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# Некоторые независимые результаты в идеальных пространствах Ротбергера

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**Аннотация**

В этой статье мы покажем, что в обычном $p$-пространстве для каждой пары непересекающихся идеального множества Ротбергера и замкнутого множества существует пара непересекающихся открытых множеств, таких, что одно содержит замкнутое множество, а дополнение другого по отношению к идеальному множеству Ротбергера находится в соответствующем подидеале. Более того, мы демонстрируем, как семейства замкнутых множеств могут быть использованы для описания идеальных пространств Ротбергера.

*Ключевые слова:* идеал по модулю Ротбергера, свойство конечного пересечения, $p$-пространство.

*Библиография:* 12 названий.

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# Some independent results on Ideal-Rothberger spaces

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**Abstract**

In this article we show that in a regular $p$-space, for every pair of disjoint ideal Rothberger set and closed set there is a pair of disjoint open sets such that one contains the closed set and other one's complement with respect to the ideal rothberger set is in the corresponding sub ideal. Moreover, we demonstrate how families of closed sets can be used to describe the ideal Rothberger spaces.

*Keywords:* Rothberger modulo an Ideal, Finite intersection property, $p$-space.

*Bibliography:* 12 titles.

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## 1. Introduction

An Ideal Rothberger Space is a concept within the realm of topology, representing a specialized class of topological spaces with distinctive covering properties. In topology, the study of sequential covering plays a crucial role in understanding the structure and behavior of topological spaces. Some covering properties of recent interest can be found in [2, 3, 4, 5, 6]. Ideal Rothberger Spaces offer a nuanced perspective on sequential covering through the integration of ideals where an ideal is a collocation of subsets of a space which is closed under union and taking subsets. This means that if a set $A$ belongs to an ideal $I$, then any superset or union of sets in $I$ also belongs to $I$.

Although Güldürdek [10] first proposed the ideal Rothberger space in 2018, Bhardwaj [7] and Güldürdek [11] have thoroughly investigated its unique features. In a topological space, every pair of compact subset and closed subset can be strongly separated by open sets. In this paper, we investigate whether analogous separation of ideal Rothberger subset and closed set is possible or not. More over a compact space is characterized by means of family of closed sets using finite intersection properties. Similarly for Lindelöf space we have countable intersection property, for Star-compact space we have modified non-finite intersection property [1], for Star-Lindelöf space we have modified non-countable intersection property [1]. Neither Bhardwaj [7] nor Güldürdek [11] has emphasized the use of a family of closed sets to depict the ideal Rothberger space. So we introduce a sequential version of finite intersection property to represent ideal Rothberger space by means of family of closed sets.

## 2. Preliminaries

In this work, we refer to a topological space $X$ equipped with the topology $\tau$ as simply 'a space'. Unless explicitly stated otherwise, separation axioms are not included. Our use of standard concepts, symbols, and nomenclature follows [8]. In this section, we state several fundamental concepts to aid the readers' understanding and convenience.

In a space $X$, a subset $\mathcal{U} \subset \tau$ is called an open cover of $X$ if $\bigcup \mathcal{U} = X$. If every open cover of a space has a finite sub cover then the space is called a compact space. A topological space is called a regular space in which for every $a \in X$ and closed set $B$ such that $a \notin B$ there exists $U, V \in \tau$ such that $x \in U$, $B \subseteq V$ and $U \cap V = \emptyset$. A p-space is a topological space in which countable intersection of open set is also open [9].

Теорема 1. *[8] If $A$ is a compact subset and $B$ is closed in a regular space $X$ such that $A \cap B = \emptyset$, then $A$ and $B$ are strongly separated.*

Definition 1. *[10] An ideal I in a topological space $(X, \tau)$ is a non empty family of subsets of $X$ which satisfies the following properties :*

*$(i)$ $X \notin I$,*

*$(ii)$ $A, B \in I \Rightarrow A \cup B \in I$,*

*$(iii)$ $A \in I$ and $B \subseteq A \Rightarrow B \in I$.*

Proposition 1. *[12] If $I$ is an ideal of subsets of $X$ and $Y \subseteq X$, then $I_Y = \{Y \cap I_1 : I_1 \in I\}$ is an ideal of subsets of $Y$.*

We call $I_Y$ a sub-ideal of $X$ with respect to $Y$

Definition 2. *[10] A subset $A$ of an ideal space $(X, \tau, I)$ is said to be Rothberger modulo I or I-Rothberger subset, if for every sequence of $\tau$-open covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of $A$ there exists a sequence of open sets $\{U_n : n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ and $A \setminus \bigcup_{i=1}^{k} U_{\alpha_i} \in I_A$. If $X$ itself is a I-Rothberger subset, then $(X, \tau, I)$ is called an I-Rothberger space.*

Definition 3. *[8] A family $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ (where $\Lambda$ is an index set) is said to have the finite intersection property if for every finite subset $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$ of $\Lambda$, $\bigcap_{i=1}^{n} F_{\alpha_i} \neq \emptyset$.*

Proposition 2. *[8] A topological space is a compact space if and only if every family of closed sets having finite intersection property have non empty intersection.*

Two general question raises that whether a I-Rothberger subset and and a closed subset in a regular space can be strongly separated, can I-Rothberger space be represented by families of closed sets. We will investigate for the answer of these questions in the next section.

## 3. Main Results

Теорема 2. *In a regular p-space $X$, for every pair $A, B$ of disjoint subsets of $X$ where $A$ is an ideal Rothberger subset and $B$ is a closed subset of $X$, there exists open sets $U$ and $V$ such that $A \setminus U \in I_A$, $B \subseteq V$ and $U \cap V = \emptyset$.*

Доказательство. Let $(X, \tau)$ be a regular p-space. Suppose, $A$ is an ideal Rothberger subset and $B$ is a closed subset of $X$ such that $A \cap B = \emptyset$. Since $X$ is a regular space, for every $a \in A$, we can find open sets $G_a$ and $H_a$ such that $a \in G_a$ and $B \subseteq H_a$ where $G_a \cap H_a \neq \emptyset$. Thus $\mathcal{G} = \{G_a : a \in A\}$ is an open cover for $A$ by the elements of $\tau$.

If we take $\mathcal{G}_n = \mathcal{G}$ for each $n \in \mathbb{N}$, than $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a sequence of open covers for $A$. But $A$ is an ideal Rothberger subset of $X$. So, there exists a sequence $\{G_n : n \in \mathbb{N}\}$ where $G_n \in \mathcal{G}_n = \mathcal{G}$ for each $n \in \mathbb{N}$ and $A \setminus (\bigcup_{n=1}^{\infty} G_n) \in I_A$.

Now, $G_n = G_a \in \tau$ for some $a \in A$ and for all $n \in \mathbb{N}$. And for each $n \in \mathbb{N}$, there exists a $H_a$ such that $B \subseteq H_a \in \tau$ and $G_a \cap H_a = \emptyset$. Let $H_n = H_a$ for that specific $a$ and specific $n \in \mathbb{N}$. Thus $G_n \cap H_n = \emptyset$ for each $n \in N$. Suppose $U = \bigcup_{n=1}^{\infty} G_n$ and $V = \bigcap_{n=1}^{\infty} H_n$. Obviously, $U \in \tau$ and since $X$ is a p-space, $V \in \tau$. Here, $A \setminus U \in I_A$, $B \subset V$ and $U \cap V = \emptyset$. □

Следствие 1. *In a Hausdörff p-space $X$, for every ideal Rothberger subset $A$ and $b \in X$such that $b \notin A$, there exists open sets $U$ and $V$ such that $A \setminus U \in I_A$, $b \in V$ and $U \cap V = \emptyset$.*

Доказательство. In a Hausdörff space, every single ton set is a closed set. So the corollary follows directly from Theorem 2. □

Definition 4. **Sequential Singletonic Intersection Module Ideal Property (SSI$^I$P):** *In an ideal space $(X, \tau, I)$, a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of families of subsets is said to have SSI$^I$ property if for every sequence $\{F_n : n \in \mathbb{N}\}$ such that $F_n \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ we get $\bigcap_{n \in \mathbb{N}} F_n \notin I$.*

Теорема 3. *Following statements are equivalent :*

*(1) $X$ is an ideal Rothberger space.*

*(2) For every sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of families of closed sets having SSI$^I$ property, there exists a $n_0 \in \mathbb{N}$ such that $\cap \mathcal{F}_{n_0} \neq \emptyset$.*

Доказательство.
(1) $\Rightarrow$ (2)
Let $(X, \tau, I)$ be an ideal Rothberger space and $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a sequence of families of closed sets having SSI$^I$ property. We also assume that $\cap \mathcal{F}_n = \emptyset$ for all $n \in \mathbb{N}$.

Therefore, $X \setminus (\cap \mathcal{F}_n) = X$ for all $n \in \mathbb{N}$.
$\Longrightarrow$ $\cup \{X \setminus F : F \in \mathcal{F}_n\} = X$ for all $n \in \mathbb{N}$.
$\Longrightarrow$ $\mathcal{U}_n = \{U = X \setminus F : F \in \mathcal{F}_n\}$ is an open cover of $X$ for all $n \in \mathbb{N}$.
$\Longrightarrow$ $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of open covers of $X$. But $X$ is an ideal Rothberger space. So there exists a sequence $\{U_n : n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for all $n \in \mathbb{N}$ and $X \setminus (\cup_{n \in \mathbb{N}} U_n) \in I$.

But $U_n = X \setminus F_n$ where $F_n \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Thus $\{F_n : n \in \mathbb{N}\}$ is a sequence such that $F_n \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ and $\cap_{n\in\mathbb{N}} F_n = \cap_{n\in\mathbb{N}}(X \setminus U_n) = X \setminus (\cup_{n\in\mathbb{N}} U_n) \in I$, which contradicts the fact that $\{\mathcal{F}_n : n \in \mathbb{N}\}$ have the $\mathrm{SSI}^I$ property. So $\cap\mathcal{F}_n$ can not be $\emptyset$ for all $n \in \mathbb{N}$. i.e. there exists atleast one $n_0 \in \mathbb{N}$ such that $\cap\mathcal{F}_{n_0} \neq \emptyset$.

(2) $\Rightarrow$ (1)

Let condition (2) holds and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of the space $X$. We take $\mathcal{F}_n = \{F = X \setminus U : U \in \mathcal{U}_n\}$ for all $n \in \mathbb{N}$. So $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a sequence of families of closed sets such that $\cap\mathcal{F}_n = \emptyset$ for all $n \in \mathbb{N}$.

By contra positivity of (2), the sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ must not have $\mathrm{SSI}^I$ property. i.e. there exists a sequence $\{F_n : n \in \mathbb{N}\}$ where $F_n \in \mathcal{F}_n$ for all $n \in \mathbb{N}$, $\cap_{n\in\mathbb{N}} F_n \in I$. But $F_n = X \setminus U_n$ where $U_n \in \mathcal{U}_n$ and for all $n \in \mathbb{N}$.

So, $\cap_{n\in\mathbb{N}} F_n \in I \implies \cap_{n\in\mathbb{N}}(X \setminus U_n) \in I$.

$\implies X \setminus (\cup_{n\in\mathbb{N}} U_n) \in I$.

Hence $X$ is an ideal Rothberger space. $\square$

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