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Высокоточный и эффективный метод исследования динамики производных разных порядков сингулярно возмущенного уравнения

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Аннотация

Целью данной статьи является построение высокоточного и эффективного численного метода исследования динамики производных различных порядков сингулярно совершенного дифференциального уравнения. В методе предварительного интегрирования высшей производной уравнения и правая часть представляются в виде конечных рядов по полиномам Чебышева первого рода с неизвестными коэффициентами разложения. Перед решением задачи выбранный ряд предварительно интегрируется и находятся выражения в виде рядов для всех низших производных и искомого решения. Неизвестные постоянные, появляющиеся при интегрировании ряда, определяются из дополнительных условий задачи. Неизвестные коэффициенты определяются из системы алгебраических уравнений и, подставляя их в нужный ряд, вычисляются производные и решение задачи.

Ключевые слова: сингулярно возмущенная задача, метод предварительного интегрирования.

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A highly accurate and efficient method for studying the dynamics of derivatives of different orders of a singularly perturbed equation

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Abstract

The purpose of this article is to construct a highly accurate and efficient numerical method for studying the dynamics of derivatives of various orders of a singularly perfected differential equation. In the method of preliminary integration of the highest derivative, the equations and the right part are represented as finite series according to Chebyshev polynomials of the first kind with unknown expansion coefficients. Before solving the problem, the selected series is pre-integrated and expressions are found in the form of series for all lower derivatives and the desired solution. Unknown constants appearing during series integration are determined from additional conditions of the problem. Unknown coefficients are determined from a system of algebraic equations and putting them in the right series, the derivatives and the solution of the problem are calculated.

Keywords: singularly perturbed problem, preliminary integration method.

Bibliography: 16 titles.

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1. Introduction

The construction and study of high-precision and efficient numerical methods for solving non-homogeneous singularly perturbed differential equations are a pressing problem in the field of numerical modeling. Despite the fact that numerous studies have been conducted to develop efficient computational algorithms, improve existing methods and create new numerical methods, intensive research is still being conducted in this direction.

In [1], a uniformly convergent algorithm is proposed for the numerical solution of a singularly perturbed second-order differential equation, where a transformation is constructed such that the dependent variable has uniformly bounded derivatives.

The paper [2] provides a critical review of recent advances and summaries of the current state of knowledge of layered structures, including descriptions of the types of layers discovered to date, with an emphasis on non-exponential layers. It presents model problems involving such layers and the conditions under which they arise, describes specific methods for creating layer-exclusive coordinate transformations and layer resolution grids, presents methods for solving specific model problems, and discusses possible applications of the methods to the numerical analysis of practical layer-related problems.

Along with difference methods, spectral [3] -[7] and spectral-grid methods [8] -[14] are successfully used to solve singularly perturbed problems. In these methods, Chebyshev polynomials are used as basis functions. In spectral and spectral-grid methods, the solution to a singularly perturbed equation is represented as a finite series in Chebyshev polynomials. All derivatives present in the equation are found by differentiating the selected finite series. Only after this are the series for the solution and its derivatives placed in a differential equation and the satisfaction of the existing additional conditions of the problem under consideration is required. The result is a system of algebraic equations with respect to the coefficients of the expansion of the solution to the problem in a finite series. Generally speaking, when differentiating series, the order of the approximating polynomials is reduced, and this affects the accuracy of the calculations.

Of particular interest in solving singularly perturbed problems is the use of the so-called preliminary integration method. The use of this method for solving homogeneous singularly perturbed differential equations is described in [15]-[17]. The essence of the method is as follows. Unlike the spectral and spectral-grid methods, in the preliminary integration method, not the solution, but the senior derivative of the differential equation is expanded into a finite series in Chebyshev polynomials. Then, before solving the problem, this series is preliminary integrated as many times as necessary. Thus, all low-order derivatives and the solution of the singularly perturbed equation are represented as finite series. With each integration of the series, a new unknown integration constant appears. Expressions for these constants are determined from satisfying the corresponding additional conditions of the differential problem. The found expressions for the constants are substituted into the finite series for solving the problem and its derivatives, and new finite series are obtained. But only with other expansion coefficients. Only after this, these series are placed in a differential equation and a system of algebraic equations with respect to the expansion coefficients is obtained. By solving the resulting system, the unknown coefficients of the series are determined. Using these coefficients, it is possible to calculate the solution and its derivatives of any order, up to the highest derivative of the singularly perturbed differential equation. It should be noted that when integrating series, the order of the approximating polynomials increases, they become smoother and the accuracy of calculating the solution and its derivatives increases. In the preliminary integration method, the order of the algebraic system being solved does not increase, i.e. the number of arithmetic operations does not increase.

The purpose of this article is to solve a boundary value problem for a non-homogeneous singularly perturbed differential equation by the method of preliminary integration and to study the dynamics of derivatives of different orders depending on the small parameter of the problem. The authors are not aware of any works aimed at solving the problem; in all the works studied, the main attention is paid to studying the dynamics of the solution itself.

2. Statement of the problem

Let us consider the following non-homogeneous singularly perturbed equation [1]

$$\varepsilon \frac{d^2u}{dy^2} + \frac{1}{2}\frac{du}{dy} = f(y), \quad y \in (-1, 1), \tag{2.1}$$

with boundary conditions

$$u(-1) = u(+1) = 0, \tag{2.2}$$

where ε is a small parameter.

The trial function u(y) for problem (2.1)–(2.2), written on the interval $[-1,1]$ for $f(y)=\frac{1}{8}(y+1)$, has the form

$$u(y)=\frac{\varepsilon-0.5}{1-e^{-\frac{1}{\varepsilon}}}\left(1-e^{\frac{-(y+1)}{2\varepsilon}}\right)-\varepsilon\frac{y+1}{2}+\frac{(y+1)^2}{8}. \tag{2.3}$$

Then the first and second derivatives of the trial function (2.3) are calculated, respectively, using the following formulas:

$$\frac{du}{dy}=\frac{(\varepsilon-\frac{1}{2})e^{-\frac{y+1}{2\varepsilon}}}{2\varepsilon(1-e^{-\frac{1}{\varepsilon}})}+\frac{y+1}{4}-\frac{\varepsilon}{2}, \tag{2.4}$$

$$\frac{d^2u}{dy^2}=\frac{1}{4}-\frac{(\varepsilon-\frac{1}{2})e^{-\frac{y+1}{2\varepsilon}}}{4\varepsilon^2(1-e^{\frac{1}{\varepsilon}})}. \tag{2.5}$$

expressions (2.3) – (2.5) will be needed to compare them with the approximate values obtained by the proposed preliminary integration method.

3. Solution method

According to the preliminary integration method, we represent the senior derivative and the right-hand side of the differential equation (2.1) in the form of the following series:

$$\frac{d^2u}{dy^2}=\sum_{i=0}^{N}a_iT_i(y), \quad f(y)=\sum_{i=0}^{N}b_iT_i(y), \tag{3.1}$$

where a_i, b_j - are unknown coefficients, $T_i(y)$ Chebyshev polynomials of the first kind, i - order. Series (3.1) for the derivative after double preliminary integration takes the form:

$$\frac{du}{dy}=\sum_{j=0}^{N+1}\sum_{i=0}^{N}f_{ji}^{(1)}a_iT_j(y)+c_1T_0(y), \tag{3.2}$$

$$u(y)=\sum_{j=0}^{N+2}\sum_{i=0}^{N}f_{ji}^{(0)}a_iT_j(y)+c_1T_1(y)+c_2T_0(y), \tag{3.3}$$

where c_1, c_2 - are unknown constants of integration. They are determined from the boundary conditions (2.2), using the following properties of Chebyshev polynomials: $T_n(\pm1)=(\pm1)^n$. And then we have:

$$u(+1)=\sum_{j=0}^{N+2}\sum_{i=0}^{N}f_{ji}^{(0)}a_i+c_1+c_2=0, \tag{3.4}$$

$$u(-1)=\sum_{j=0}^{N+2}\sum_{i=0}^{N}f_{ji}^{(0)}a_i-c_1+c_2=0. \tag{3.5}$$

First adding and then subtracting equations (3.4) and (3.5), we have expressions for determining the constants c_1 and c_2 of the following form:

$$c_2=-\frac{1}{2}\sum_{i=0}^{N}\left[\sum_{j=0}^{N+2}\left(f_{ji}^{(0)}+(-1)^jf_{ji}^{(0)}\right)\right]a_i,$$

$$c_1=\frac{1}{2}\sum_{i=0}^{N}\left[\sum_{j=0}^{N+2}\left((-1)^jf_{ji}^{(0)}-\sum_{j=0}^{N+2}f_{ji}^{(0)}\right)\right]a_i,$$

To simplify the writing of these formulas, we introduce the following notations:

$$\delta_i^{(0)} = \sum_{j=0}^{N+2} f_{ji}^{(0)}, \quad \overline{\delta}_i^{(0)} = \sum_{j=0}^{N+2} (-1)^j f_{ji}^{(0)}.$$

Then, to determine the constants c_1 and c_2 we have the formulas:

$$c_1 = \frac{1}{2} \sum_{i=0}^{N} [\overline{\delta}_i^{(0)} + \delta_i^{(0)}] a_i. \tag{3.6}$$

$$c_2 = -\frac{1}{2} \sum_{i=0}^{N} \left[\delta_i^{(0)} + \overline{\delta}_i^{(0)}\right] a_i, \tag{3.7}$$

Formulas (3.2) and (3.3), taking into account the expressions for constants (3.6), (3.7), can be written in the following general form:

$$u^{(\beta)}(y) = \sum_{j=0}^{N+2-\beta} \sum_{i=0}^{N} g_{ji}^{(\beta)} a_i T_j(y), \quad \beta = 0, 1, \tag{3.8}$$

where

$$g_{ji}^{(1)} = f_{ji}^{(1)} + \delta_{j,0} \frac{1}{2} \left(\overline{\delta}_i^{(0)} - \delta_i^{(0)}\right), \tag{3.9}$$

$$g_{ji}^{(1)} = f_{ji}^{(0)} + \delta_{j,1} \frac{1}{2} \left(\overline{\delta}_i^{(0)} - \delta_i^{(0)}\right) - \delta_{j,0} \frac{1}{2} (\delta_i^{(0)} - \overline{\delta}_i^{(0)}), \tag{3.10}$$

and β indicates the order of the derivative.

In these formulas

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad \text{is Kronecker symbol.}$$

Now, substituting series (3.1), (3.8) into equation (2.1) and equating the coefficients of the same degrees of polynomials, we have a linear algebraic system for determining the unknown expansion coefficients $a_0, a_1, \ldots, a_N$:

$$\sum_{k=0}^{N} \left[\varepsilon \delta_{ik} + \frac{1}{2} g_{ik}^{(1)}\right] a_k = b_i, \quad i = 0, 1, 2, \ldots, n. \tag{3.11}$$

The right hand sides b_i in system (3.11) are defined as follows. First, the connection is established

$$f(y) = \frac{1}{8}(y+1) = \sum_{i=0}^{N} b_i T_i(y),$$

then by the inverse transformation [8, 11, 12, 15] b_i is calculated:

$$b_i = \frac{1}{4NC_i} \sum_{l=0}^{N} \frac{2}{C_l} (y_l + 1) T_i(y_l), \quad i = 0, 1, 2, \ldots, N, \quad \text{where} \quad C_0 = C_N = 2, C_l = 1,$$

If $l \neq 0, N$ where $y_l = \cos\frac{\pi l}{N}$ are collocation nodes for Chebyshev polynomials of the first kind.

Let us present an algorithm for calculating constants $\delta_i^{(\beta)}, \overline{\delta_i}^{(\beta)}$ at $\beta = 0; 1$ in formulas (3.8)-(3.10).

These constants are calculated using the formulas:

$$\delta_i^{(\beta)} = \sum_{j=0}^{N+2-\beta} f_{ji}^{(\beta)}, \quad \overline{\delta_i}^{(\beta)} = \sum_{j=0}^{N+2-\beta} (-1)^j f_{ji}^{(\beta)}, \quad \beta = 0, 1,$$

where

$$f_{ji}^{(1)} = \delta_{j,i+1}\beta_i^{(1)} + \delta_{j,i-1}\zeta_i^{(1)},$$
$$f_{ji}^{(0)} = \delta_{j,i+2}\beta_i^{(0)} + \delta_{j,i}\zeta_i^{(0)} + \delta_{j,i-2}\mu_i^{(0)}.$$

Here the values of the constants are determined by the formulas:

$$\beta_i^{(1)} = \frac{C_i}{2(i+1)}, \quad i \geq 0; \quad \beta_i^{(0)} = \frac{\beta_i^{(1)}}{2(i+2)}, \quad i \geq 0; \ \zeta_i^{(1)} = \frac{-1}{2(i-1)}, \quad i \geq 2;$$
$$\zeta_i^{(0)} = \frac{\zeta_i^{(1)} - \beta_i^{(1)}}{2i}, \quad i \geq 1; \ \mu_i^{(0)} = \frac{-\zeta_i^{(1)}}{2(i-2)}, \qquad i \geq 3.$$

4. Calculation results

We present the results of numerical calculations obtained by the preliminary integration method for studying derivatives of different orders, when the value of the small parameter is equal to for different numbers of approximating polynomials N.

Table 1 shows the results of calculations, the values of the Chebyshev polynomials calculated at the nodes $y_l = \cos\frac{\pi l}{N}$ at $l = 0, 1, 2, \ldots, N$ when the number of polynomials is relatively small $N = 10$. For the selected value of the small parameter ($\varepsilon = 10^{-2}$) a strong difference between the value of the solution and its derivatives is observed.

Таблица 1: Numerical values of the solution for, $\varepsilon = 10^{-2}, N = 10$.

Nodes of polynomials y_l at 1	Exact values	Approximate values	Absolute error
1	-0.448	0.5657	1.0132
3	-0.4708	0.4792	0.9500
5	-0.37	0.5751	0.9451
7	-0.1828	0.7589	0.9417
9	-0.0239	0.8997	0.9236

From the results given in Table 1 it is evident that with a small number of polynomials the exact (2.3) and approximate solution (3.8) differ greatly. This difference is clearly shown in Figure 1.

Table 2 shows the numerical results of calculating the first derivative for the same values of parameters and N, calculated using formulas (2.4) and (3.8).

Таблица 2: Numerical values of the first derivative at, $\varepsilon = 10^{-2}, N = 10$.

Nodes of polynomials y_i at 1	Exact values	Approximate values	Absolute error
1	-24.505	-30.8403	6.3352
3	0.0981	1.0006	0.9025
5	0.2450	1.3939	1.1489
7	0.3919	1.7127	1.3207
9	0.4828	2.7067	2.2239

Graphical representations of the results of Table 2 are shown in Figure 2.

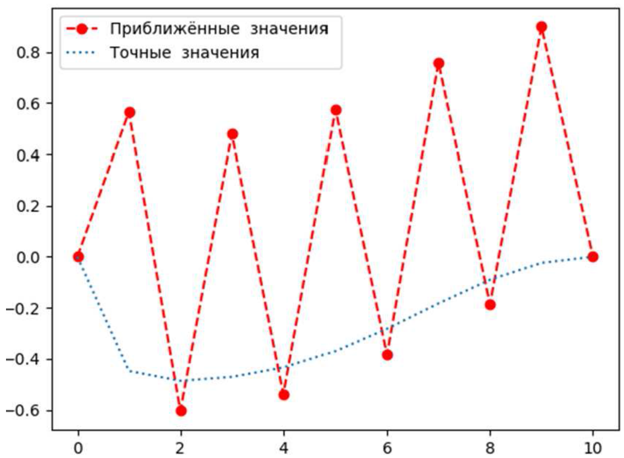

Рис. 1:
Dynamics of the solution for $\varepsilon = 10^{-2}, N = 10$.

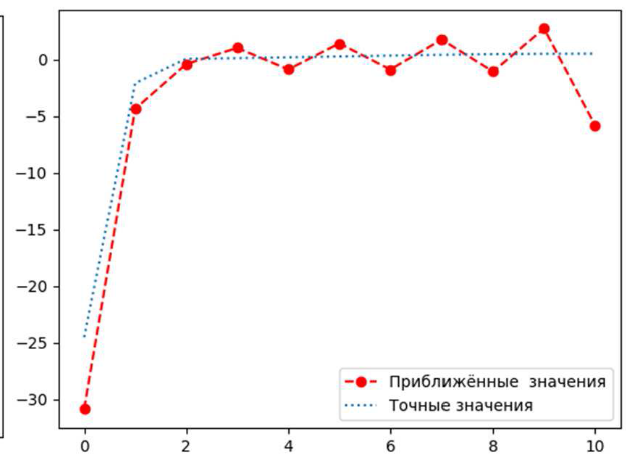

Рис. 2: Dynamics of the first derivative at, $\varepsilon = 10^{-2}, N = 10$.

Table 3 shows the results of calculating the second derivative; the exact values of the second derivative were calculated using formula (2.5), and the approximate values were determined using formula (3.1) using the preliminary integration method.

Таблица 3: Numerical values of the second derivative for $\varepsilon = 10^{-2}, N = 10$.

Nodes of polynomials y_i at 1	Exact values	Approximate values	Absolute error
1	106.2588	219.1197	112.8609
3	0.2500	-44.8771	45.1271
5	0.25	-57.1973	57.4473
7	0.25	-65.7870	66.0370
9	0.25	-110.9445	111.1945

The results of Table 3 are most clearly presented in Figure: 3

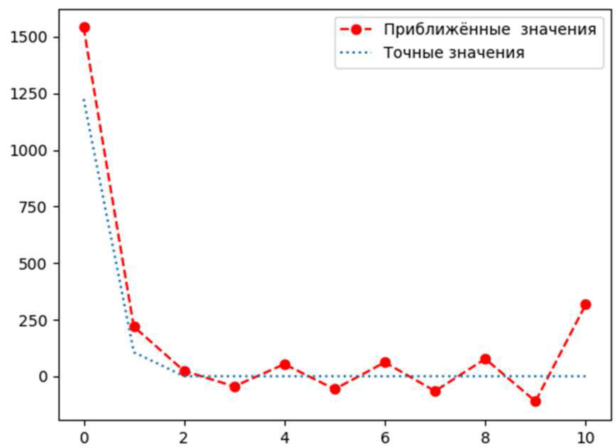

Рис. 3: Dynamics of the second derivative at $\varepsilon = 10^{-2}, N = 10$.

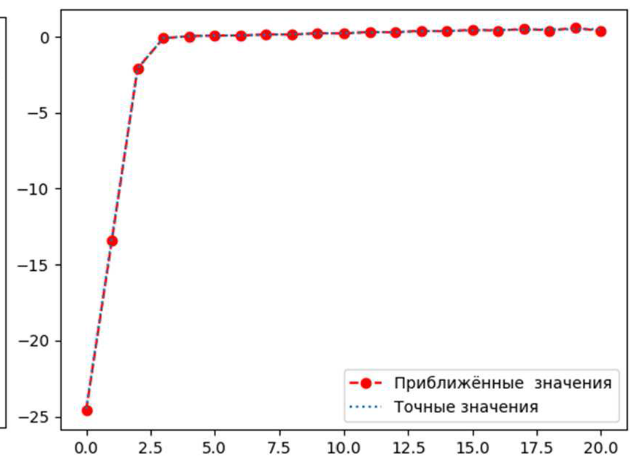

Рис. 4: Dynamics of the first derivative at, $\varepsilon = 10^{-2}, N = 20$.

From the results presented in Tables 1 -3 and Fig. 1-3 it is evident that with a small number of Chebyshev polynomials both in the solution and in the derivatives sawtooth jumps of high amplitude appear. It should be noted that the scale of change of the values of the second derivative increases sharply.

Now, leaving the value of the small parameter $\varepsilon = 10^{-2}$ unchanged, we gradually increase the number of approximating polynomials N.

Let us consider the case $N = 20$. In this case, the absolute error of the solution for the selected values of the parameters ε and N is a value of the order of 10^{-2}.

Table 4 shows the results of calculating the first derivative.

Таблица 4: Numerical values of the first derivative for $\varepsilon = 10^{-2}, N = 20$.

Nodes of polynomials y_i at 1	Exact values	Approximate values	Absolute error
1	13.2399	-13.4111	0.1712
6	0.0981	0.0800	0.0181
11	0.2841	0.3180	0.0339
16	0.4473	0.3990	0.0483

The results of Table 4 are graphically presented in Figure 4.

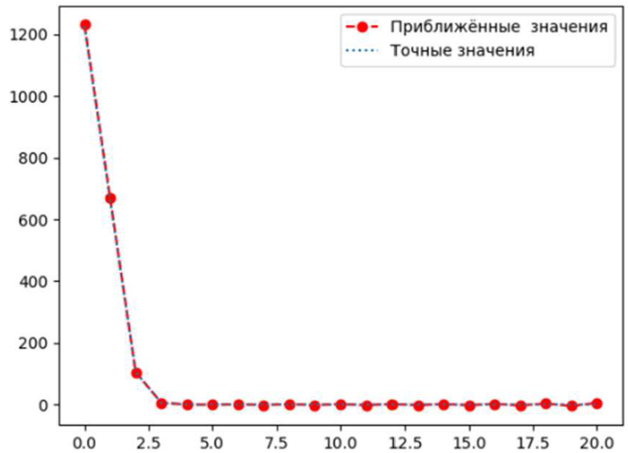

Рис. 5: Dynamics of the second derivative at $\varepsilon = 10^{-2}, N = 20$.

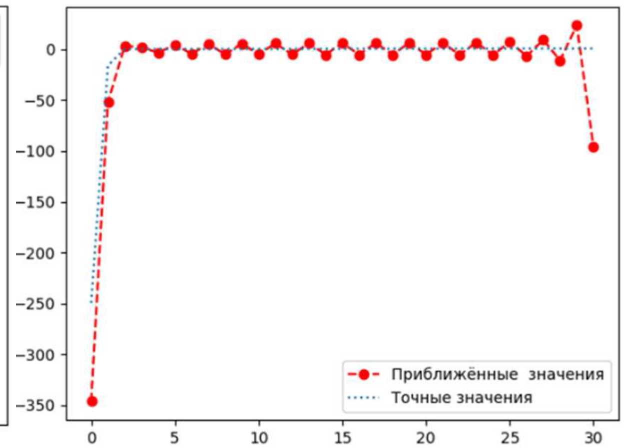

Рис. 6: Dynamics of the first derivative at, $\varepsilon = 10^{-3}, N = 30$.

Table 5 shows the results of calculations of the second derivative for the same values of parameters ε and N.

Таблица 5: Numerical values of the second derivative at $\varepsilon = 10^{-2}, N = 20$.

Nodes of polynomials y_i at 1	Exact values	Approximate values	Absolute error
1	662.1491	670.7086	8.5595
6	0.250001	1.1557	0.9057
11	0.25	-1.4457	1.6957
16	0.25	2.6602	2.4150

The results presented in Table 5 are graphically presented in Figure 5.

From the results of Table 4-5 and Fig. 4-5 it is evident that the exact and approximate values of the first and second order derivatives are very close. With a further increase in the number of approximating polynomials, these values are determined with high accuracy and it becomes practically impossible to distinguish them in graphical form. Therefore, we present the resulting Table 6, which includes the maximum absolute error for the first and second order derivatives with different numbers of polynomials.

Таблица 6: Maximum absolute error at $\varepsilon = 10^{-2}, N = 30\text{div}50$.

Number of polynomials N	Maximum absolute error		
	Solution	First derivative	Second derivative
30	$2.2 \cdot 10^{-5}$	$4.6 \cdot 10^{-4}$	$2.3 \cdot 10^{-3}$
40	$3.5 \cdot 10^{-5}$	$1.1 \cdot 10^{-6}$	$5.7 \cdot 10^{-5}$
50	$1.8 \cdot 10^{-15}$	$9.8 \cdot 10^{-11}$	$4.9 \cdot 10^{-9}$

In the further calculation results presented, the value of the small parameter ε is reduced by 10 times and the case is considered $\varepsilon = 10^{-3}$.

Figure 6 shows a comparison of the exact and approximate values of the first derivative for $\varepsilon = 10^{-3}, N = 30$.

The results of comparison of exact and approximate values of the second derivative for the same values of characteristic parameters ε and N are shown in Figure 7. It is evident that the scale of change of the second derivative increases greatly and becomes equal to $175 \cdot 10^3$.

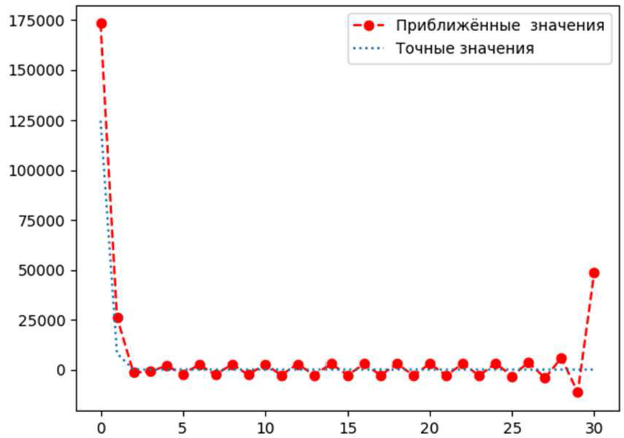

Рис. 7: Dynamics of the second derivative at $\varepsilon = 10^{-3}, N = 30$.

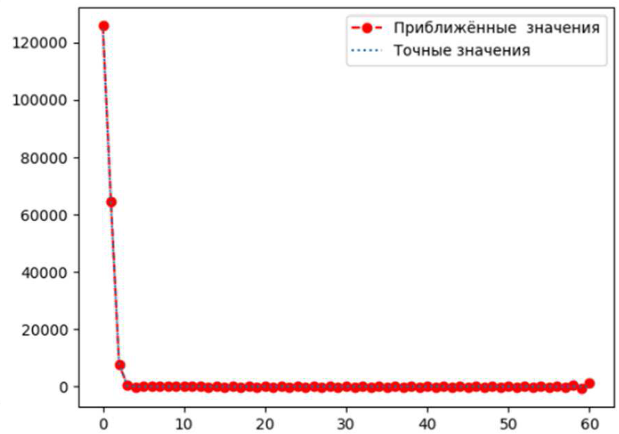

Рис. 8: Dynamics of the second derivative at, $\varepsilon = 10^{-3}, N = 60$.

From Figure 6-7 it is evident that decreasing the value of the small parameter ε strongly influences the dynamics of the first and second derivatives.

Figure 8 shows the exact and approximate values of the second derivative when the number of approximating polynomials is $N = 60$.

It is evident that increasing the number of polynomials significantly improves the accuracy of calculations. Finally, we present the resulting table 7, illustrating the change in the maximum absolute error depending on the number of approximating polynomials N at $\varepsilon = 10^{-3}$.

Таблица 7: Maximum absolute error at $\varepsilon = 10^{-3}$ и $N = 60\text{div}120$.

Number of polynomials N	Maximum absolute error		
	Solution	First derivative	Second derivative
60	$5.5 \cdot 10^{-3}$	0.5	240.4
70	$1.0 \cdot 10^{-6}$	$7.1 \cdot 10^{-3}$	238.3
80	$1.0 \cdot 10^{-6}$	$3.8 \cdot 10^{-3}$	19.1
90	$2.9 \cdot 10^{-5}$	$1.3 \cdot 10^{-3}$	6.9
100	$4.1 \cdot 10^{-6}$	$7.1 \cdot 10^{-4}$	0.3
120	$4.8 \cdot 10^{-10}$	$5.8 \cdot 10^{-6}$	$9.1 \cdot 10^{-3}$

From Table 7 it is evident that the application of the preliminary integration method allows us to study the dynamics of the solution of a non-homogeneous singularly perturbed differential equation and its derivatives with high accuracy for different values of the small parameter ε.

5. Conclusion

1. A highly accurate and effective method for studying inhomogeneous singularly perturbed differential equations has been constructed.

2. An algorithm for the preliminary integration method has been constructed.

3. The proposed method has been used to study the dynamics of derivatives of the solution of a singularly perturbed equation for different values of the small parameter.

4. The results of numerical calculations are presented in tabular and graphical form, illustrating the high accuracy and efficiency of the method.

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