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МОДИФИКАЦИЯ ТЕОРЕМЫ МИШУ

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Аннотация

В 2007 г. Г. Мишу доказал совместную теорему униврсальности для дзета-функции Римана $\zeta(s)$ и дзета-функции Гурвица $\zeta(s,\alpha)$ с трансцендентным параметром α об одновременном приближении пары функций из широкого класса аналитических функций сдвигами $(\zeta(s+i\tau),\zeta(s+i\tau,\alpha))$, $\tau\in\mathbb{R}$. Он получил, что множество таких сдвигов, приближающих данную пару аналитических функций, имеет положительную нижнюю плотность. В статье получено, что множество таких сдвигов имеет положительную плотность для всех $\varepsilon>0$, за исключением счетного множества значений ε, где ε – точность приближения.

Результаты аналогичного типа также получены для сложных функций $F(\,\zeta(s),\zeta(s,\alpha))$ для некоторых классов операторов F в пространстве аналитических функций.

Ключевые слова: дзета-функция Гурвица, дзета-функция Римана, пространство аналитических функций, универсальность.

Библиография: 21 названий.

MODIFICATION OF THE MISHOU THEOREM

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Abstract

The Mishou theorem asserts that a pair of analytic functions from a wide class can be approximated by shifts of the Riemann zeta and Hurwitz zeta-functions $(\zeta(s+i\tau),\zeta(s+i\tau,\alpha))$ with transcendental α, $\tau\in\mathbb{R}$, and that the set of such τ has a positive lower density. In the paper, we prove that the above set has a positive density for all but at most countably many $\varepsilon>0$, where ε is the accuracy of approximation. We also obtain similar results for composite functions $F(\zeta(s),\zeta(s,\alpha))$ for some classes of operator F.

Keywords: Hurwitz zeta-function, Riemann zeta-function, space of analytic functions, universality.

Bibliography: 21 titles.

1. Introduction

Let $\zeta(s)$, $s=\sigma+it$, be the Riemann zeta-function. In 1975, S. M. Voronin discovered [21] the universality property of $\zeta(s)$ which means that a wide class of non-vanishing analytic functions can be approximated by shifts $\zeta(s+i\tau)$, $\tau\in\mathbb{R}$. The non-vanishing of approximated functions is connected to the existence of Euler's product over primes for $\zeta(s)$.

Now let $0<\alpha\leqslant 1$ be a fixed parameter, and $\zeta(s,\alpha)$ denotes the Hurwitz zeta-function which is defined, for $\alpha>1$, by the series

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},$$

and can be meromorphically continued to the whole complex plane. Clearly, $\zeta(s,1)=\zeta(s)$, and

$$\zeta\left(s,\frac{1}{2}\right)=(2^s-1)\zeta(s).$$

For other values of the parameter α, the function $\zeta(s,\alpha)$ has no Euler product. It is well known that the Hurwitz zeta-function with transcendental or rational $\neq 1, \frac{1}{2}$ parameter α is also universal in the above sense, however, its shifts $\zeta(s+i\tau,\alpha)$ approximate not necessarily non-vanishing analytic functions. The universality of $\zeta(s,\alpha)$ with algebraic irrational α is an open problem.

Some other zeta-functions are also universal in the Voronin sense. The universality for zeta-functions of certain cusp forms was obtained in [12], for periodic zeta-functions was studied in [20] and [15], while the works [2], [4] and [5] are devoted to periodic Hurwitz zeta-functions. Universality theorems for Lerch zeta-functions can be found in [11]. A very good survey on universality of zeta-functions is given in [16].

In [19], H. Mishou began to study the so-called mixed joint universality. In this case, a collection of analytic functions are simultaneously approximated by shifts of a collection of zeta-functions consisting from functions having the Euler product and having no such a product. H. Mishou considered the pair $(\zeta(s),\zeta(s,\alpha))$ with transcendental α. For the statement of the Mishou theorem, we need some notation. Let $D=\left\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\right\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements. Moreover, let $H(K)$, $K\in\mathcal{K}$, be the class of continuous functions on K which are analytic in the interior of K, and let $H_0(K)$, $K\in\mathcal{K}$, be the subclass of $H(K)$ consisting from non-vanishing functions on K. Denote by meas$A$ the Lebesgue measure of a measurable set $A\subset\mathbb{R}$. Then H. Mishou proved [19] the following theorem.

THEOREM 1. *Suppose that α is transcendental number. Let $K_1,K_2\in\mathcal{K}$, and $f_1(s)\in H_0(K_1)$, $f_2(s)\in H(K_2)$. Then, for every $\varepsilon>0$*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\Big\{\tau\in[0;T]:\sup_{s\in K_1}|\zeta(s+i\tau)-f_1(s)|<\varepsilon,\\ \sup_{s\in K_2}|\zeta(s+i\tau,\alpha)-f_2(s)|<\varepsilon\Big\}>0.$$

Mixed joint universality theorems are also proved in [3], [7] and [10].

Our aim is to replace "lim inf"in Theorem 1 by "lim". In the case of the function $\zeta(s)$, this was done in [13] and [18], and, in the case of $\zeta(s,\alpha)$, a similar theorem was obtained in [14]. Let $\mathbb{P}$ be the set of all prime numbers, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and

$$L(\alpha,\mathbb{P})=\{(\log(m+\alpha):m\in\mathbb{N}_0),(\log p:p\in\mathbb{P})\}.$$

THEOREM 2. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K_1,K_2\in\mathcal{K}$, and $f_1(s)\in H_0(K_1)$, $f_2(s)\in H(K_2)$. Then the limit*

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\Big\{\tau\in[0;T]:\sup_{s\in K_1}|\zeta(s+i\tau)-f_1(s)|<\varepsilon,\sup_{s\in K_2}|\zeta(s+i\tau,\alpha)-f_2(s)|<\varepsilon\Big\}>0$$

exists for all but at most countably many $\varepsilon>0$.

For example, if α is transcendental, then the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$.

Let $H(G)$ be the space of analytic functions on G equipped with the topology of uniform convergence on compacta. In [9], universality theorems were proved for the functions $F(\zeta(s),\zeta(s,\alpha))$ with some operators $F:H^2(D)\to H(D)$. Let

$$S=\{g\in H(D):g(s)\neq 0 \text{ or } g(s)\equiv 0\}.$$

Then, for example in [9], the following assertion was obtained.

THEOREM 3. *Suppose that α is transcendental, and that $F : H^2(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap (S \times H(D))$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(D)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0;T] : \sup_{s\in K} |F(\zeta(s+i\tau), \zeta(s+i\tau,\alpha)) - f(s)| < \varepsilon\Big\} > 0.$$

More general results are obtained in [10].

Clearly, the transcendence of α in Theorem 3 can be replaced by a linear independence over $\mathbb{Q}$ of the set $L(\alpha, \mathbb{P})$. Therefore, we will prove the following theorem.

THEOREM 4. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, and that F, K and $f(s)$ are the same as in Theorem 3. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0;T] : \sup_{s\in K} |F(\zeta(s+i\tau), \zeta(s+i\tau,\alpha)) - f(s)| < \varepsilon\Big\} > 0 \qquad (1)$$

exists for all but at most countably many $\varepsilon > 0$.

Now, let V be an arbitrary positive number, $D_V = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1,\ |t| < \mathrm{V}\right\}$ and

$$S_V = \{g \in H(D_V) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

For brevity, we use the notation $H^2(D_V, D) = H(D_V) \times H(D)$.

THEOREM 5. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, and that K and $f(s)$ are the same as in Theorem 3, and $V > 0$ is such that $K \subset D_V$. Let $F : H^2(D_V, D) \to H(D_V)$ be a continuous operator such that , for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S_V \times H(D_V))$ is non-empty. Then the limit (1) exists for all but at most countably many $\varepsilon > 0$.*

For example, Theorem 5 implies the modified universality of the functions

$$c_1\zeta(s) + c_2\zeta(s,\alpha) \text{ and } c_1\zeta'(s) + c_2\zeta'(s,\alpha) \quad \text{with } c_1, c_2 \in \mathbb{C} \setminus \{0\}.$$

Let $a_1, ..., a_r$ be arbitrary distinct complex numbers, and

$$H_{a_1,...,a_r}(D) = \left\{g \in H(D) : (g(s) - a_j)^{-1} \in H(D),\ j = 1, ..., r\right\}.$$

THEOREM 6. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $F : H^2(D) \to H(D)$ is a continuous operator such that $F(S \times H(D)) \supset H_{a_1,...,a_r}(D)$. When $r = 1$, let $K \in \mathcal{K}$, and $f(s) \in H(K)$ and $f(s) \neq a_1$ on K. Then the limit (1) exists for all but at most countably many $\varepsilon > 0$. If $r \geqslant 2$, $K \subset D$ is an arbitrary compact subset, and $f(s) \in H_{a_1,...,a_r}(D)$, then the limit (1) exists for all but at most countably many $\varepsilon > 0$.*

The case $r = 1$ with $a_1 = 0$ shows that, for $F(g_1(s), g_2(s)) = e^{g_1(s)+g_2(s)}$, the limit (1) exists for all but at most countably many $\varepsilon > 0$. If $r = 2$ and $a_1 = 1$, $a_2 = -1$, then, for example, for $F(g_1(s), g_2(s)) = \cos(g_1(s) + g_2(s))$ and $f(s) \in H_{1,-1}(D)$, the limit (1) exists for all but at most countably many $\varepsilon > 0$.

THEOREM 7. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, $F : H^2(D) \to H(D)$ is a continuous operator, $K \subset D$ is a compact subset, and $f(s) \in F(S \times H(D))$. Then the limit (1) exists for all but at most countably many $\varepsilon > 0$.*

2. Lemmas

In this section, we present probabilistic theorems on the weak convergence of probability measures in the space of analytic functions.

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and

$$\Omega_1 = \prod_p \gamma_p \quad \text{and} \quad \Omega_2 = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, the tori Ω_1 and Ω_2 with the product topology and operation of pointwise multiplication are compact topological Abelian groups. Similarly, $\Omega = \Omega_1 \times \Omega_2$ is also a compact topological Abelian group. Therefore, denoting by $\mathcal{B}(X)$ the Borel σ-field of the space X, we have that, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega_1(p)$ and $\omega_2(m)$ the projections of $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ to the coordinate spaces γ_p, $p \in \mathbb{P}$, and γ_m, $m \in \mathbb{N}_0$, respectively, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^2(D)$-valued random element $\underline{\zeta}(s, \omega)$, $\omega = (\omega_1, \omega_2) \in \Omega$, by the formula

$$\underline{\zeta}(s, \alpha, \omega) = \left(\zeta(s, \omega_1), \zeta(s, \alpha, \omega_2)\right),$$

where

$$\zeta(s, \omega_1) = \prod_p \left(1 - \frac{\omega_1(p)}{p^s}\right)^{-1}$$

and

$$\zeta(s, \alpha, \omega_2) = \sum_{m=0}^{\infty} \frac{\omega_2(m)}{(m+\alpha)^s}.$$

Moreover, let

$$P_{\underline{\zeta}}(A) = m_H\left(\omega \in \Omega : \underline{\zeta}(s, \alpha, \omega) \in A\right), \quad A \in \mathcal{B}(H^2(D)),$$

i.e., $P_{\underline{\zeta}}$ is the distribution of the random element $\underline{\zeta}(s, \omega)$. We set $\underline{\zeta}(s, \alpha) = \big(\zeta(s),\, \zeta(s, \alpha)\big)$, and

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \underline{\zeta}(s + i\tau, \alpha) \in A\right\}, \quad A \in \mathcal{B}(H^2(D)).$$

LEMMA 1. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$. Then P converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$.*

PROOF. The lemma for transcendental α is proved in [19], Theorem 1, however, the transcendence of α is used only for the linear independence of the set $L(\alpha, \mathbb{P})$.

□

Let X_1 and X_2 be two metric spaces, and let the function $u : X_1 \to X_2$ be $(\mathcal{B}(X_1), \mathcal{B}(X_2))$-measurable. Then every probability measure P on $(X_1, \mathcal{B}(X_1))$ induces on $(X_2, \mathcal{B}(X_2))$ the unique probability measure $Pu^{-1}(A)$ given by the formula

$$Pu^{-1} = P(u^{-1}A), \quad A \in \mathcal{B}(X_2).$$

It is well known that if u is a continuous function, then it is $(\mathcal{B}(X_1), \mathcal{B}(X_2))$-measurable.

In the sequel, the following property of weakly convergent probability measures will be very useful.

LEMMA 2. *Suppose that P_n, $n \in \mathbb{N}$, and P are probability measures on $(X_1,\, \mathcal{B}(X_1))$, the function $u : X_1 \to X_2$ is continuous, and P_n converges weakly to P as $n \to \infty$. Then $P_n u^{-1}$ also converges weakly to Pu^{-1} as $n \to \infty$.*

The lemma is Theorem 5.1 from [1].

LEMMA 3. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $F : H^2(D) \to H(D)$ is a continuous operator. Then*

$$P_{T,F}(A) \overset{\text{def}}{=} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : F\left(\underline{\zeta}(s+i\tau,\alpha)\right) \in A\right\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{\underline{\zeta}}F^{-1}$ as $T \to \infty$.

PROOF. The definitions of P_T and $P_{T,F}$ imply that $P_{T,F} = P_T F^{-1}$. Therefore, the continuity of F and Lemmas 1 and 2 prove the lemma.

□

Let $V > 0$, and, for $A \in \mathcal{B}\left(H^2(D_V, D)\right)$,

$$P_{T,V}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\zeta}(s+i\tau,\alpha) \in A\right\},$$

$$P_{\underline{\zeta},V}(A) = m_H\left(\omega \in \Omega : \underline{\zeta}(s,\alpha,\omega) \in A\right).$$

LEMMA 4. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $F : H^2(D_V, D) \to H(D_V)$ is a continuous operator. Then*

$$P_{T,F,V}(A) \overset{\text{def}}{=} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : F\left(\underline{\zeta}(s+i\tau,\alpha)\right) \in A\right\}, \quad A \in \mathcal{B}(H(D_V)),$$

converges weakly to $P_{\underline{\zeta},V}F^{-1}$ as $T \to \infty$.

PROOF. Clearly, the function $u_V : H^2(D) \to H^2(D_V, D)$ given by the formula

$$u_V(g_1(s), g_2(s)) = \left(g_1(s)\Big|_{s\in D_V}, g_2(s)\right), \; g_1, g_2 \in H(D),$$

is continuous, and, $P_{T,V} = P_T u_V^{-1}$. Therefore, Lemmas 1 and 2 imply that $P_{T,V}$ converges weakly to $P_{\underline{\zeta},V}$ as $T \to \infty$. Since $P_{T,F,V} = P_{T,V}F^{-1}$, we have that $P_{T,F,V}$ converges weakly to $P_{\underline{\zeta},V}F^{-1}$ as $T \to \infty$.

□

Now we consider the supports of the limit measures $P_{\underline{\zeta}}$, $P_{\underline{\zeta}}F^{-1}$, $P_{\underline{\zeta},V}$ and $P_{\underline{\zeta},V}F^{-1}$.

LEMMA 5. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$. Then the support of the measure $P_{\underline{\zeta}}$ is the set $S \times H(D)$.*

PROOF. Denote by m_{1H} and m_{2H} the probability Haar measures on $(\Omega_1, \mathcal{B}(\Omega_1))$ and $(\Omega_2, \mathcal{B}(\Omega_2))$, respectively. Then we have that m_H is the product of m_{1H} and m_{2H}, i.e., if $A = A_1 \times A_2$, where $A_1 \in \mathcal{B}(\Omega_1)$ and $A_2 \in \mathcal{B}(\Omega_2)$, then

$$m_H(A) = m_{1H}(A_1)m_{2H}(A_2). \tag{2}$$

The space $H^2(D)$ is separable, therefore, $\mathcal{B}(H^2(D)) = \mathcal{B}(H(D)) \times \mathcal{B}(H(D))$. Thus, it suffices to consider the measure $P_{\underline{\zeta}}$ on the sets $A = A_1 \times A_2$, $A_1, A_2 \in H(D)$.

It is known [20] that the support of the measure

$$m_{1H}\left(\omega_1 \in \Omega_1 : \zeta(s,\omega_1) \in A\right), \quad A \in \mathcal{B}(H(D)) \tag{3}$$

is the set S. The linear independence of $L(\alpha,\mathbb{P})$ implies that of the set $L(\alpha)=\{\log(m+\alpha): m\in\mathbb{N}_0\}$. Therefore, the case $r=1$ of Theorem 11 from [6] gives that the support of the measure

$$m_{2H}\left(\omega_2\in\Omega_2:\zeta(s,\alpha,\omega_2)\in A\right),\quad A\in\mathcal{B}(H(D)), \tag{4}$$

is the set $H(D)$. Since

$$P_{\underline{\zeta}}(A)=m_H\left(\omega\in\Omega:\underline{\zeta}(s,\alpha,\omega)\in A\right),\ \ A\in\mathcal{B}(H^2(D)),$$

in view of (2), we have that, for $A=A_1\times A_2$,

$$P_{\underline{\zeta}}(A)=m_{1H}\left(\omega_1\in\Omega_1:\zeta(s,\omega_1)\in A_1\right)m_{2H}\left(\omega_2\in\Omega_2:\zeta(s,\alpha,\omega_2)\in A_2\right).$$

Therefore, the lemma follows from remarks on supports of the measures (3) and (4), and minimality property of a support.

□

LEMMA 6. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $F: H^2(D)\to H(D)$ is a continuous operator such that, for every open set $G\subset H(D)$, the set $(F^{-1}G)\cap(S\times H(D))$ is non-empty. Then the support of the measure $P_{\underline{\zeta}}F^{-1}$ is the whole of $H(D)$.*

PROOF. We apply standard arguments. Let $g\in H(D)$ be an arbitrary element, and G be its any open neighborhood. Since the operator F is continuous, the set $F^{-1}G$ is open, too. Therefore, by the hypothesis of the lemma, $F^{-1}G$ is an open neighborhood of a certain element of the set $S\times H(D)$. Hence, by Lemma 5, $P_{\underline{\zeta}}(F^{-1}G)>0$. Therefore,

$$P_{\underline{\zeta}}F^{-1}(G)=P_{\underline{\zeta}}(F^{-1}G)>0.$$

Since g and G are arbitrary, this proves the lemma.

□

In what follows, the Mergelyan theorem on the approximation of analytic functions by polynomials will be exceptionally useful [17].

LEMMA 7. *Suppose that $K\subset\mathbb{C}$ is a compact subset with connected complement, and $f(s)$ is a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon>0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K}|f(s)-p(s)|<\varepsilon.$$

LEMMA 8. *Suppose that the set $L(\alpha,\mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $V>0$. Then the support of $P_{\underline{\zeta},V}$ is the set $S_V\times H(D)$.*

PROOF. Let g be an arbitrary element of $S_V\times H(D)$, and G be its open neighborhood. The function u_V defined in the proof of Lemma 4 is continuous. Therefore, by the definition of u_V, the set $u_V^{-1}G$ is open and non-empty. Really, it is well known, see, for example, [8], that the approximation in the space $H(D)$ coincides with the uniform approximation on compact sets with connected complements. Therefore, by Lemma 7, there exists a polynomial $p(s)$ such that $p(s)\in G$. Since the polynomial $p(s)$ is an entire function, $p(s)$ also belongs to $u_V^{-1}G$. Thus, the set $u_V^{-1}G$ is non-empty, and is an open neighborhood of an element from $S\times H(D)$. Therefore, by Lemma 5, $P_{\underline{\zeta}}(u_V^{-1}G)>0$. Hence, $P_{\underline{\zeta},V}(G)=P_{\underline{\zeta}}u_V^{-1}(G)=P_{\underline{\zeta}}(u_V^{-1}G)>0$. Clearly, if $(g_1,g_2)\in S\times H(D)$, then also $(g_1,g_2)\in S_V\times H(D)$. Therefore,

$$m_H\left(\omega\in\Omega:\underline{\zeta}(s,\alpha,\omega)\in S_V\times H(D)\right)\geqslant m_H\left(\omega\in\Omega:\underline{\zeta}(s,\alpha,\omega)\in S\times H(D)\right)=1.$$

Hence,

$$P_{\underline{\zeta},V}\left(S_V\times H(D)\right)=1.$$

□

LEMMA 9. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$. Let $F : H^2(D_V, D) \to H(D_V)$ be a continuous operator such that, for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S_V \times H(D))$ is non-empty. Then the support of the measure $P_{\underline{\zeta},V} F^{-1}$ is the whole of $H(D_V)$.*

PROOF. Let g be an arbitrary element of $H(D_V)$, and G be its arbitrary open neighbourhood. Then, by Lemma 7, there exists a polynomial $p(s) \in G$. Therefore, the hypotheses of the lemma imply that the set $F^{-1}G$ is open and contains an element of the set $S_V \times H(D)$. Thus, in virtue of Lemma 8, $P_{\underline{\zeta},V}(F^{-1}G) > 0$. From this, it follows that

$$P_{\underline{\zeta},V} F^{-1}(G) = P_{\underline{\zeta},V}(F^{-1}G) > 0,$$

and the lemma is proved because g and G are arbitrary.

□

LEMMA 10. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, and the operator $F : H^2(D) \to H(D)$ satisfies the hypotheses of Theorem 6. Then the support of the measure $P_{\underline{\zeta}} F^{-1}$ contains the closure of the set $H_{a_1,\dots,a_r}(D)$.*

PROOF. Since $F(S \times H(D)) \supset H_{a_1,\dots,a_r}(D)$, for each element $g \in H_{a_1,\dots,a_r}(D)$, there exists an element $(g_1, g_2) \in S \times H(D))$ such that $F(g_1, g_2) = g$. If G is an arbitrary open neighborhood of g, then we have that the open set $F^{-1}G$ is an open neighborhood of a certain element of $S \times H(D)$. Therefore, in view of Lemma 5, $P_{\underline{\zeta}}(F^{-1}G) > 0$. Hence,

$$P_{\underline{\zeta}} F^{-1}(G) = P_{\underline{\zeta}}(F^{-1}G) > 0.$$

This shows that the element g lies in the support of the measure $P_{\underline{\zeta}} F^{-1}$. Since g is an arbitrary element of $H_{a_1,\dots,a_r}(D)$, we have that the support of $P_{\underline{\zeta}} F^{-1}$ contains the set $H_{a_1,\dots,a_r}(D)$. However, the support is a closed set, therefore, it contains the closure of $H_{a_1,\dots,a_r}(D)$.

□

LEMMA 11. *Suppose that the set $L(\alpha, \mathbb{P})$ is linearly independent over $\mathbb{Q}$, and $F : H^2(D) \to H(D)$ is a continuous operator. Then the support of $P_{\underline{\zeta}} F^{-1}$ is the closure of $F(S \times H(D))$.*

PROOF. Let g be an arbitrary element of $F(S \times H(D))$, and G is its any neighborhood. Then, by Lemma 5, $P_{\underline{\zeta}}(F^{-1}G) > 0$. Hence, $P_{\underline{\zeta}} F^{-1}(G) > 0$. Moreover, by Lemma 5 again,

$$P_{\underline{\zeta}} F^{-1} \left(F(S \times H(D)) \right) = P_{\underline{\zeta}} \left(S \times H(D) \right) = 1.$$

Therefore, the support of $P_{\underline{\zeta}} F^{-1}$ is the closure of $F(S \times H(D))$.

□

3. Proof of universality theorems

We will apply the equivalent of the weak convergence of probability measures in terms of continuity sets. We remind that $A \in \mathcal{B}(X)$ is a continuity set of the probability measure P on $(X, \mathcal{B}(X))$ if $P(\partial A) = 0$, where ∂A is the boundary of A.

LEMMA 12. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(X, \mathcal{B}(X))$. Then P_n, as $n \to \infty$, converges weakly to P if and only if, for every continuity set A of P,*

$$\lim_{n \to \infty} P_n(A) = P(A).$$

A proof of the lemma can be found in[1], Theorem 2.1.

Proof of Theorem 2. Put

$$G_\varepsilon = \Big\{(g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |g_2(s) - f_2(s)| < \varepsilon\Big\}.$$

Then G_ε is an open set in $H^2(D)$. Moreover,

$$\begin{aligned}\partial G_\varepsilon = &\Big\{(g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |g_2(s) - f_2(s)| = \varepsilon\Big\}\\ &\cup \Big\{(g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| = \varepsilon, \sup_{s\in K_2} |g_2(s) - f_2(s)| < \varepsilon\Big\}\\ &\cup \Big\{(g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - f_1(s)| = \varepsilon, \sup_{s\in K_2} |g_2(s) - f_2(s)| = \varepsilon\Big\}.\end{aligned}$$

Therefore, if $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_1 \neq \varepsilon_2$, then $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \varnothing$. Hence, we have that $P_{\underline{\zeta}}(\partial G_\varepsilon) > 0$ for at most a countable set of values of $\varepsilon > 0$. This means that the set G_ε is a continuity set of $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon > 0$. Therefore, by Lemmas 1 and 12,

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0;T] : \underline{\zeta}(s + i\tau) \in G_\varepsilon\Big\} = P_{\underline{\zeta}}(G_\varepsilon),$$

or, by the definition of G_ε,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0;T] : &\sup_{s\in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon,\\ &\sup_{s\in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon\Big\} = P_{\underline{\zeta}}(G_\varepsilon)\end{aligned} \tag{5}$$

for all but at most countably many $\varepsilon > 0$. By Lemma 7, there exist polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s\in K_1} \big|f_1(s) - e^{p_1(s)}\big| < \frac{\varepsilon}{2} \tag{6}$$

and

$$\sup_{s\in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon}{2}. \tag{7}$$

In view of Lemma 5, $\{e^{p_1(s)}, p_2(s)\}$ is and element of the support of the measure $P_{\underline{\zeta}}$. Therefore, putting

$$\hat{G}_\varepsilon = \Big\{(g_1, g_2) \in H^2(D) : \sup_{s\in K_1} |g_1(s) - e^{p_1(s)}| < \frac{\varepsilon}{2}, \sup_{s\in K_2} |g_2(s) - p_2(s)| < \frac{\varepsilon}{2}\Big\},$$

we obtain that $P_{\underline{\zeta}}(\hat{G}_\varepsilon) > 0$. Inequalities (6) and (7) show, that for $(g_1, g_2) \in \hat{G}_\varepsilon$,

$$\sup_{s\in K_1} |g_1(s) - f_1(s)| < \varepsilon$$

and

$$\sup_{s\in K_2} |g_2(s) - f_2(s)| < \varepsilon.$$

Thus, we have that $\hat{G}_\varepsilon \subset G_\varepsilon$. Hence, $P_{\underline{\zeta}}(G_\varepsilon) \geqslant P_{\underline{\zeta}}(\hat{G}_\varepsilon) > 0$. This together with (5) proves the theorem.

□

Proof of Theorem 4. Define the set

$$G_{1,\varepsilon}=\left\{g\in H(D):\sup_{s\in K}|g(s)-f(s)|<\varepsilon\right\}.$$

Then we have that $G_{1,\varepsilon}$ is a continuity set of the measure $P_{\underline{\zeta}}F^{-1}$ for all but at most countably many $\varepsilon>0$.Hence, in view of Lemmas 3 and 12,

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0;T]:F\left(\underline{\zeta}(s+i\tau,\alpha)\right)\in G_{1,\varepsilon}\right\}\\ =&\lim_{T\to\infty}\frac{1}{T}\text{meas}\Big\{\tau\in[0;T]:\sup_{s\in K}|F(\zeta(s+i\tau),\zeta(s+i\tau,\alpha))-f(s)|<\varepsilon\Big\}\\ =&P_{\underline{\zeta}}F^{-1}(G_{1,\varepsilon})\end{aligned}\tag{8}$$

for all but at most countably many $\varepsilon>0$. By Lemma 7, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}|f(s)-p(s)|<\frac{\varepsilon}{2}.\tag{9}$$

Define

$$\hat{G}_{1,\varepsilon}=\left\{g\in H(D):\sup_{s\in K}|g(s)-f(s)|<\frac{\varepsilon}{2}\right\}.$$

The polynomial $p(s)$, by Lemma 6, is an element of the support of the measure $P_{\underline{\zeta}}F^{-1}$. Hence, $P_{\underline{\zeta}}(\hat{G}_{1,\varepsilon})>0$. Obviously, for $g\in\hat{G}_{1,\varepsilon}$, by (9),

$$\sup_{s\in K}|g(s)-f(s)|<\varepsilon.$$

Therefore, $\hat{G}_{1,\varepsilon}\subset G_{1,\varepsilon}$, $P_{\underline{\zeta}}F^{-1}(G_{1,\varepsilon})\geqslant P_{\underline{\zeta}}F^{-1}(\hat{G}_{1,\varepsilon})>0$, and the theorem follows from (8).

□

Proof of Theorem 5. We follow the proof of Theorem 4, and use Lemma 4 in place of Lemma 3, and Lemma 9 in place of Lemma 6.

□

Proof of Theorem 6. The case $r=1$. By Lemma 7, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}|f(s)-p(s)|<\frac{\varepsilon}{4}.\tag{10}$$

By hypotheses of the theorem, $f(s)\neq a_1$ on K. Therefore, in view of (10), $p(s)\neq a_1$ on K as well if ε is small enough. Thus, we can define a continuous branch of $\log(p(s)-a_1)$ which will be an analytic function in the interior of K. Using Lemma 7 once more, we find a polynomial $p_1(s)$ such that

$$\sup_{s\in K}|p(s)-a_1-e^{p_1(s)})|<\frac{\varepsilon}{4}.\tag{11}$$

Now we put $f_1(s)=e^{p_1(s)}+a_1$. Then $f_1(s)\in H(D)$ and $f_1(s)\neq a_1$. Therefore, by Lemma 10, $f_1(s)$ is an element of the support of the measure $P_{\underline{\zeta}}F^{-1}$. Define

$$\mathcal{G}_{1,\varepsilon}=\left\{g\in H(D):\sup_{s\in K}|g(s)-f_1(s)|<\frac{\varepsilon}{2}\right\}.$$

Then $\mathcal{G}_{1,\varepsilon}$ is an open neighborhood of $f_1(s)$, thus, $P_{\underline{\zeta}}F^{-1}(\mathcal{G}_{1,\varepsilon}) > 0$. Now consider the set

$$\hat{\mathcal{G}}_{1,\varepsilon} = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Similarly as in the proof of the above theorems, we observe that $\mathcal{G}_{1,\varepsilon}$ is an continuity set of the measure $P_{\underline{\zeta}}F^{-1}$ for all but at most countably many $\varepsilon > 0$. Therefore, taking into account Lemmas 3 and 12, we have that

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{ \tau \in [0;T] : F\left(\underline{\zeta}(s+i\tau,\alpha)\right) \in \hat{\mathcal{G}}_{1,\varepsilon} \right\} \\ &= \lim_{T\to\infty} \frac{1}{T}\text{meas}\Big\{ \tau \in [0;T] : \sup_{s\in K} |F(\underline{\zeta}(s+i\tau,\alpha)) - f(s)| < \varepsilon \Big\} = P_{\underline{\zeta}}F^{-1}(\hat{\mathcal{G}}_{1,\varepsilon}). \end{aligned} \tag{12}$$

Clearly, by (10) and (11),

$$\sup_{s\in K} |f(s) - f_1(s)| < \frac{\varepsilon}{2}.$$

Therefore, if $g \in \mathcal{G}_{1,\varepsilon}$, then $g \in \hat{\mathcal{G}}_{1,\varepsilon}$, i.e., $\mathcal{G}_{1,\varepsilon} \subset \hat{\mathcal{G}}_{1,\varepsilon}$. Since $P_{\underline{\zeta}}F^{-1}(\mathcal{G}_{1,\varepsilon}) > 0$, we have that $P_{\underline{\zeta}}F^{-1}(\hat{\mathcal{G}}_{1,\varepsilon}) > 0$. This inequality together with (12) proves the theorem in the case $r = 1$.

Now let $r \geqslant 2$. Define

$$\mathcal{G}_{2,\varepsilon} = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Since $f(s) \in H_{a_1,\dots,a_r}(D)$, we have by Lemma 10, that $f(s)$ is an element of the support of $P_{\underline{\zeta}}F^{-1}$. Moreover, $\mathcal{G}_{2,\varepsilon}$ is an open neighborhood of $f(s)$. Therefore,

$$P_{\underline{\zeta}}F^{-1}(\mathcal{G}_{2,\varepsilon}) > 0. \tag{13}$$

On the other hand, $\mathcal{G}_{2,\varepsilon}$ is a continuity set of the measure $P_{\underline{\zeta}}F^{-1}$ for all but at most countably many $\varepsilon > 0$. Therefore, in view of Lemmas 3 and 12, and (12)

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T}\text{meas}\Big\{ \tau \in [0;T] : \sup_{s\in K} |F(\underline{\zeta}(s+i\tau,\alpha)) - f(s)| < \varepsilon \Big\} \\ &= \lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{ \tau \in [0;T] : F\left(\underline{\zeta}(s+i\tau,\alpha)\right) \in \mathcal{G}_{2,\varepsilon} \right\} = P_{\underline{\zeta}}F^{-1}(\mathcal{G}_{2,\varepsilon}) > 0. \end{aligned}$$

□

Proof of Theorem 7. We repeat the proof of the case $r \geqslant 2$ of Theorem 6, and, in place of Lemma 10, we apply Lemma 11.

□

4. Conclusions

It was well known that the Riemann zeta-function $\zeta(s)$ and Hurwitz zeta-function $\zeta(s,\alpha)$ with transcendental or rational parameter α are universal in the Voronin sense, i.e., their shifts $\zeta(s+i\tau)$ and $\zeta(s+i\tau,\alpha)$, $\tau \in \mathbb{R}$, approximate functions from wide classes. H. Mishou obtained a joint universality theorem for $\zeta(s)$ and $\zeta(s,\alpha)$. He proved that the set of shifts $(\zeta(s+i\tau), \zeta(s+i\tau,\alpha))$ with transcendental α approximating a pair of given analytic functions has a positive lower density.

In the paper, it is observed that the set of the above shifts has a positive density for all but at most countably many values of $\varepsilon > 0$, where ε is accuracy of approximation.

Also, it is obtained that composite functions $F(\zeta(s), \zeta(s,\alpha))$ for some classes of operators F in the space of analytic functions $H(D)$ has a similar approximation property, namely, the set of shifts $F(\zeta(s+i\tau), \zeta(s+i\tau,\alpha))$ approximating a given analytic function with accuracy $\varepsilon > 0$ has a positive density for all but at most countably many values of ε.

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