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Уравнение Гарри Дима со специальным самосогласованным источником

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Аннотация

Работа посвящена изучению интегрирования уравнения Гарри Дима с самосогласованным источником. Источник состоит из комбинации собственных функций и линейно-независимого решения с теми же собственными функциями соответствующей спектральной задачи для уравнения струны, не имеющего спектральных особенностей. При рассмотрении источника точки дискретного спектра уравнения струны есть функции от времени. Выведены временные характеристики данных рассеяния уравнения струны, которые позволяют интегрировать задачу Коши для уравнения Гарри Дима со специальным самосогласованным источником в классе быстроубывающих функций методом обратной задачи рассеяния.

Ключевые слова: уравнение струны, уравнение Гарри Дима, метод обратной задачи рассеяния, данные рассеяния, солитоны.

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Harry Dym equation with a special self-consistent source

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Abstract

The work is concerned with studying the integration of the Harry Dym equation with the self-consistent source. The source consists of the combination of the eigenfunctions and linear independent solution with the same eigenfunctions of the corresponding spectral problem for the string equation which has not spectral singularities. While considering the source, the points of the discrete spectrum of the string equation have been as the functions of time. Deduced the time performance of the scattering data of the string equation which allows to integrate the Cauchy problem for the Harry Dym equation with the special self-consistent source in the class of the rapidly decreasing functions via the inverse scattering method.

Keywords: string equation, Harry Dym equation, inverse scattering method, scattering data, solitons.

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1. Introduction

The Harry Dym equation is an integrable nonlinear evolution equation which has the applications in the hydrodynamics [1] and has strong relation with the solution of the Korteweg-de Vries equation [2], [3], [4]. This equation firstly was appeared in the works [5]-[7], which was represented as

$$u_t = -\frac{1}{2}u^3 u_{xxx}$$

for real-valued function $u(x,t)$ and related to the classical string problem [8].

In the work [9] shown that Harry Dym equation can be transformed to the modified Korteweg-de Vries equation. The significantly important results for the Harry Dym equation on construction finite-gap solutions was studied in [10]. In the paper [11] worked out the reciprocal transformations generated by the adjoint eigenfunctions, which are useful to construct the new type solutions of the Harry Dym hierarchy.

The present work is based on integrating of the Harry Dym equation with a special self-consistent source using the techniques in [12]-[14]. There are also another interesting works on the integration of the Harry Dym equation with the source in the various class of functions [15], [16] as well as Harry Dym hierarchy with self-consistent sources [17], [18].

We consider the following system of equations:

$$q_t(x,t) = 2\left(\frac{1}{\sqrt{1+q(x,t)}}\right)_{xxx} - 2\sum_{n=1}^{N}(1+q(x,t))\frac{\partial}{\partial x}(f_n(x,t))g_n(x,t)) - \\ - q_x(x,t)\sum_{n=1}^{N} f_n(x,t)g_n(x,t) \tag{1}$$

$$f_n''(x,t) - \chi_n^2 f_n(x,t)q(x,t) = \chi_n^2 f_n(x,t) \tag{2}$$

$$g_n''(x,t) - \chi_n^2 g_n(x,t)q(x,t) = \chi_n^2 g_n(x,t) \tag{3}$$

with initial data

$$q(x,0) = q_0(x) \tag{4}$$

which has following properties

- $$\int_{-\infty}^{\infty}(1+x^2)\left(|q_0(x)| + \left|1 - \frac{1}{1+q_0(x)}\right|\right)dx < \infty, \tag{5}$$

- The operator $L(0) := \frac{d^2}{dx^2} + \lambda^2 q_0(x)$ possesses exactly N simple eigenvalues $-\chi_1^2(0) >$ $> -\chi_2^2(0) > ... > -\chi_N^2(0)$ without spectral singularities.

Here, the prime means the derivative with respect to variable x, while dot means the derivate by variable t, $f_n(x,t)$ is eigenfunction corresponding to the eigenvalue $-\chi_n^2$, while $g_n(x,t)$ is linear independent solution with $f_n(x,t)$

$$W(f_n(x,t), g_n(x,t)) = f_n(x,t)g_n'(x,t) - {f_n}'(x,t)g_n(x,t) = \omega_n(t), \tag{6}$$

where $\omega_n(t)$ is ahead given continuous function satisfying the condition

$$\omega_1(t) < \omega_2(t) < ... < \omega_N(t) \tag{7}$$

for any $t \geq 0$ and $q(x,t)$ is assumed to be sufficiently smooth and sufficiently rapidly tends to zero as $|x| \to \infty$:

$$\int_{-\infty}^{\infty}(1+x^2)\left(|q(x,t)| + \left|1 - \frac{1}{1+q(x,t)}\right|\right)dx < \infty. \tag{8}$$

Let us put

$$L = \frac{d^2}{dx^2} + \lambda^2 q(x,t)$$

$$B = 2\lambda^2\left[\frac{2}{\sqrt{1+q(x,t)}}\frac{\partial}{\partial x} - \left(\frac{1}{\sqrt{1+q(x,t)}}\right)_x\right], \tag{9}$$

then the equation (1) can be represented with the Lax form

$$L_t = [B, L] + G, \tag{10}$$

$$G = -2\lambda^2 \sum_{n=1}^{N} (1 + q(x,t)) \frac{\partial}{\partial x} (f_n(x,t)) g_n(x,t)) - \lambda^2 q_x(x,t) \sum_{n=1}^{N} f_n(x,t) g_n(x,t) \tag{11}$$

We have concerned on determining the time evolution equations of the scattering data with approach of the inverse scattering method for the operator $L(t)$ in order to find the solution $\{q(x,t), f_n(x,t), g_n(x,t)\}$ of the problem (1)-(6) under the assumption of existence in the class of decreasing functions (8).

2. Facts from the scattering theory

In this section, we will present the necessary information concerning the direct and inverse scattering problem for the equation [8]

$$Ly \equiv y'' + \lambda^2 q(x) y = -\lambda^2 y. \tag{12}$$

Lemma 1. *If $X(x,\lambda)$ and $Y(x,\mu)$ are solutions of $LX = -\lambda^2 X$ and $LY = -\mu^2 Y$ respectively, then the following identity holds*

$$\frac{dW(X(x,\lambda), Y(x,\mu))}{dx} = (1 + q(x))(\lambda^2 - \mu^2) X(x,\lambda), Y(x,\mu), \tag{13}$$

where $W(X(x,\lambda), Y(x,\mu)) = X(x,\lambda) Y'(x,\mu) - X'(x,\lambda) Y(x,\mu)$.

The equation (12) has the Jost solutions $\psi(x,\lambda)$ and $\varphi(x,\lambda)$ with the following asymptotics

$$\psi(x,\lambda) \to e^{-i\lambda x} \text{ as } x \to -\infty, \tag{14}$$

$$\varphi(x,\lambda) \to e^{i\lambda x} \text{ as } x \to \infty, \tag{15}$$

and for real $\lambda \neq 0$ parameters the pairs $\{\varphi(x,\lambda),\, \varphi(x,-\lambda)\}$ and $\{\psi(x,\lambda),\, \psi(x,-\lambda)\}$ form the fundamental solutions of the equation (12) and therefore, at any $\lambda \neq 0$ we have representations

$$\psi(x,\lambda) = a(\lambda)\varphi(x,-\lambda) + b(\lambda)\varphi(x,\lambda), \tag{16}$$

where,

$$a(\lambda) = \frac{W(\psi(x,\lambda), \varphi(x,\lambda))}{2i\lambda}, \tag{17}$$

$$b(\lambda) = \frac{W(\psi(x,-\lambda), \varphi(x,\lambda))}{2i\lambda}. \tag{18}$$

Here, $a(\lambda)$ admits an analytic continuation in the upper half plane $Im\lambda > 0$ and $\varphi(x,\lambda)e^{-i\lambda x}$ and $\psi(x,\lambda)e^{i\lambda x}$ can be analytic for $Im\lambda \geqslant 0$. From this yields that in the upper half plane $Im\lambda > 0$

$$\begin{gathered} \psi(x,\lambda)e^{i\lambda x} \to a(\lambda), \text{ as } x \to \infty, \\ \varphi(x,\lambda)e^{-i\lambda x} \to a(\lambda), \text{ as } x \to -\infty. \end{gathered} \tag{19}$$

We assume that $a(\lambda)$ has a finite number of simple zeros in the upper half plane $Im\lambda > 0$ such $\lambda_n = i\chi_n, n = 1, 2, \ldots,$

N, which corresponds to the eigenvalues $-\chi_n^2$, $(\chi_n > 0)$ $n = \overline{1, N}$ of L and the follwoing relation holds for the Jost solutions

$$\psi(x, \lambda_n) = c_n \varphi(x, \lambda_n), \ \ n = 1,\ 2,\ ...,\ N. \tag{20}$$

We define reflection coefficient by the formula

$$R(\lambda) = \frac{b(\lambda)}{a(\lambda)}. \tag{21}$$

The following integral representation is valid for the Jost function $f(x, \lambda)$

$$\varphi(x, \lambda) = e^{i\lambda(x+\varepsilon_+)} + e^{i\lambda\varepsilon_+} \int_x^\infty K(x, s) e^{i\lambda s} ds, \tag{22}$$

where

$$\varepsilon_+ = \int_x^\infty \sigma_{-1}\, dx, \quad \sigma_{-1} = 1 - \sqrt{1+q}$$

and the kernel K is assumed to satisfy

$$\lim_{s\to\infty} K(x, s) = 0 \tag{23}$$

and have relation with $q(x)$ in this form

$$1 + q(x) = [1 - K(x, x)]^{-4}. \tag{24}$$

For $x \leqslant y$, $K(x, y)$ kernel satisfies the following Gelfand-Levitan-Marchenko equation

$$K(x, y) - \Omega(x + y) - \int_x^\infty K(x, s)\Omega'(s + y) ds = 0.$$

Here

$$\begin{aligned} \Omega(z) &= -\sum_{n=1}^N c_n \frac{e^{-\chi_n(z+2\varepsilon_+(x))}}{\chi_n} + \frac{1}{2\pi}\int_{-\infty}^\infty R(\lambda)\frac{e^{i\lambda(z+2\varepsilon_+(x))}}{i\lambda} d\lambda, \\ \Omega'(z) &= \sum_{n=1}^N c_n e^{-\chi_n(z+2\varepsilon_+(x))} + \frac{1}{2\pi}\int_{-\infty}^\infty R(\lambda) e^{i\lambda(z+2\varepsilon_+(x))} d\lambda. \end{aligned} \tag{25}$$

Definition 1. *The set of quantities* $\{R(\lambda),\ c_n,\ -\chi_n^2,\ n = \overline{1, N}\}$ *is called the scattering data associated to the equation* (12).

3. Evolution equations

3.1. Evolution equations for $a(\lambda)$ and $b(\lambda)$

Let $y(x, t)$ be a solution of the equation

$$Ly = y''(x, t) + \lambda^2 y(x, t) q(x, t) = -\lambda^2 y(x, t) \tag{26}$$

and let for $F(x, \lambda)$ it is valid the equation

$$\frac{\partial F(x, \lambda)}{\partial x} = (1 + q(x, t)) f_n(x, t) y(x, t). \tag{27}$$

Introduce the function

$$H = \dot{y} - By - \lambda^2 \sum_{n=1}^{N} g_n(x,t) F(x,\lambda). \tag{28}$$

For $\lambda \in \mathbb{R}$ function (28) will be a solution of the equation

$$LH + \lambda^2 H = -\lambda^2 \sum_{n=1}^{N} (1 + q(x,t)) g_n(x,t) \hat{H},$$

where

$$\hat{H} = \left(\chi_n^2 + \lambda^2\right) F(x,\lambda) + W(f_n(x,t), y(x,t))).$$

The following functions

$$\begin{aligned} F^-(x,\lambda) &= -\int_{-\infty}^{x} (1 + q(\tau,t)) f_n(\tau,t) \psi(\tau,\lambda) d\tau, \\ F^+(x,\lambda) &= \int_{x}^{\infty} (1 + q(\tau,t)) f_n(\tau,t) \varphi(\tau,\lambda) d\tau, \end{aligned} \tag{29}$$

which are defined by the Jost solutions, satisfy the equation (27), consequently, it is easy to show that the following functions

$$\begin{aligned} H_0^-(\lambda) &= \dot{\psi}(x,\lambda) - B\psi(x,\lambda) - \lambda^2 \sum_{n=1}^{N} g_n(x,t) F^-(x,\lambda), \\ H_0^+(\lambda) &= \dot{\varphi}(x,\lambda) - B\varphi(x,\lambda) - \lambda^2 \sum_{n=1}^{N} g_n(x,t) F^+(x,\lambda) \end{aligned} \tag{30}$$

satisfy the equation (26). In fact, taking into account the identity (13), it yields that $\dfrac{\partial \hat{H}}{\partial x} = 0$. With the help of the asymptotes of the Jost solutions and taking account of (29), for $Im\lambda \geqslant 0$, we have $\hat{H}^{\pm} \to 0$ as $x \to \pm\infty$, then it yields that for $Im\lambda \geqslant 0$ and at any for $x \in (-\infty, \infty)$ it is valid $\hat{H}^{\pm} \equiv 0$. Hence, $LH^+ = -\lambda^2 H^+$ and $LH^- = -\lambda^2 H^-$.

At $\lambda = i\chi_j$ it holds that

$$f_j(x,t) = c_j^+ \varphi(x,\lambda_j) = c_j^- \psi(x,\lambda_j), \tag{31}$$

and the following asymptotics for the solutions $g_j^{\pm}(x,t)$

$$\begin{aligned} g_j^+ &= \frac{\omega_j}{2\chi_j c_j^+} e^{\chi_j x}, \quad x \to +\infty, \\ g_j^- &= -\frac{\omega_j}{2\chi_j c_j^-} e^{-\chi_j x}, \quad x \to -\infty \end{aligned} \tag{32}$$

are valid.

Lemma 2. *$a(\lambda)$ and $b(\lambda)$ coefficients satisfy the following differential equations*

$$\begin{aligned} \dot{b}\left(\lambda\right) &= 8i\lambda^3 b\left(\lambda\right), \\ \dot{a}\left(\lambda\right) &= -\lambda^3 \sum_{n=1}^{N} \frac{i\omega_n}{\left(\lambda^2 + \chi_n^2\right)\chi_n} a\left(\lambda\right). \end{aligned} \tag{33}$$

Proof. Using the result for $H^{\pm}(\lambda)$ we introduce the following auxiliary function for $\lambda \neq 0$

$$S = H_0^-(\lambda) - a(\lambda)\, H_0^+(-\lambda) - b(\lambda)\, H_0^+(\lambda). \tag{34}$$

With the help of the relation (16) and representations (29) for $F^-(x,\lambda)$, $F^+(x,\lambda)$ in equalities (30) and substituting them into (34), we obtain

$$S = \dot a(\lambda)\,\varphi(x,-\lambda) + \dot b(\lambda)\,\varphi(x,\lambda). \tag{35}$$

In the other hand, taking account the uniqueness of the Jost solutions and the representation (9) and asymptotics (32), we obtain the following representations

$$H_0^-(\lambda) = 4i\lambda^3\psi(x,\lambda) - \lambda^2\sum_{n=1}^{N}\frac{i\omega_n}{2\chi_n(\lambda+i\chi_n)}\psi(x,\lambda), \tag{36}$$

$$H_0^+(\lambda) = -4\lambda^3 i\varphi(x,\lambda) - \lambda^2\sum_{n=1}^{N}\frac{i\omega_n}{2\chi_n(\lambda+i\chi_n)}\varphi(x,\lambda). \tag{37}$$

Substituting expressions (36), (37) into (34) and using the relation (16), we have

$$S = 8i\lambda^3 b(\lambda)\,\varphi(x,\lambda) - \lambda^3\sum_{n=1}^{N}\frac{i\omega_n}{(\lambda^2+\chi_n^2)\,\chi_n}a(\lambda)\,\varphi(x,-\lambda). \tag{38}$$

Comparing (35) and (38) we have relations (33) for $\lambda \neq 0$. Lemma is proved.

From the relations (33) and according to (21), we have the equality

$$\dot R(\lambda,t) = i\lambda^3\left(8 + \sum_{n=1}^{N}\frac{\omega_n}{(\lambda^2+\chi_n^2)\,\chi_n}\right)R(\lambda,t). \tag{39}$$

3.2. Evolution equations of the discrete spectrum and normalization constants

LEMMA 3. *The discret spectrum and normalization constants satisfy the following differential eqautions*

$$\frac{d\chi_j(t)}{dt} = \frac{\chi_j(t)\omega_j(t)}{2}, \quad j = \overline{1,N}. \tag{40}$$

$$\dot c_j(t) = c_j(t)\left(8\chi_j^3(t) + i\beta_j\frac{\omega_j(t)\chi_j(t)}{2}\right), \quad j = \overline{1,N}. \tag{41}$$

Proof. Now, we determine the function

$$h(i\chi_j) = h^-(i\chi_j) - c_j(t)h^+(i\chi_j). \tag{42}$$

Here, functions $h^-(i\chi_j)$ and $h^+(i\chi_j)$ are defined as

$$h^-(i\chi_j(t)) = \dot\psi(x,i\chi_j(t)) - B\psi(x,i\chi_j(t)) + \chi_j^2(t)\sum_{n=1}^{N}g_n^-(x,t)F^-(x,i\chi_j(t)),$$

$$h^+(i\chi_j(t)) = \dot\varphi(x,i\chi_j(t)) - B\varphi(x,i\chi_j) + \chi_j^2(t)\sum_{n=1}^{N}g_n^+(x,t)F^+(x,i\chi_j(t)).$$

Using the asymptotics (14) and (15) of the Jost soluitons $\varphi(x, i\chi_j)$ and $\psi(x, i\chi_j)$ we have

$$h^-(i\chi_j(t)) = 4\chi_j^3(t)\psi(x, i\chi_j(t)),$$
$$h^+(i\chi_j(t)) = -4\chi_j^3(t)\varphi(x, i\chi_j(t)).$$

Due to the relation (20) we show that

$$h(i\chi_j(t)) = 8\chi_j^3(t)c_j(t)\varphi(x, i\chi_j(t)). \tag{43}$$

In the other hand, differentiating the equality (20) by t we have

$$\dot{\psi}(x, i\chi_j(t)) = \dot{c}_j(t)\varphi\left(x, i\chi_j(t)\right) + c_j(t)\dot{\varphi}(x, i\chi_j(t)) - i\tilde{\varphi}(x, i\chi_j(t))\frac{d\chi_j(t)}{dt}, \tag{44}$$

where

$$\tilde{\varphi}(x, i\chi_j(t)) = \frac{d}{d\lambda}\left(\psi(t, \lambda) - c_j(t)\varphi(t, \lambda)\right)\Big|_{\lambda=i\chi_j(t)}.$$

Using (44) in the expression (42) we receive

$$h(i\chi_j(t)) = \dot{c}_j(t)\varphi\left(x, i\chi_j(t)\right) - i\tilde{\varphi}(x, i\chi_j(t))\frac{d\chi_j(t)}{dt}$$
$$+ \chi_j^2(t)\sum_{n=1}^{N} g_n(x,t)(F^-(x, i\chi_j(t)) - c_j(t)F^+(x, i\chi_j(t))),$$

where

$$F^-(x, i\chi_j(t)) - c_j(t)F^+(x, i\chi_j(t)) = \int_{-\infty}^{\infty}\left(1 + q(x,t)\right) f_n(x,t)c_j(t)\varphi(x, i\chi_j(t))d\tau. \tag{45}$$

Using the orthogonality of the eigenfunctions corresponding to the different eigenvalues for $n \neq j$, we can show that the integral in the right-hand side of equality (45) equals to zero, therefore,

$$\begin{gathered} h(i\chi_j(t)) = \dot{c}_j(t)\varphi\left(x, i\chi_j(t)\right) - i\tilde{\varphi}(x, i\chi_j(t))\frac{d\chi_j(t)}{dt} - \\ + \chi_j^2(t)g_j^+(x,t)\int_{-\infty}^{\infty}\left(1 + q(x,t)\right) f_j(x,t)c_j(t)\varphi(x, i\chi_j(t))d\tau. \end{gathered} \tag{46}$$

As the $\tilde{\varphi}(x, i\chi_j)$ being the solution of the equation (2) i.e. $(L - \chi_j^2)\tilde{\varphi}\left(x, i\chi_j\right) = 0$, this solution can be represented by

$$\tilde{\varphi}(x, i\chi_j) = \alpha_j g_j(x,t) + \beta_j\psi(x, i\chi_j). \tag{47}$$

By virtue of (14), (15) and (19), we find the following asymptotics for the function $\tilde{\varphi}(x, i\chi_j(t))$

$$\tilde{\varphi}(x, i\chi_j(t)) \sim \left(\frac{da(\lambda)}{d\lambda}\right)\Big|_{\lambda=i\chi_j(t)} e^{\chi_j x}, \quad x \to +\infty.$$

$$\tilde{\varphi}(x, i\chi_j(t)) \sim -c_j\left(\frac{da(\lambda)}{d\lambda}\right)\Big|_{\lambda=i\chi_j(t)} e^{-\chi_j x}, \quad x \to -\infty.$$

Taking into account (19), (32), (47) and the last asymptotics, we can write the following representation

$$\tilde{\varphi}(x, i\chi_j(t)) = \frac{2\chi_j(t)c_j^+(t)}{\omega_j}\left(\frac{da(\lambda)}{d\lambda}\right)\Big|_{\lambda=i\chi_j(t)} g_j^+(x,t) + \beta_j c_j(t)\varphi(x, i\chi_j(t)) \tag{48}$$

is valid. It is easy to show that the following equality holds

$$\left(\frac{da(\lambda)}{d\lambda}\right)\Bigg|_{\lambda=i\chi_j} = -i\int_{-\infty}^{+\infty}(1+q(t,\tau))\psi(\tau,i\chi_j)\varphi(\tau,i\chi_j)d\tau. \tag{49}$$

Using the equalities (48) and (49) in (46) we have

$$\begin{aligned} h(i\chi_j(t)) = \dot{c}_j(t)\varphi(x,i\chi_j(t)) - i\frac{d\chi_j(t)}{dt}\frac{2\chi_j(t)c_j^+(t)}{\omega_j(t)}\dot{a}(i\chi_j)g_j^+(x,t)- \\ - i\frac{d\chi_j(t)}{dt}\beta_j c_j(t)\varphi(x,i\chi_j(t)) + i\chi_j^2(t)g_j^+(x,t)c_j^+(t)\dot{a}(i\chi_j(t)). \end{aligned} \tag{50}$$

Comparing (43) and (50) we get

$$\begin{aligned} &\dot{c}_j(t) - i\frac{d\chi_j(t)}{dt}\beta_j c_j(t) = 8\chi_j^3(t)c_j(t), \\ &\frac{2\chi_j(t)}{\omega_j(t)}\frac{d\chi_j(t)}{dt} = \chi_j^2(t), \quad j=\overline{1,N}. \end{aligned} \tag{51}$$

Hence, we obtain the time evolution (40) for $\chi_j(t)$ and subtituting it to the differential equation for $c_j(t)$ in (51) we have (41). Here, β_j can be defined by (47). Lemma is proved.

4. Multi-soliton solution

We use the (A,B,C) triplet matrix technique [20] for providing the explicit form of the multi-soliton solution. As we are focusing on the the reflectionless case $R(\lambda,t)=0$ and that $L(t)$ operator has simple eigenvalues, we take (A,B,C) triplet matrix in the following form

$$A = \begin{pmatrix} \chi_1(t) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \chi_N(t) \end{pmatrix}, B = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, C = \begin{pmatrix} c_1(t) & \dots & c_N(t) \end{pmatrix},$$

where $\chi_j(t)$ and $c_j(t)$, $j=\overline{1,N}$ satisfy the equations (40) and (41). Hence, the kernel of Gelfand-Levitan-Marchenko equation can be written as

$$\Omega(x+y,t) = -Ce^{-A(x+y+2\varepsilon_+)}A^{-1}B.$$

Inserting this into the Gelfand-Levitan equation and solving it we obtain

$$K(x,x,t) = -Ce^{-A(x+2\varepsilon_+)}A^{-1}\Gamma^{-1}(x,t)e^{-Ax}B,$$

and, by the formula (24) we obtain

$$q(x,t) = [1 + Ce^{-A(x+2\varepsilon_+)}A^{-1}\Gamma^{-1}(x,t)e^{-Ax}B]^{-4} - 1,$$

where

$$\Gamma(x,t) = I - e^{-2A\varepsilon_+}Q(x)$$
$$Q(x) = \int_x^{\infty} e^{-As}BCe^{-As}ds.$$

For one soliton case, we have

$$q(x,t) = tanh^4\left[\chi_1(t)(x+\varepsilon_+(x)) - \frac{1}{2}\ln\frac{c_1(t)}{2\chi_1(t)}\right] - 1, \tag{52}$$

where

$$\varepsilon_+ = \frac{1}{\chi_1(t)}\left\{1 - tanh\left[\chi_1(t)(x+\varepsilon_+(x)) - \frac{1}{2}\ln\frac{c_1(t)}{2\chi_1(t)}\right]\right\}.$$

Using the integral expression for the Jost solution $\varphi(x,\lambda)$ and taking $c_j^+ = 1$ in (31), we can find the eigenfunction

$$f(x,i\chi_1) = \sqrt{\frac{2\chi_1(t)}{c_1(t)}}\frac{1}{2\sin h\left(\chi_1(t)(x+\varepsilon_+(x)) - \frac{1}{2}\ln\frac{c_1(t)}{2\chi_1(t)}\right)}. \tag{53}$$

By solving the differential equation (6) with (53), we get the representation for the solution $g(x,i\chi_1)$:

$$g(x,i\chi_1) = \sqrt{\frac{2c_1(t)}{\chi_1(t)}}\frac{1}{2\sin h\left(\chi_1(t)(x+\varepsilon_+(x)) - \frac{1}{2}\ln\frac{c_1(t)}{2\chi_1(t)}\right)} \times\left(\omega_1(t)\int \sin h^2\left(\chi_1(t)(x+\varepsilon_+(x)) - \frac{1}{2}\ln\frac{c_1(t)}{2\chi_1(t)}\right)dx\right). \tag{54}$$

5. Conclusions

The equations (39), (40) and (41) allow completely determine the time evolution of all scattering data for the eigenvalue problem of the form of (2). Then, we can integrate the problem (1)-(8) by reducing it into solving the integral Gelfand-Levitan equation with the obtained results (39)–(41). Due to the condition (7) for the considering problem (1)-(8), there is no effect of the creation and anhilation of the solitons differing from the KdV equation [12].

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