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FREE COMMUTATIVE g-DIMONOIDS

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Abstract

A dialgebra is a vector space equipped with two binary operations $\dashv$ and $\vdash$ satisfying the following axioms:

(D1) $(x \dashv y) \dashv z = x \dashv (y \dashv z)$,

(D2) $(x \dashv y) \dashv z = x \dashv (y \vdash z)$,

(D3) $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,

(D4) $(x \dashv y) \vdash z = x \vdash (y \vdash z)$,

(D5) $(x \vdash y) \vdash z = x \vdash (y \vdash z)$.

This notion was introduced by Loday while studying periodicity phenomena in algebraic K-theory. Leibniz algebras are a non-commutative variation of Lie algebras and dialgebras are a variation of associative algebras. Recall that any associative algebra gives rise to a Lie algebra by $[x, y] = xy - yx$. Dialgebras are related to Leibniz algebras in a way similar to the relationship between associative algebras and Lie algebras. A dialgebra is just a linear analog of a dimonoid. If operations of a dimonoid coincide, the dimonoid becomes a semigroup. So, dimonoids are a generalization of semigroups.

Pozhidaev and Kolesnikov considered the notion of a 0-dialgebra, that is, a vector space equipped with two binary operations $\dashv$ and $\vdash$ satisfying the axioms $(D2)$ and $(D4)$. This notion have relationships with Rota-Baxter algebras, namely, the structure of Rota-Baxter algebras appearing on 0-dialgebras is known.

The notion of an associative 0-dialgebra, that is, a 0-dialgebra with two binary operations $\dashv$ and $\vdash$ satisfying the axioms $(D1)$ and $(D5)$, is a linear analog of the notion of a g-dimonoid. In order to obtain a g-dimonoid, we should omit the axiom $(D3)$ of inner associativity in the definition of a dimonoid. Axioms of a dimonoid and of a g-dimonoid appear in defining identities of trialgebras and of trioids introduced by Loday and Ronco.

The class of all g-dimonoids forms a variety. In the paper of the second author the structure of free g-dimonoids and free n-nilpotent g-dimonoids was given. The class of all commutative g-dimonoids, that is, g-dimonoids with commutative operations, forms a subvariety of the variety of g-dimonoids. The free dimonoid in the variety of commutative dimonoids was constructed in the paper of the first author.

In this paper we construct a free commutative g-dimonoid and describe the least commutative congruence on a free g-dimonoid.

Keywords: dimonoid, g-dimonoid, commutative g-dimonoid, free commutative g-dimonoid, semigroup, congruence.

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СВОБОДНЫЕ КОММУТАТИВНЫЕ g-ДИМОНОИДЫ

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Аннотация

Диалгеброй называется векторное пространство, снабжённое двумя бинарными операциями $\dashv$ и $\vdash$, удовлетворяющими следующим аксиомам:

(D1) $(x \dashv y) \dashv z = x \dashv (y \dashv z)$,

(D2) $(x \dashv y) \dashv z = x \dashv (y \vdash z)$,

(D3) $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,

(D4) $(x \dashv y) \vdash z = x \vdash (y \vdash z)$,

(D5) $(x \vdash y) \vdash z = x \vdash (y \vdash z)$.

Это понятие было введено Лодэ во время изучения феномена периодичности в алгебраической K-теории. Алгебры Лейбница являются некоммутативной версией алгебр Ли, а диалгебры – версией ассоциативных алгебр. Напомним, что любая ассоциативная алгебра даёт алгебру Ли, если положить $[x, y] = xy - yx$. Диалгебры связаны с алгебрами Лейбница аналогично тому как связаны между собой ассоциативные алгебры и алгебры Ли. Диалгебра является линейным аналогом димоноида. Если операции димоноида совпадают, то он превращается в полугруппу. Таким образом, димоноиды обобщают полугруппы.

Пожидаев и Колесников рассмотрели понятие 0-диалгебры, то есть векторного пространства, снабжённого двумя бинарными операциями $\dashv$ и $\vdash$, удовлетворяющими аксиомам $(D2)$ и $(D4)$. Это понятие имеет связи с алгебрами Рота-Бакстера, а именно известна структура алгебр Рота-Бакстера, возникающих на 0-диалгебрах.

Понятие ассоциативной 0-диалгебры, то есть 0-диалгебры с двумя бинарными операциями $\dashv$ и $\vdash$, удовлетворяющими аксиомам $(D1)$ и $(D5)$, является линейным аналогом понятия g-димоноида. Для того, чтобы получить g-димоноид, мы должны опустить аксиому $(D3)$ внутренней ассоциативности в определении димоноида. Аксиомы димоноида и g-димоноида появляются в тождествах триалгебр и триоидов, введенных Лодэ и Ронко.

Класс всех g-димоноидов образует многообразие. Строение свободных g-димоноидов и свободных n-нильпотентных g-димоноидов было описано в статье второго автора. Класс всех коммутативных g-димоноидов, то есть g-димоноидов с коммутативными операциями, образует подмногообразие многообразия g-димоноидов. Свободный димоноид в многообразии коммутативных димоноидов был построен в статье первого автора.

В этой статье мы строим свободный коммутативный g-димоноид, а также описываем наименьшую коммутативную конгруэнцию на свободном g-димоноиде.

Ключевые слова: димоноид, g-димоноид, коммутативный g-димоноид, свободный коммутативный g-димоноид, полугруппа, конгруэнция.

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1. Introduction and preliminaries

Pozhidaev [1] and Kolesnikov [2] considered the notion of a 0-dialgebra. This notion have relationships with associative dialgebras [3–6] and Rota-Baxter algebras [1]. The notion of an associative 0-dialgebra, that is, a 0-dialgebra with two binary associative operations, is a linear analog of the notion of a g-dimonoid. In order to obtain a g-dimonoid, we should omit the axiom of inner associativity in the definition of a dimonoid [7]. The class of all g-dimonoids forms a variety. Free g-dimonoids and free n-nilpotent g-dimonoids were constructed in [8, 9] and [9], respectively. Axioms of a g-dimonoid also appear in defining identities of trialgebras and of trioids [10–12].

The class of all commutative g-dimonoids, that is, g-dimonoids with commutative operations, forms a subvariety of the variety of g-dimonoids. The free dimonoid in the variety of commutative dimonoids was constructed in [13]. In this paper we construct a free commutative g-dimonoid (Theorem 1) and describe the least commutative congruence on a free g-dimonoid (Theorem 2).

To make the paper almost self-contained, we recall basic definitions that will be used later.

A nonempty set equipped with two binary operations $\dashv$ and $\vdash$ satisfying the axioms $(D1)$–$(D5)$ is called a dimonoid. For a general introduction and basic theory see [3, 7, 14]. A nonempty set equipped with two binary operations $\dashv$ and $\vdash$ satisfying the axioms $(D1)$, $(D2)$, $(D4)$, $(D5)$ is called a generalized dimonoid or simply a g-dimonoid for short. It is obvious that any dimonoid is a g-dimonoid. Other examples of g-dimonoids can be found in [3, 7–9, 13–15]. Independence of axioms of a g-dimonoid follows from independence of axioms of a dimonoid [7].

If $f : D_1 \to D_2$ is a homomorphism of g-dimonoids, then the corresponding congruence on D_1 will be denoted by Δ_f.

2. The main result

In this section we construct a free commutative g-dimonoid.

A g-dimonoid $(D, \dashv, \vdash)$ will be called commutative, if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative. A g-dimonoid which is free in the variety of commutative g-dimonoids will be called a free commutative g-dimonoid.

Now we give a new example of a g-dimonoid. Let A be an arbitrary nonempty set and $\overline{A} = \{\overline{x} \,|\, x \in A\}$. For every $x \in A$ assume $\widetilde{\overline{x}} = x$ and introduce a map $\alpha = \alpha_A : A \cup \overline{A} \to A$ by the following rule:

$$y\alpha = \begin{cases} y, \ y \in A, \\ \widetilde{y}, \ y \in \overline{A}. \end{cases}$$

Let further S be an arbitrary semigroup. Define operations $\dashv$ and $\vdash$ on $S \cup \overline{S}$ by

$$a \dashv b = (a\alpha_S)(b\alpha_S), \quad a \vdash b = \overline{(a\alpha_S)(b\alpha_S)}$$

for all $a, b \in S \cup \overline{S}$. Denote $(S \cup \overline{S}, \dashv, \vdash)$ by $S^{(\alpha)}$.

LEMMA 1. *$S^{(\alpha)}$ is a g-dimonoid but not a dimonoid.*

Proof. The proof follows by a routine verification. □

Evidently, if S is commutative, then $S^{(\alpha)}$ is a commutative g-dimonoid. If X is a generating set for a semigroup S, then, obviously, $S^{(\alpha)} \backslash \overline{X}$ is a g-subdimonoid of $S^{(\alpha)}$ generated by X. Denote by $FCgD(X)$ the g-dimonoid $S^{(\alpha)} \backslash \overline{X}$ in which S is the free commutative semigroup on X.

THEOREM 1. *$FCgD(X)$ is the free commutative g-dimonoid.*

Proof. Show that $FCgD(X)$ is free in the variety of commutative g-dimonoids.

Let $(G, \dashv', \vdash')$ be an arbitrary commutative g-dimonoid, $\psi : X \to G$ be an arbitrary map and $x_i, y_j \in X$, $i \in \{1, 2, ..., m\}$, $j \in \{1, 2, ..., n\}$. Define a map

$$\xi : FCgD(X) \to (G, \dashv', \vdash') : w \mapsto w\xi, \quad \text{assuming}$$

$$w\xi = \begin{cases} x_1\psi \dashv' ... \dashv' x_m\psi, \ w = x_1...x_m, m \geq 1, \\ x_1\psi \vdash' ... \vdash' x_m\psi, \ w = \overline{x_1...x_m}, m > 1. \end{cases}$$

Further prove that ξ is a homomorphism.

Let $w, u \in FCgD(X)$. In the case $w = \overline{x_1...x_m}$, $u = \overline{y_1...y_n}$ obtain

$$(w \dashv u)\xi = x_1\psi \dashv' ... \dashv' x_m\psi \dashv' y_1\psi \dashv' ... \dashv' y_n\psi =$$
$$= (x_1\psi \dashv' ... \dashv' x_m\psi) \dashv' (y_1\psi \vdash' ... \vdash' y_n\psi) =$$
$$= (y_1\psi \vdash' ... \vdash' y_n\psi) \dashv' (x_1\psi \dashv' ... \dashv' x_m\psi) =$$
$$= (y_1\psi \vdash' ... \vdash' y_n\psi) \dashv' (x_1\psi \vdash' ... \vdash' x_m\psi) =$$
$$= (x_1\psi \vdash' ... \vdash' x_m\psi) \dashv' (y_1\psi \vdash' ... \vdash' y_n\psi) =$$
$$= \overline{x_1...x_m}\xi \dashv' \overline{y_1...y_n}\xi = w\xi \dashv' u\xi.$$

For $w = \overline{x_1...x_m}$, $u = y_1...y_n$ get

$$(w \dashv u)\xi = x_1\psi \dashv' ... \dashv' x_m\psi \dashv' y_1\psi \dashv' ... \dashv' y_n\psi =$$
$$= (y_1\psi \dashv' ... \dashv' y_n\psi) \dashv' (x_1\psi \dashv' ... \dashv' x_m\psi) =$$
$$= (y_1\psi \dashv' ... \dashv' y_n\psi) \dashv' (x_1\psi \vdash' ... \vdash' x_m\psi) =$$
$$= (x_1\psi \vdash' ... \vdash' x_m\psi) \dashv' (y_1\psi \dashv' ... \dashv' y_n\psi) =$$
$$= \overline{x_1...x_m}\xi \dashv' (y_1...y_n)\xi = w\xi \dashv' u\xi.$$

The remaining two cases are considered in a similar way. So, $(w \dashv u)\xi = w\xi \dashv' u\xi$ for all $w, u \in FCgD(X)$.

Similarly, one can check that $(w \vdash u)\xi = w\xi \vdash' u\xi$ for all $w, u \in FCgD(X)$.

Consequently, ξ is a homomorphism and $FCgD(X)$ is the free commutative g-dimonoid. □

If N_+ is the additive semigroup of all positive integers, obviously, $N_+^{(\alpha)} \backslash \{\overline{1}\}$ is the free commutative g-dimonoid of rank 1.

It is not difficult to see that the automorphism group of the free commutative g-dimonoid $FCgD(X)$ is isomorphic to the symmetric group on X and semigroups of $FCgD(X)$ are isomorphic.

We conclude this section with some additional property of g-dimonoids.

LEMMA 2. *Operations of a g-dimonoid $(D, \dashv, \vdash)$ with a commutative idempotent operation $\dashv$ (respectively, $\vdash$) coincide.*

Proof. For all $x, y, z \in D$ we have

$$x \vdash y = (x \vdash y) \dashv (x \vdash y) = (x \vdash y) \dashv (x \dashv y) =$$
$$= (x \dashv y) \dashv (x \vdash y) = (x \dashv y) \dashv (x \dashv y) = x \dashv y$$

according to the idempotency, the commutativity of $\dashv$ and the axioms $(D1)$, $(D2)$ of a g-dimonoid. The case with the operation $\vdash$ is proved similarly. □

From Lemma 2 it follows that there do not exist commutative g-dimonoids with different idempotent operations.

3. The least commutative congruence on a free g-dimonoid

In this section we present the least commutative congruence on a free g-dimonoid.

If ρ is a congruence on a g-dimonoid $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/\rho$ is a commutative g-dimonoid, we say that ρ is a commutative congruence.

In our next result we need the following construction.

Let X be an arbitrary nonempty set and let w be an arbitrary word in the alphabet X. The length of w will be denoted by $l(w)$. Let further T be the free monoid on the two-element set $\{a, b\}$, $\theta \in T$ be an empty word and $*$ denotes the operation on T. By definition, $l(\theta) = 0$. For every $u \in T \backslash \{\theta\}$ denote the last letter of u by $u^{(1)}$. Define operations $\dashv$ and $\vdash$ on T, assuming

$$u_1 \dashv u_2 = u_1 * a^{l(u_2)+1}, \quad u_1 \vdash u_2 = u_2 * b^{l(u_1)+1}$$

for all $u_1, u_2 \in T$. The obtained algebra is denoted by $T_a^b(1)$.

Let $F[X]$ be the free semigroup on X and

$$XT_a^b(1) = \{(w, u) \in F[X] \times T_a^b(1) \mid l(w) - l(u) = 1\}.$$

By Theorem 1 from [9] $XT_a^b(1)$ is the free g-dimonoid.

THEOREM 2. *Let $XT_a^b(1)$ be the free g-dimonoid and $FCgD(X)$ be the free commutative g-dimonoid. A map*

$$\beta : XT_a^b(1) \to FCgD(X) :$$

$$(w, u) \mapsto (w, u)\beta = \begin{cases} \overline{w}, & u^{(1)} = b, \\ w & otherwise \end{cases}$$

is an epimorphism inducing the least commutative congruence on $XT_a^b(1)$.

Proof. Take arbitrary elements (w_1, u_1), $(w_2, u_2) \in XT_a^b(1)$. We have

$$((w_1, u_1) \dashv (w_2, u_2))\beta = (w_1 w_2, u_1 * a^{l(u_2)+1})\beta =$$

$$= w_1 w_2 = (w_1, u_1)\beta \dashv (w_2, u_2)\beta,$$

$$((w_1, u_1) \vdash (w_2, u_2))\beta = (w_1 w_2, u_2 * b^{l(u_1)+1})\beta =$$

$$= \overline{w_1 w_2} = (w_1, u_1)\beta \vdash (w_2, u_2)\beta.$$

Thus, β is a homomorphism.

Let $FC[X]$ be the free commutative semigroup on X and $\omega, x \in FC[X]$, where $l(\omega) > 1$ and $l(x) = 1$. For elements $\omega, \overline{\omega}, x \in FCgD(X)$ there exist elements (ω, ua), (ω, ub), $(x, \theta) \in XT_a^b(1)$, where $u \in T$, such that

$$(\omega, ua)\beta = \omega, \quad (\omega, ub)\beta = \overline{\omega}, \quad (x, \theta)\beta = x.$$

So, β is surjective. By Theorem 1 $FCgD(X)$ is the free commutative g-dimonoid. Then Δ_β is the least commutative congruence on $XT_a^b(1)$. □

Let α be an arbitrary fixed congruence on $F[X]$. Define a relation α' on $XT_a^b(1)$ by

$$(w_1, u_1)\alpha'(w_2, u_2) \Leftrightarrow w_1 \,\alpha\, w_2$$

for all (w_1, u_1), $(w_2, u_2) \in XT_a^b(1)$.

It is not hard to prove the following lemma.

Lemma 3. *The relation α' is a congruence on the free g-dimonoid $XT_a^b(1)$. Besides, operations of $XT_a^b(1)/\alpha'$ coincide.*

From Lemma 3 we obtain

Corollary 1. *If α is a diagonal of $F[X]$, then $XT_a^b(1)/\alpha'$ is the free semigroup.*

4. Conclusions

In this paper we consider g-dimonoids which are sets with two binary associative operations satisfying additional axioms. Dimonoids in the sense of Loday are examples of g-dimonoids. The main result of this paper is the construction of a free commutative g-dimonoid. We also present the least commutative congruence on a free g-dimonoid.

REFERENCES

1. Pozhidaev, A. P. 2009, “0-dialgebras with bar-unity and nonassociative Rota-Baxter algebras”, *Sib. Math. J.*, vol. 50, no. 6, pp. 1070–1080.

2. Kolesnikov, P. S. 2008, “Varieties of dialgebras and conformal algebras”, *Sib. Math. J.*, vol. 49, no. 2, pp. 257–272.

3. Loday, J.-L. 2001, “Dialgebras. In: Dialgebras and related operads”, *Lecture Notes in Math., Berlin: Springer-Verlag*, vol. 1763, pp. 7–66.

4. Frabetti, A. 2001, “Dialgebra (co)homology with coefficients. In: Dialgebras and related operads”, *Lecture Notes in Math., Berlin: Springer-Verlag*, vol. 1763, pp. 67–103.

5. Bokut, L. A., Chen, Y. & Liu, C. 2010, “Gröbner-Shirshov bases for dialgebras”, *Int. J. Algebra Comput.*, vol. 20, no. 3, pp. 391–415.

6. Kolesnikov, P. S. & Voronin, V. Yu. 2013, “On the special identities for dialgebras”, *Linear and Multilinear Algebra*, vol. 61, no. 3, pp. 377–391.

7. Zhuchok, A. V. 2011, "Dimonoids", *Algebra and Logic*, vol. 50, no. 4, pp. 323–340.

8. Movsisyan, Y., Davidov, S. & Safaryan, Mh. 2014, "Construction of free g-dimonoids", *Algebra and Discrete Math.*, vol. 18, no. 1, pp. 138–148.

9. Zhuchok, Yul. V. 2014, "On one class of algebras", *Algebra and Discrete Math.*, vol. 18, no. 2, pp. 306–320.

10. Loday, J.-L. & Ronco, M. O. 2004, "Trialgebras and families of polytopes", *Contemp. Math.*, vol. 346, pp. 369–398.

11. Casas, J. M. 2006, "Trialgebras and Leibniz 3-algebras", *Boletín de la Sociedad Matemática Mexicana*, vol. 12, no. 2, pp. 165–178.

12. Zhuchok, A. V. 2014, "Semiretractions of trioids", *Ukr. Math. J.*, vol. 66, no. 2, pp. 218–231.

13. Zhuchok, A. V. 2010, "Free commutative dimonoids", *Algebra and Discrete Math.*, vol. 9, no. 1, pp. 109–119.

14. Zhuchok, A. V. 2014, "Elements of dimonoid theory", *Mathematics and its Applications. Proceedings of Institute of Mathematics of NAS of Ukraine, Kiev*, vol. 98, 304 p. (in Ukrainian).

15. Zhuchok, A. V. 2011, "Semilattices of subdimonoids", *Asian-Eur. J. Math.*, vol. 4, no. 2, pp. 359–371.

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Pozhidaev A. P. 0-dialgebras with bar-unity and nonassociative Rota-Baxter algebras // Sib. Math. J. 2009. Vol. 50, no. 6. P. 1070–1080.

2. Kolesnikov P. S. Varieties of dialgebras and conformal algebras // Sib. Math. J. 2008. Vol. 49, no. 2. P. 257–272.

3. Loday J.-L. Dialgebras. In: Dialgebras and related operads // Lecture Notes in Math. Berlin: Springer-Verlag. 2001. Vol. 1763. P. 7–66.

4. Frabetti A. Dialgebra (co)homology with coefficients. In: Dialgebras and related operads // Lecture Notes in Math. Berlin: Springer-Verlag. 2001. Vol. 1763. P. 67–103.

5. Bokut L. A., Chen Y., Liu C. Gröbner-Shirshov bases for dialgebras // Int. J. Algebra Comput. 2010. Vol. 20, no. 3. P. 391–415.

6. Kolesnikov P. S., Voronin V. Yu. On the special identities for dialgebras // Linear and Multilinear Algebra. 2013. Vol. 61, no. 3. P. 377–391.

7. Zhuchok A. V. Dimonoids // Algebra and Logic. 2011. Vol. 50, no. 4. P. 323–340.

8. Movsisyan Y., Davidov S., Safaryan Mh. Construction of free g-dimonoids // Algebra and Discrete Math. 2014. Vol. 18, no. 1. P. 138–148.

9. Zhuchok Yul. V. On one class of algebras // Algebra and Discrete Math. 2014. Vol. 18, no. 2. P. 306–320.

10. Loday J.-L., Ronco M. O. Trialgebras and families of polytopes // Contemp. Math. 2004. Vol. 346. P. 369–398.

11. Casas J. M. Trialgebras and Leibniz 3-algebras // Boletín de la Sociedad Matemática Mexicana. 2006. Vol. 12, no. 2. P. 165–178.

12. Zhuchok A. V. Semiretractions of trioids // Ukr. Math. J. 2014. Vol. 66, no. 2. P. 218–231.

13. Zhuchok A. V. Free commutative dimonoids // Algebra and Discrete Math. 2010. Vol. 9, no. 1. P. 109–119.

14. Жучок А. В. Елементи теорії дімоноїдів // Математика та її застосування. Праці Інституту математики НАН України, Київ. 2014. Т. 98. 304 с.

15. Zhuchok A. V. Semilattices of subdimonoids // Asian-Eur. J. Math. 2011. Vol. 4, no. 2. P. 359–371.

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