [10] and in $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ [7] were investigated.

The Theorem 1 can be improved for $p$-adic leading polynomials. Such a polynomial $P \in \mathcal{P}_n$ satisfies

$$|a_n|_p \gg 1.$$  

**Theorem 2.** Let $\omega \in \mathbb{Z}_p$ and $P \in \mathcal{P}_n(H)$ be a $p$-adic leading polynomial. Then if

$$|P(\omega)|_p < H^{-w}$$ 

for $w > 2n - 2$, and for sufficiently large $H > H_0(n)$, it follows that the root $\gamma_1$ of $P$ belongs to $\mathbb{Q}_p$ and

$$|\omega - \gamma_1|_p < 1.$$ 

**Remark 1.** If $D(P) \neq 0$ then we have that the root $\gamma_1$ of $P$ is closest to $\omega \in \mathbb{Z}_p$. The above theorem will be proved using a general method of V.I. Bernik which was developed in [11,12].
2. Preliminary setup and auxilliary Lemmas

Let $P \in \mathcal{P}_n$ have roots $\gamma_1, \gamma_2, \ldots, \gamma_n$ in $\mathbb{Q}_p^*$, where $\mathbb{Q}_p^*$ is the smallest field containing $\mathbb{Q}_p$ and all algebraic numbers. Then, from (1) it follows that

$$|\gamma_i|_p \ll 1, \quad i = 1, \ldots, n;$$

i.e. the roots are bounded. This follows from Lemma 4 in ([13], p.85).

Define the sets

$$T_p(\gamma_k) = \{ \omega \in \mathbb{Z}_p : |\omega - \gamma_k|_p = \min_{1 \leq i \leq n} |\omega - \gamma_i|_p, \ 1 \leq k \leq n. \}$$

Consider the set $T_p(\gamma_k)$ for a fixed $k$ and for ease of notation assume that $k = 1$. Next, reorder the other roots so that

$$|\gamma_1 - \gamma_2|_p \leq |\gamma_1 - \gamma_3|_p \leq \ldots \leq |\gamma_1 - \gamma_n|_p.$$

Fix $\epsilon > 0$ where $\epsilon$ is sufficiently small and suppose that $\epsilon_1 = \epsilon N^{-1}$ where $N = N(n) > 0$ is sufficiently large. Let $T = [\epsilon_1^{-1}]$.

For a polynomial $P \in \mathcal{P}_n(H)$ define the real numbers $\rho_j$ by

$$|\gamma_1 - \gamma_j|_p = H^{-\rho_j}, \quad 2 \leq j \leq n, \quad \rho_2 \geq \rho_3 \geq \ldots \geq \rho_n.$$

Define the integers $m_j$, $2 \leq j \leq n$, such that

$$\frac{m_j - 1}{T} \leq \rho_j < \frac{m_j}{T}, m_2 \geq m_3 \geq \ldots \geq m_n \geq 0.$$

Further define numbers $s_i$ such that

$$s_i = \frac{m_{i+1} + \ldots + m_n}{T}, \quad (1 \leq i \leq n - 1), \quad s_n = 0.$$

The first Lemma is a $p$-adic analogue of the Lemma, which was proved by Bernik in [14] and is a generalisation of Sprindžuk’s Lemma ([13], p.77).

**Lemma 1.** ([15]) Let $\omega \in T_p(\gamma_1)$. Then

$$|\omega - \gamma_1|_p \leq \min_{1 \leq j \leq n} (|P(\omega)|_p|P'(\gamma_1)|_p^{-1} \prod_{k=2}^j |\gamma_1 - \gamma_k|_p^{1/j}).$$

The following Lemma is often referred to as Gelfond’s Lemma.

**Lemma 2** ([16], Lemma A.3). Let $P_1, P_2, \ldots, P_k$ be polynomials of degree $n_1, \ldots, n_k$ respectively, and let $P = P_1P_2 \ldots P_k$. Let $n = n_1 + n_2 + \ldots + n_k$. Then

$$2^{-n}H(P_1)H(P_2) \ldots H(P_k) \leq H(P) \leq 2^n H(P_1)H(P_2) \ldots H(P_k).$$

In the proof of theorem we will refer to the following statement known as Hensel’s Lemma.

**Lemma 3** ([4], p. 134). Let $P$ be a polynomial with coefficients in $\mathbb{Z}_p$, let $\xi = \xi_0 \in \mathbb{Z}_p$ and $|P(\xi)|_p < |P'(\xi)|_p^2$. Then as $n \to \infty$ the sequence

$$\xi_{n+1} = \xi_n - \frac{P(\xi_n)}{P'(\xi_n)}$$

tends to some root $\beta \in \mathbb{Q}_p$ of the polynomial $P$ and

$$|\beta - \xi|_p \leq |P(\xi)|_p/|P'(\xi)|_p^2 < 1.$$
3. Proof of Theorem 2

Two cases must be dealt with separately: \(D(P) \neq 0\) and \(D(P) = 0\).

3.1. Case I: \(D(P) \neq 0\)

First consider a polynomial \(P \in \mathcal{P}_n(H)\) satisfying \(D(P) \neq 0\) and (2), and assume that \(|P'(\omega)|_p^2 \leq |P(\omega)|_p\). We will obtain a contradiction. Using (4), we get \(|P'(\omega)|_p < H^{-w/2}\).

It is well known that \(|D(P)| = \frac{|\Delta|}{|a_n|}\) where

\[
\Delta = \begin{pmatrix}
a_n & a_{n-1} & a_{n-2} & \ldots & a_1 & a_0 & 0 & \ldots & 0 \\
0 & a_n & a_{n-1} & a_{n-2} & \ldots & a_1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \ldots & a_1 & 0 & \ldots & 0 \\
0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \ldots & a_1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \ldots & a_1 \\
\end{pmatrix}
\]

Hence the determinant,

\[
|\Delta| \leq |a_n|((2n-2)!nH)^{2n-2} + n(2n-2)!nH^{2n-2})
= |a_n|(2n-2)!nH^{2n-2} \leq 2n^{2n-1}(2n-2)!H^{2n-2}|a_n|,
\]

using the fact that \(|a_i| \leq H, i = 0, 1, \ldots, n\). Thus, \(|D(P)| \leq 2n^{2n-1}(2n-2)!H^{2n-2}\). This implies that

\[
|D(P)|_p \geq 2^{-1}n^{1-2n}(2n-2)!^{-1}H^{-2n+2}. \tag{5}
\]

Using Lemma 1, \(|a_n|_p \gg 1\) and (2),

\[
|\omega - \gamma_1|_p \leq \min_{1 \leq j \leq n} |(P(\omega)|_p|P'(\gamma_1)|_p^{-1} \prod_{k=2}^{n} |\gamma_1 - \gamma_k|_p|^{1/j}
\leq \min_{1 \leq j \leq n} |H^{-w}|_{a_n}^{-1} \prod_{k=1}^{n} |\gamma_1 - \gamma_k|_{p}^{-1}|^{1/j}
\leq \min_{1 \leq j \leq n} |H^{-w}|_{a_n}^{-1} H^{(j)}_{w+1} |^{1/j}
\leq \min_{1 \leq j \leq n} |H^{-w}|_{j}^{-1}.
\]

Define \(\sigma(P)\) as the cylinder of points \(\omega\) satisfying

\[
|\omega - \gamma_1|_p \leq \min_{1 \leq j \leq n} H^{-w+1}.
\]

Let \(\theta_j = \frac{w-s_j}{j}\) and denote by \(\theta_0\) the maximum value of \(\theta_j, j = 1, \ldots, n\).

Now the polynomial \(P'\) is expanded as a Taylor series and each term is estimated on \(\sigma(P)\). Thus

\[
P'(\omega) = P'(\gamma_1) + \sum_{j=2}^{n} ((j-1)!-1)P^{(j)}(\gamma_1)(\omega - \gamma_1)^{j-1};
\]

\[
|P^{(j)}(\gamma_1)(\omega - \gamma_1)^{j-1}|_p \leq H^{-s_j+(n-j)\epsilon_1} H^{-\theta_0(j-1)}.
\]

As \(\theta_0 \geq \theta_j\), this implies that

\[
|P^{(j)}(\gamma_1)(\omega - \gamma_1)^{j-1}|_p \leq H^{-s_j+(n-j)\epsilon_1} \frac{1}{H^{-1}}(-w+s_j) \leq H^{-w/2+(n-2)\epsilon_1} \quad \text{for} \ 2 \leq j \leq n.
\]

Thus,

\[
|P'(\gamma_1)|_p \leq \max_{1 \leq j \leq n} \{|P^{(j)}(\gamma_1)(\omega - \gamma_1)^{j-1}|_p\} \leq H^{-w/2+(n-2)\epsilon_1}
\]
for $H > H_0(n)$. 

Expressing the discriminant $D(P)$ in the form

$$|D(P)|_p = |a_n|_p^{2n-2} \prod_{1 \leq i < j \leq n} |\gamma_i - \gamma_j|_p^2 = |a_n|_p^{2n-4} |P'(\gamma_1)|_p^2 \prod_{2 \leq i < j \leq n} |\gamma_i - \gamma_j|_p^2$$

and using the facts that $|\gamma_i|_p \ll 1$ and $|a_n|_p \leq 1$, we obtain

$$|D(P)|_p \ll |P'(\gamma_1)|_p^2.$$ 

This contradicts (5) for $w > 2n - 2 + 2(n - 2)\epsilon_1$ and sufficiently large $H$. Therefore, $|P'(\omega)|_p^2 > |P(\omega)|_p$ holds for $w > 2n - 2 + 2(n - 2)\epsilon_1$, and case I follows immediately from Lemma 3. Hence, there exists a root $\gamma_1 \in \mathbb{Q}_p$ of $P$ such that $|\omega - \gamma_1|_p \leq |P(\omega)|_p/|P'(\omega)|_p^2 < 1$. 

### 3.2. Case II: $D(P) = 0$

Consider the polynomial $P \in \mathbb{P}_n$ satisfying $D(P) = 0$. First, $P$ is decomposed into irreducible polynomials $T_i(\omega) \in \mathbb{Z}[\omega]$, i.e.

$$P(\omega) = \prod_{i=1}^k T_i^{s_i}(\omega).$$

It will be shown that for some index $j$, $1 \leq j \leq k$,

$$|T_j(\omega)|_p < 2^{nw/2}H^{-w}(T_j). \quad (6)$$

Assume the opposite, so that

$$|T_j(\omega)|_p \geq 2^{nw/2}H^{-w}(T_j) \text{ for all } j, 1 \leq j \leq k.$$

Then, by Lemma 2,

$$|P(\omega)|_p \geq \prod_{j=1}^k (2^{nw/2}H^{-w}(T_j))^{s_j} \geq 2^{nw(\sum_j s_j/2 - 1)} H(P)^{-w} \geq H(P)^{-w}$$

which contradicts (2). Thus (6) holds.

Hence, applying the same method as in Case I for $T_j$, $D(T_j) \neq 0$, which satisfies (6), it follows that there exists a $p$-adic number $\gamma_1$ such that $|\omega - \gamma_1|_p < 1$ and $T_j(\gamma_1) = 0$. This implies $P(\gamma_1) = 0$. 

\begin{flushright}$\Box$\end{flushright}

### REFERENCES


