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PROBLEM OF NESTERENKO AND METHOD OF BERNIK

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Dedicated to Yuri Valentinovich Nesterenko and Vasilii Ivanovich Bernik on their 70th birthdays

Abstract

In this article we prove that, if integer polynomial P satisfies $|P(\omega)|_p < H^{-w}$, then for $w > 2n-2$ and sufficiently large H the root γ belongs to the field of p-adic numbers.

Keywords: integer polynomials, discriminants of polynomials.

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1. Introduction

Throughout this paper, p is a prime number, $\mathbb{Q}_p$ is the field of p-adic numbers,

$$P(x) = a_n x^n + \ldots + a_1 x + a_0$$

is an integer polynomial with degree $\deg P(x) = n$ and height $H(P) = \max_{0 \leqslant j \leqslant n} |a_j|$. We denote by $\mathcal{P}_n$ the set of integer polynomials of degree n. Let $\mathcal{P}_n(H) = \{P \in \mathcal{P}_n \,:\, H(P) = H\}$.

In this paper, a result originally considered by Y. V. Nesterenko is examined. In [1] Y.V. Nesterenko discussed the solvability of the equation $P(x) = 0$ in the ring of p-adic integers $\mathbb{Z}_p$ and proved the following result.

Theorem 1. *Let x be an integer and $P \in \mathcal{P}_n(H)$. If*

$$|P(x)|_p \leqslant e^{-8n^2} H^{-4n},$$

then there exists a p-adic number γ such that

$$P(\gamma) = 0, \quad |x - \gamma|_p < 1.$$

Note that a similar problem was considered in [2] and there was given a criteria for when the closest root of a polynomial to a real point belongs to the field of real numbers. Knowledge of the nature of the roots is very important in the problems of Diophantine approximations for construction of regular systems [3,4]. Numerous applications of this concept arose when obtaining estimates for the Hausdorff measure and Hausdorff dimension of Diophantine sets [5] and proving analogues of the Khintchine theorem [6,7]. Using the regular systems, the exact theorems on approximation of real numbers by real algebraic [6], by algebraic integers [8], of complex numbers by complex algebraic [9] were obtained, and similar problems in the field of p-adic numbers [10] and in $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ [7] were investigated.

The Theorem 1 can be improved for p-adic leading polynomials. Such a polynomial $P \in \mathcal{P}_n$ satisfies

$$|a_n|_p \gg 1. \tag{1}$$

Theorem 2. *Let $\omega \in \mathbb{Z}_p$ and $P \in \mathcal{P}_n(H)$ be a p-adic leading polynomial. Then if*

$$|P(\omega)|_p < H^{-w} \tag{2}$$

for $w > 2n-2$, and for sufficiently large $H > H_0(n)$, it follows that the root γ_1 of P belongs to $\mathbb{Q}_p$ and

$$|\omega - \gamma_1|_p < 1. \tag{3}$$

Remark 1. *If $D(P) \neq 0$ then we have that the root γ_1 of P is closest to $\omega \in \mathbb{Z}_p$.* The above theorem will be proved using a general method of V.I. Bernik which was developed in [11,12].

2. Preliminary setup and auxilliary Lemmas

Let $P \in \mathcal{P}_n$ have roots $\gamma_1, \gamma_2, \ldots, \gamma_n$ in $\mathbb{Q}_p^*$, where $\mathbb{Q}_p^*$ is the smallest field containing $\mathbb{Q}_p$ and all algebraic numbers. Then, from (1) it follows that

$$|\gamma_i|_p \ll 1, \quad i = 1, \ldots, n; \tag{4}$$

i.e. the roots are bounded. This follows from Lemma 4 in ([13], p.85).

Define the sets

$$T_p(\gamma_k) \;=\; \{\omega \in \mathbb{Z}_p : |\omega - \gamma_k|_p = \min_{1 \leqslant i \leqslant n} |\omega - \gamma_i|_p\}, \;\; 1 \leqslant k \leqslant n.$$

Consider the set $T_p(\gamma_k)$ for a fixed k and for ease of notation assume that $k = 1$. Next, reorder the other roots so that

$$|\gamma_1 - \gamma_2|_p \;\leqslant\; |\gamma_1 - \gamma_3|_p \;\leqslant \ldots \leqslant\; |\gamma_1 - \gamma_n|_p.$$

Fix $\epsilon > 0$ where ϵ is sufficiently small and suppose that $\epsilon_1 = \epsilon N^{-1}$ where $N = N(n) > 0$ is sufficiently large. Let $T = [\epsilon_1^{-1}]$.

For a polynomial $P \in \mathcal{P}_n(H)$ define the real numbers ρ_j by

$$|\gamma_1 - \gamma_j|_p \;=\; H^{-\rho_j}, \;\; 2 \leqslant j \leqslant n, \quad \rho_2 \geqslant \rho_3 ... \geqslant \rho_n.$$

Define the integers m_j, $2 \leqslant j \leqslant n$, such that

$$\frac{m_j - 1}{T} \leqslant \rho_j < \frac{m_j}{T}, m_2 \geqslant m_3 \geqslant ... \geqslant m_n \geqslant 0.$$

Further define numbers s_i such that

$$s_i \;=\; \frac{m_{i+1} + \ldots + m_n}{T}, \quad (1 \leqslant i \leqslant n-1), \;\; s_n = 0.$$

The first Lemma is a p-adic analogue of the Lemma, which was proved by Bernik in [14] and is a generalisation of Sprindžuk's Lemma ([13], p.77).

Lemma 1. *[15] Let $\omega \in T_P(\gamma_1)$. Then*

$$|\omega - \gamma_1|_p \leqslant \min_{1 \leqslant j \leqslant n} (|P(\omega)|_p |P'(\gamma_1)|_p^{-1} \prod_{k=2}^{j} |\gamma_1 - \gamma_k|_p)^{1/j}.$$

The following Lemma is often referred to as Gelfond's Lemma.

Lemma 2 ([16], Lemma A.3). *Let $P_1, P_2, \ldots, P_k$ be polynomials of degree $n_1, \ldots, n_k$ respectively, and let $P = P_1 P_2 \ldots P_k$. Let $n = n_1 + n_2 + \ldots + n_k$. Then*

$$2^{-n} H(P_1) H(P_2) \ldots H(P_k) \leqslant H(P) \leqslant 2^n H(P_1) H(P_2) \ldots H(P_k).$$

In the proof of theorem we will refer to the following statement known as Hensel's Lemma.

Lemma 3 ([4], p. 134). *Let P be a polynomial with coefficients in $\mathbb{Z}_p$, let $\xi = \xi_0 \in \mathbb{Z}_p$ and $|P(\xi)|_p < |P'(\xi)|_p^2$. Then as $n \to \infty$ the sequence*

$$\xi_{n+1} = \xi_n - \frac{P(\xi_n)}{P'(\xi_n)}$$

tends to some root $\beta \in \mathbb{Q}_p$ of the polynomial P and

$$|\beta - \xi|_p \leqslant |P(\xi)|_p / |P'(\xi)|_p^2 < 1.$$

3. Proof of Theorem 2

Two cases must be dealt with separately: $D(P) \neq 0$ and $D(P) = 0$.

3.1. Case I: $D(P) \neq 0$

First consider a polynomial $P \in \mathcal{P}_n(H)$ satisfying $D(P) \neq 0$ and (2), and assume that $|P'(\omega)|_p^2 \leqslant |P(\omega)|_p$. We will obtain a contradiction. Using (4), we get $|P'(\omega)|_p < H^{-w/2}$.

It is well known that $|D(P)| = \frac{|\Delta|}{|a_n|}$, where

$$\Delta = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 & 0 & \dots & 0 \\ 0 & a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \\ na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & a_1 & \dots & 0 & \dots & 0 \\ 0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & a_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & a_1 \end{pmatrix}.$$

Hence the determinant,

$$\begin{aligned} |\Delta| &\leqslant |a_n|((2n-2)!(nH)^{2n-2} + n(2n-2)!(nH)^{2n-2}) \\ &= |a_n|(2n-2)!(n+1)(nH)^{2n-2} \leqslant 2n^{2n-1}(2n-2)!H^{2n-2}|a_n|, \end{aligned}$$

using the fact that $|a_i| \leqslant H$, $i = 0, 1, \dots, n$. Thus, $|D(P)| \leqslant 2n^{2n-1}(2n-2)!H^{2n-2}$. This implies that

$$|D(P)|_p \geqslant 2^{-1}n^{1-2n}((2n-2)!)^{-1}H^{-2n+2}. \tag{5}$$

Using Lemma 1, $|a_n|_p \gg 1$ and (2),

$$\begin{aligned} |\omega - \gamma_1|_p &\leqslant \min_{1\leqslant j\leqslant n}(|P(\omega)|_p|P'(\gamma_1)|_p^{-1}\textstyle\prod_{k=2}^{j}|\gamma_1 - \gamma_k|_p)^{1/j} \\ &< \min_{1\leqslant j\leqslant n}(H^{-w}|a_n|_p^{-1}\textstyle\prod_{k=j+1}^{n}|\gamma_1 - \gamma_k|_p^{-1})^{1/j} \\ &\leqslant \min_{1\leqslant j\leqslant n}(H^{-w}|a_n|_p^{-1}H^{s_j})^{1/j} \\ &\ll \min_{1\leqslant j\leqslant n} H^{\frac{-w+s_j}{j}}. \end{aligned}$$

Define $\sigma(P)$ as the cylinder of points w satisfying

$$|\omega - \gamma_1|_p \ll \min_{1\leqslant j\leqslant n} H^{\frac{-w+s_j}{j}}.$$

Let $\theta_j = \frac{w-s_j}{j}$ and denote by θ_0 the maximum value of θ_j, $j = 1, \dots, n$.

Now the polynomial P' is expanded as a Taylor series and each term is estimated on $\sigma(P)$. Thus

$$\begin{aligned} P'(\omega) &= P'(\gamma_1) + \sum_{j=2}^{n}((j-1)!)^{-1}P^{(j)}(\gamma_1)(\omega-\gamma_1)^{j-1}, \\ |P^{(j)}(\gamma_1)(\omega-\gamma_1)^{j-1}|_p &\ll H^{-s_j+(n-j)\epsilon_1}H^{-\theta_0(j-1)}. \end{aligned}$$

As $\theta_0 \geqslant \theta_j$, this implies that

$$|P^{(j)}(\gamma_1)(\omega-\gamma_1)^{j-1}|_p \ll H^{-s_j+(n-j)\epsilon_1+\frac{j-1}{j}(-w+s_j)} \leqslant H^{-w/2+(n-2)\epsilon_1} \quad \text{for } 2 \leqslant j \leqslant n.$$

Thus,

$$|P'(\gamma_1)|_p \leqslant \max_{1\leqslant j\leqslant n}\{|P^{(j)}(\gamma_1)(\omega-\gamma_1)^{j-1}|_p\} \ll H^{-w/2+(n-2)\epsilon_1}$$

for $H > H_0(n)$.

Expressing the discriminant $D(P)$ in the form

$$|D(P)|_p = |a_n|_p^{2n-2} \prod_{1\leqslant i<j\leqslant n} |\gamma_i - \gamma_j|_p^2 = |a_n|_p^{2n-4} |P'(\gamma_1)|_p^2 \prod_{2\leqslant i<j\leqslant n} |\gamma_i - \gamma_j|_p^2$$

and using the facts that $|\gamma_i|_p \ll 1$ and $|a_n|_p \leqslant 1$, we obtain

$$|D(P)|_p \ll |P'(\gamma_1)|_p^2.$$

This contradicts (5) for $w > 2n-2+2(n-2)\epsilon_1$ and sufficiently large H. Therefore, $|P'(\omega)|_p^2 > |P(\omega)|_p$ holds for $w > 2n-2+2(n-2)\epsilon_1$, and case I follows immediately from Lemma 3. Hence, there exists a root $\gamma_1 \in \mathbb{Q}_p$ of P such that $|\omega - \gamma_1|_p \leqslant |P(\omega)|_p / |P'(\omega)|_p^2 < 1$.

3.2. Case II: $D(P) = 0$

Consider the polynomial $P \in \mathcal{P}_n$ satisfying $D(P) = 0$. First, P is decomposed into irreducible polynomials $T_i(\omega) \in \mathbb{Z}[\omega]$, i.e.

$$P(\omega) = \prod_{i=1}^{k} T_i^{s_i}(\omega).$$

It will be shown that for some index j, $1 \leqslant j \leqslant k$,

$$|T_j(\omega)|_p < 2^{nw/2} H^{-w}(T_j). \tag{6}$$

Assume the opposite, so that

$$|T_j(\omega)|_p \geqslant 2^{nw/2} H^{-w}(T_j) \text{ for all } j,\ 1 \leqslant j \leqslant k.$$

Then, by Lemma 2,

$$|P(\omega)|_p \geqslant \prod_{j=1}^{k} (2^{nw/2} H^{-w}(T_j))^{s_j} \geqslant 2^{nw(\sum_{j=1}^{k} s_j/2-1)} H(P)^{-w} \geqslant H(P)^{-w}$$

which contradicts (2). Thus (6) holds.

Hence, applying the same method as in Case I for T_j, $D(T_j) \neq 0$, which satisfies (6), it follows that there exists a p-adic number γ_1 such that $|\omega - \gamma_1|_p < 1$ and $T_j(\gamma_1) = 0$. This implies $P(\gamma_1) = 0$. □

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