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СМЕШАННАЯ СОВМЕСТНАЯ ФУНКЦИОНАЛЬНАЯ НЕЗАВИСИМОСТЬ ДЛЯ ДЗЕТА-ФУНКЦИИ РИМАНА И ПЕРИОДИЧЕСКИХ ДЗЕТА-ФУНКЦИИЙ ГУРВИЦА

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Аннотация

Функциональная независимость дзета-функций является интересной проблемой современности и восходит к Д. Гилберту. В 1990, выступая с докладом на Международном конгресе математиков в Париже, он выдвинул гипотезу, что дзета-функция Римана не удовлетворяет никакому алгебраическому дифференциальному уравнению. Эта гипотеза была доказана А. Островским. В 1975 г. С.М. Воронин доказал функциональную независимость дзета-функции Римана. С тех пор многими авторами была получена функциональная независимость ряда дзета и L-функций.

В настоячей статье получена совместная функциональная независимость дзета-функции Римана и периодических дзета-функциий Гурвица с параметрами, алгебраически независимыми над полем рациональных чисел. Такая функциональная независимость иногда называется смешанной, поскольку дзета-функция Римана имеет эйлеровое произведение по простым числам, а периодические дзета-функции Гурвица такого произведения не имеет.

Ключевые слова: дзета-функция Римана, дзета-функция Гурвица, периодические коэффициенты, функциональная независимость, универсальность.

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THE MIXED JOINT FUNCTIONAL INDEPENDENCE OF THE RIEMANN ZETA- AND PERIODIC HURWITZ ZETA-FUNCTIONS

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Abstract

The functional independence of zeta-functions is an interesting nowadays problem. This problem comes back to D. Hilbert. In 1900, at the International Congress of Mathematicians in Paris, he conjectured that the Riemman zeta-function does not satisfy any algebraic-differential equation. This conjecture was solved by A. Ostrowski. In 1975, S.M. Voronin proved the functional independence of the Riemann zeta-function. After that many mathematicians obtained the functional independence of certain zeta- and L-functions.

In the present paper, the joint functional independence of a collection consisting of the Riemann zeta-function and several periodic Hurwitz zeta-functions with parameters algebraically independent over the field of rational numbers is obtained. Such type of functional independence is called as “mixed functional independence” since the Riemann zeta-function has Euler product expansion over primes while the periodic Hurwitz zeta-functions do not have Euler product.

Keywords: functional independence, Hurwitz zeta-function, periodic coefficients, Riemann zeta-function, universality.

Bibliography: 17 titles.

1. Introduction

The functional independence of certain functions has a long history and is relevant in nowadays. Let us recall some important facts.

Denote by $s = \sigma + it$ a complex variable, and by $\mathbb{N}, \mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ the sets of all positive integers, primes, rational numbers, real numbers and complex numbers, respectively.

In 1887, O. Hölder proved [1] the algebraic-differential independence for the gamma-function $\Gamma(s)$, i.e., that there exists no polynomial $P \not\equiv 0$ such that

$$P\Big(s, \Gamma(s), \Gamma'(s), ..., \Gamma^{(n-1)}(s)\Big) = 0$$

for all $s \in \mathbb{C}$, $n \in \mathbb{N}$.

In 1900, D. Hilbert noted [2] that the Riemann zeta-function $\zeta(s)$ does not satisfy any algebraic differential equation, also. He proposed a more general problem, i.e., to prove that the function

$$\zeta(s, x) = \sum_{m=1}^{\infty} \frac{x^m}{m^s}$$

does not satisfy any algebraic-differential equation. In 1920, this problem was solved by A. Ostrowski [3], and later generalized by A.G. Postnikov [5], [4].

In 1973, S.M. Voronin obtained [6] the functional independence of the Riemann zeta-function. He proved that the Riemann zeta-function $\zeta(s)$ does not satisfy any differential equation

$$\sum_{j=0}^{n} s^j F_j\Big(\zeta(s), \zeta'(s), ..., \zeta^{(N-1)}(s)\Big) \equiv 0,$$

where F_j are continuous functions, not all identically zero.

The Voronin result was generalized by R. Garunkštis, A. Laurinčikas, K. Matsumoto, H. Mishou, J. Steuding and many other mathematicians (see, for example, [7], [8]).

The mixed joint functional independence of Riemann zeta-function and Hurwitz zeta-function was obtained by H. Mishou in 2007 [9]. Later, R. Kačinskaitė and A. Laurinčikas generalized Mishou's result for the periodic zeta- and periodic Hurwitz zeta-functions [10].

The aim of the paper is to prove the mixed joint functional independence of a collection consisting of the Riemann zeta-function and several periodic Hurwitz zeta-functions, i.e., to extend the collection of zeta-functions into more general case as in the above mentioned results.

Since one of functions under investigation is the Riemann zeta-function $\zeta(s)$, we recall its definition and main properties.

In 1859, B. Riemann introduced [11] the zeta-function $\zeta(s)$ as complex-variable function. For $\sigma > 1$, it is given by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}.$$

The function $\zeta(s)$ can be written by Euler product over primes as

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1,$$

The function $\zeta(s)$ is analytically continuable to the whole complex plane $\mathbb{C}$, except for a simple pole at the point $s = 1$ with residue 1.

The second zeta-function under our interest is the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$. In 2006, it was introduced by A. Javtokas and A. Laurinčikas [12]. Let $\mathfrak{a}=\{a_m : a_m \in \mathbb{N}\cup\{0\}\}$ be a periodic sequence of complex numbers with a minimal period $k \in \mathbb{N}$. The function $\zeta(s,\alpha;\mathfrak{a})$ with a fixed parameter α, $0<\alpha\leqslant 1$, is defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s,\alpha;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m}{(m+\alpha)^s}.$$

If $a_m\equiv 1$, the function $\zeta(s,\alpha;\mathfrak{a})$ reduces to the classical Hurwitz zeta-function

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s}, \quad \sigma>1.$$

From the periodicity of the sequence $\mathfrak{a}$, we have that, for $\sigma>1$,

$$\begin{aligned}\zeta(s,\alpha;\mathfrak{a}) &= \sum_{r=0}^{k-1}\sum_{m=0}^{\infty}\frac{a_r}{(mk+r+\alpha)^s}=\frac{1}{k^s}\sum_{r=0}^{k-1}a_r\sum_{m=0}^{\infty}\frac{1}{\left(m+\left(\frac{r+\alpha}{k}\right)\right)^s}\\ &= \frac{1}{k^s}\sum_{r=0}^{k-1}a_r\zeta\Big(s;\frac{r+\alpha}{k}\Big). \end{aligned} \tag{1}$$

It is known that the Hurwitz zeta-function $\zeta(s,\alpha)$ has an analytic continuation to the whole s-plane except for a simple pole at the point $s=1$ with residue 1. Then the equality (1) gives an analytic continuation of the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$ to the whole s-plane, except, maybe, for a simple pole $s=1$ with residue

$$a:=\frac{1}{k}\sum_{r=0}^{k-1}a_r.$$

If $a=0$, then $\zeta(s,\alpha;\mathfrak{a})$ is an entire function.

The joint functional independence of a collection of periodic Hurwitz zeta-functions with parameters algebraically independent over the field of rational numbers $\mathbb{Q}$ was obtained by A. Laurinčikas in [13].

Suppose that $0<\alpha_j\leq 1$ is a fixed parameter, $j=1,...,r$. Let, for positive integer l_j, $\mathfrak{a}_{jl}=\{a_{mjl} : m\in\mathbb{N}\cup\{0\}\}$ be a periodic sequence of complex numbers a_{mjl} with a minimal period $k_{jl}\in\mathbb{N}$, and let $\zeta(s,\alpha_j;\mathfrak{a}_{jl})$ denote the corresponding periodic Hurwitz zeta-function, $j=1,...,r$, $l=1,...,l_j$. Moreover, let k_j be the least multiple of the periods $k_{j1},...,k_{jl_j}$, and

$$\mathfrak{A}_j=\begin{pmatrix} a_{1j1} & a_{1j2} & \dots & a_{1jl_j}\\ a_{2j1} & a_{2j2} & \dots & a_{2jl_j}\\ \dots & \dots & \dots & \dots\\ a_{k_jj1} & a_{k_jj2} & \dots & a_{k_jjl_j}\end{pmatrix}, \quad j=1,...,r.$$

Denote $\kappa=l_1+l_2+...+l_r+1$.

The main purpose of the paper is to prove the mixed joint functional independence of the functions $\zeta(s)$ and $\zeta(s,\alpha_j;\mathfrak{a}_{jl})$, $j=1,...,r$, $l=1,...,l_j$.

For the statement of our result, we need a certain condition for the parameters in the definition of the functions $\zeta(s,\alpha_j;\mathfrak{a}_{jl})$, $j=1,...,r$, $l=1,...,l_j$. Recall that the numbers $\alpha_1,\alpha_2,...,\alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$, if there exists no polynomial $p(x_1,x_2,...,x_r)\not\equiv 0$ with rational coefficients such that $p(\alpha_1,\alpha_2,...,\alpha_r)=0$.

The main result of the paper is the following theorem.

THEOREM 1. *Suppose that* $\alpha_1, \alpha_2, ..., \alpha_r$ *are numbers algebraically independent over* $\mathbb{Q}$, $\mathrm{rank}\mathfrak{A}_j = l_j$, $j = 1, ..., r$, *and* $F_g : \mathbb{C}^{\kappa N} \to \mathbb{C}$ *is a continuous function for each* $g = 0, 1, ..., n$. *If, for* $N \in \mathbb{N}$, *the function*

$$\sum_{g=0}^{n} s^g \cdot F_g\Big(\begin{aligned}[t] &\zeta(s), \zeta'(s), ..., \zeta^{(N-1)}(s),\\ &\zeta(s, \alpha_1; \mathfrak{a}_{11}), \zeta'(s, \alpha_1; \mathfrak{a}_{11}), ..., \zeta^{(N-1)}(s, \alpha_1; \mathfrak{a}_{11}), ...,\\ &\zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \zeta'(s, \alpha_1; \mathfrak{a}_{1l_1}), ..., \zeta^{(N-1)}(s, \alpha_1; \mathfrak{a}_{1l_1}), ...,\\ &\zeta(s, \alpha_r; \mathfrak{a}_{r1}), \zeta'(s, \alpha_r; \mathfrak{a}_{r1}), ..., \zeta^{(N-1)}(s, \alpha_r; \mathfrak{a}_{r1}), ...,\\ &\zeta(s, \alpha_r; \mathfrak{a}_{rl_r}), \zeta'(s, \alpha_r; \mathfrak{a}_{rl_r}), ..., \zeta^{(N-1)}(s, \alpha_r; \mathfrak{a}_{rl_r})\Big) \end{aligned}$$

is identically equal to zero, then $F_g \equiv 0$ *for* $g = 0, ..., n$.

For the proof of the mixed joint functional independence for the functions $\zeta(s)$ and $\zeta(s, \alpha_j; \mathfrak{a}_{jl})$, $j = 1, ..., r$, $l = 1, ..., l_j$, first of all, we need the joint universality theorem in the Voronin sense. We need a denseness lemma as well. Both of them in the next section are given.

2. Auxiliary results

For the statement of auxiliary results, we need some notation and definitions.

Let S be any space. Denote by $\mathcal{B}(S)$ the set of all Borel subset of S, and by $\mathrm{meas}A$ denote the Lebesgue measure of the measurable set $A \subset \mathbb{R}$. Let $H(G)$ be the space of analytic on a certain region G functions. By D denote the strip $\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ of the complex plane $\mathbb{C}$.

2.1. The mixed joint universality theorem

Suppose that K is a compact subset of $\mathbb{C}$. Denote by $H^c(K)$ the set of all $\mathbb{C}$-valued functions defined on K, continuous on K and analytic in the interior of K. Let $H_0^c(K)$ be the subset of $H^c(K)$, consisting of all elements of $H^c(K)$ which are non-vanishing on K. Then, for the functions $\zeta(s)$ and $\zeta(s, \alpha_j; \mathfrak{a}_{jl})$, $j = 1, ..., r$, $l = 1, ..., l_j$, the following mixed joint universality property is true.

THEOREM 2. *Suppose that all hypotheses on* α_j *and* $\mathrm{rank}\mathfrak{A}_j$, $j = 1, ..., r$, *of* Theorem 1 hold. Let K_1 and K_{jl} be a compact subsets of the strip D with connected complements, $j = 1, ..., r$, $l = 1, ..., l_j$, and that $f_1(s) \in H_0^c(K_1)$ ir $f_{jl}(s) \in H^c(K_{jl})$, $j = 1, ..., r$, $l = 1, ..., l_j$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0, T] : \begin{aligned}[t] &\sup_{s\in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon,\\ &\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\Big\} > 0. \end{aligned}$$

This is Theorem 3 from [14].

2.2. A denseness lemma

We already mentioned that, for the proof of the mixed joint functional independence of the functions $\zeta(s)$ and $\zeta(s, \alpha_j; \mathfrak{a}_{jl})$, $j = 1, ..., r$, $l = 1, ..., l_j$, we use a denseness lemma as well.

Define the function $u : \mathbb{R} \to \mathbb{C}^{\kappa N}$ by the formula

$$u(t) = (\begin{aligned}[t] &\zeta(\sigma + it), \zeta'(\sigma + it), ..., \zeta^{(N-1)}(\sigma + it),\\ &\zeta(\sigma + it, \alpha_1; \mathfrak{a}_{11}), \zeta'(\sigma + it, \alpha_1; \mathfrak{a}_{11}), ..., \zeta^{(N-1)}(\sigma + it, \alpha_1; \mathfrak{a}_{11}), ...,\\ &\zeta(\sigma + it, \alpha_1; \mathfrak{a}_{1l_1}), \zeta'(\sigma + it, \alpha_1; \mathfrak{a}_{1l_1}), ..., \zeta^{(N-1)}(\sigma + it, \alpha_1; \mathfrak{a}_{1l_1}), ..., \end{aligned}$$

$$\zeta(\sigma+it,\alpha_r;\mathfrak{a}_{r1}),\zeta'(\sigma+it,\alpha_r;\mathfrak{a}_{r1}),...,\zeta^{(N-1)}(\sigma+it,\alpha_r;\mathfrak{a}_{r1})),...,$$
$$\zeta(\sigma+it,\alpha_r;\mathfrak{a}_{rl_r}),\zeta'(\sigma+it,\alpha_r;\mathfrak{a}_{rl_r}),...,\zeta^{(N-1)}(\sigma+it,\alpha_r;\mathfrak{a}_{rl_r})),$$

where $\frac{1}{2}<\sigma<1$.

LEMMA 1. *Suppose that* α_j *and* $\mathrm{rank}\mathfrak{A}_j=l_j$, $j=1,...,r$, *are the same as in* Theorem 1. Then the image of $\mathbb{R}$ by u is everywhere dense in $\mathbb{C}^{\kappa N}$.

PROOF. We will prove that, for any $\varepsilon>0$, there exists a sequence $\{\tau_k:\tau_k\in\mathbb{R}\}$, $\lim_{k\to\infty}\tau_k=+\infty$, such that

$$|u(\tau_k)-\underline{s}|_{\mathbb{C}^{\kappa N}}<\varepsilon,$$

where $\underline{s}=(s_{10},...,s_{1N-1},s_{111},...,s_{1l_11},...,s_{1l_1,N-1},...,s_{rl_r1},...,s_{rl_r,N-1})$ is an arbitrary point on $\mathbb{C}^{\kappa N}$, and $|\cdot|_{\mathbb{C}^{\kappa N}}$ denotes the distance in the space $\mathbb{C}^{\kappa N}$. To show this, it is sufficient to prove that there exists a sequence $\{\tau_k:\tau_k\in\mathbb{R}\}$, $\lim_{k\to\infty}\tau_k=+\infty$, such that, for $g=0,...,N-1$ and every $\varepsilon>0$, the inequalities

$$|\zeta^{(g)}(\sigma+i\tau_k)-s_{1g}|<\frac{\varepsilon}{N},$$

and

$$|\zeta^{(g)}(\sigma+i\tau_k,\alpha_j;\mathfrak{a}_{jl})-s_{dg}|<\frac{\varepsilon}{(\kappa-1)N},$$

hold with $j=1,...,r$, $l=1,...,l_j$ and d (here and thereinafter d means the pair of j and l).

Define a polynomial

$$p_{dN}(s)=\sum_{g=0}^{N-1}\frac{s_{dg}s^g}{g!},$$

where $d=1$ for the function $\zeta(s)$, and d is in the above described sense for the function $\zeta(s,\alpha_j;\mathfrak{a}_{jl})$, $j=1,...,r$, $l=1,...,l_j$. Then, for $g=0,...,N-1$, we have that

$$p_{dN}^{(g)}(0)=s_{dg}.$$

Now we fix a number σ_0, $\frac{1}{2}<\sigma_0<1$. Let K be a compact subset of the strip D such that σ_0 is an interior point of K. Then, by Theorem 2, there exists a sequence $\{\tau_k:\tau_k\in\mathbb{R}\}$, $\lim_{k\to\infty}\tau_k=+\infty$, such that

$$\sup_{s\in K_1}|\zeta(s+i\tau_k)-p_{1N}(s-\sigma_0)|<\frac{\varepsilon\delta^N}{2^{N+1}N!},$$
$$\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|\zeta(s+i\tau_k,\alpha_j;\mathfrak{a}_{jl})-p_{dN}(s-\sigma_0)|<\frac{\varepsilon\delta^N}{2^NN!N(\kappa-1)},$$

where $\delta=\min(\delta_1,\delta_{jl})$, $j=1,...,r$, $l=1,...,l_j$ (here δ_1 and δ_{jl} are the distances of σ_0 from the boundaries of the sets K_1 and K_{jl}, respectively). By the integral Cauchy formula, for $g=0,1,...,N-1$, we have that

$$|\zeta^{(g)}(\sigma_0+i\tau_k)-s_{1g}|$$
$$=\Big|\frac{g!}{2\pi i}\int_{|s-\sigma_0|=\frac{\delta}{2}}\frac{\zeta(s+i\tau_k)-p_{1N}(s-\sigma_0)}{(s-\sigma_0)^{g+1}}ds\Big|<\frac{\varepsilon}{N}.$$

Similarly,

$$|\zeta^{(g)}(\sigma_0+i\tau_k,\alpha_j;\mathfrak{a}_{jl})-s_{dg}|$$
$$=\Big|\frac{g!}{2\pi i}\int_{(s-\sigma_0)=\frac{\delta}{2}}\frac{\zeta(s+i\tau_k,\alpha_j;\mathfrak{a}_{jl})-p_{dN}(s-\sigma_0)}{(s-\sigma_0)^{g+1}}ds\Big|<\frac{\varepsilon}{N(\kappa-1)}$$

for $j=1,...,r$, $l=1,...,l_j$. This proves the lemma.

3. Proof of Theorem 1

Now we are ready to complete the proof of Theorem 1. The proof uses Theorem 2 and Lemma 1. We first prove that in Theorem 1 the function $F_g \equiv 0$, $g = 1, ..., n$.

Instead of the function F_g, we will investigate the general function F. Let $F : \mathbb{C}^{\kappa N} \to \mathbb{C}$ be a continuous function, and

$$F\Big(\begin{array}{l} \zeta(s), \zeta'(s), ..., \zeta^{(N-1)}(s), \\ \zeta(s,\alpha_1;\mathfrak{a}_{11}), \zeta'(s,\alpha_1;\mathfrak{a}_{11}), ..., \zeta^{(N-1)}(s,\alpha_1;\mathfrak{a}_{11}), ..., \\ \zeta(s,\alpha_1;\mathfrak{a}_{1l_1}), \zeta'(s,\alpha_1;\mathfrak{a}_{1l_1}), ..., \zeta^{(N-1)}(s,\alpha_1;\mathfrak{a}_{1l_1}), ..., \\ \zeta(s,\alpha_r;\mathfrak{a}_{r1}), \zeta'(s,\alpha_r;\mathfrak{a}_{r1}), ..., \zeta^{(N-1)}(s,\alpha_r;\mathfrak{a}_{r1}), ..., \\ \zeta(s,\alpha_r;\mathfrak{a}_{rl_r}), \zeta'(s,\alpha_r;\mathfrak{a}_{rl_r}), ..., \zeta^{(N-1)}(s,\alpha_r;\mathfrak{a}_{rl_r})\Big) \equiv 0. \end{array}$$

We will prove that $F \equiv 0$. Let $\frac{1}{2} < \sigma < 1$. Suppose, on the contrary, that $F \not\equiv 0$. Then there exists a point $\underline{a} \in \mathbb{C}^{\kappa N}$ such that $F(\underline{a}) \neq 0$. Since the function F is continuous, we can find a bounded region $G \subset \mathbb{C}^{\kappa N}$, $\underline{a} \in G$, and such that, for all $\underline{s} \in G$, the inequality

$$|F(\underline{s})| \geq c > 0 \tag{2}$$

holds. Then, by Lemma 1, there exists a sequence $\{\tau_m : \tau_m \in \mathbb{R}\}$, $\lim\limits_{m\to\infty} \tau_m = \infty$, such that

$$\Big(\begin{array}{l} \zeta(\sigma+i\tau_k), \zeta'(\sigma+i\tau_k), ..., \zeta^{(N-1)}(\sigma+i\tau_k), \\ \zeta(\sigma+i\tau_k,\alpha_1;\mathfrak{a}_{11}), \zeta'(\sigma+i\tau_k,\alpha_1;\mathfrak{a}_{11}), ..., \zeta^{(N-1)}(\sigma+i\tau_k,\alpha_1;\mathfrak{a}_{11}), ..., \\ \zeta(\sigma+i\tau_k,\alpha_1;\mathfrak{a}_{1l_1}), \zeta'(\sigma+i\tau_k,\alpha_1;\mathfrak{a}_{1l_1}), ..., \zeta^{(N-1)}(\sigma+i\tau_k,\alpha_1;\mathfrak{a}_{1l_1}), ..., \\ \zeta(\sigma+i\tau_k,\alpha_r;\mathfrak{a}_{r1}), \zeta'(\sigma+i\tau_k,\alpha_r;\mathfrak{a}_{r1}), ..., \zeta^{(N-1)}(\sigma+i\tau_k,\alpha_r;\mathfrak{a}_{r1}), ..., \\ \zeta(\sigma+i\tau_k,\alpha_r;\mathfrak{a}_{rl_r}), \zeta'(\sigma+i\tau_k,\alpha_r;\mathfrak{a}_{rl_r}), ..., \zeta^{(N-1)}(\sigma+i\tau_k,\alpha_r;\mathfrak{a}_{rl_r})\Big) \in G. \end{array}$$

However, this together with inequality (2) contradicts the hypothesis that $F \not\equiv 0$.

Similarly, we can show that every continuous function $F_g \equiv 0$, $g = 1, ..., n$.

The proof of Theorem 1 is complete.

4. Conclusions

Analogue results on the joint mixed functional independence of certain zeta-functions we may obtain in the same way as Theorem 1 if the function $\zeta(s)$ will be replaced by certain zeta-functions, namely by the zeta-functions of normalized Hecke cusp forms, the zeta-functions of newforms with a Dicirhlet characters, the L-functions from the Selberg class, and etc. In the proof of the functional independence, the main role is played by the joint mixed universality for the collection consisting of one before mentioned zeta-function and several periodic Hurwitz zeta-functions like in Theorem 1. Such theorems are proved (or only stated) in [15], [16] and [17], respectively.

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